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A REMARK ON GELFAND DUALITY

SHU-HAO SUN

In this paper, we prove a Gelfand-Mulvey type of duality for a certain class of rings which includes the Gelfand rings. We also show that the Maximal Ideal Theorem (MIT) can be replaced by the Prime Ideal Theorem (PIT) in the original Gelfand-Mulvey duality.

1. INTRODUCTION

The Gelfand duality theorem states that the functor from the category of compact T_2 spaces to the category of commutative C^*-algebras, obtained by assigning to each compact T_2 space X the commutative C^*-algebra $C(X)$ of continuous complex functions on X, determines a duality between these categories. The dual functor is that obtained by assigning to each commutative C^*-algebra A the compact T_2 space $\operatorname{Max} A$ consisting of the maximal ideals of A endowed with the hull-kernel topology (of which the subsets of $\operatorname{Max} A$ of the form $D(a) = \{M \in \operatorname{Max} A \mid a \notin M\}$ for each $a \in A$ form a basis of the topology). It may be remarked that the maximal ideal space of a commutative ring A is generally neither T_2 nor functorial on the category of commutative rings. The existence of this functor therefore depends on particular preperties of commutative C^*-algebras. To extend the Gelfand duality to (not necessarily commutative) rings, Mulvey [4] introduced the following notion: a ring A is called *Gelfand* if for any two distinct maximal right ideals M and M', there exist elements $a \notin M$ and $a' \notin M'$ such that $aAa' = 0$. It was shown in [4] that although the definition might appear to be that of a right Gelfand ring, the condition would turn out to be equivalent to that in terms of maximal left ideals and that the maximal ideal space of any Gelfand ring is compact T_2, and that the assignment to each ring A of the maximal ideal space $\operatorname{Max} A$ is functorial on the category of Gelfand rings and ring homomorphisms. In fact, Mulvey obtained the following:

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THEOREM. [4]. *The functor from the category of compact local ringed spaces to the category of Gelfand rings, obtained by assigning to each ringed space (X, O_X) the ring of sections $O_x(X)$, determines a duality between these categories.*

It may be remarked here that, although Mulvey's definition of Gelfand rings is sensible only in the presence of the maximal ideal Theorem (MIT) — whose validity for all rings is logically equivalent to the Axiom of Choice (AC) — it is possible to give another description of Mulvey's duality by using only the Prime Ideal Theorem (PIT) – which is strictly weaker than AC. It is also possible to extend Gelfand-Mulvey's duality to a wider class of rings. These are the main purposes of this paper.

In general, the inverse image of a maximal (right) ideal under a ring homomorphism need not necessarily be a maximal (right) ideal; for rings which are not commutative, the inverse image need not even be a prime ideal. For Gelfand rings, Mulvey has shown that the inverse image of a maximal right ideal determines a unique maximal ideal since each maximal right ideal is completely prime (an ideal P of a ring R is called completely prime if $ab \in P$ implies that $a \in P$ or $b \in P$) and the inverse of a completely prime ideal under a ring homomorphism is clearly completely prime. But a maximal ideal is not necessarily completely prime, even for strongly harmonic rings (a ring R is called *strongly harmonic* if for any two distinct maximal 2-sided ideals M and M', there exist elements $a \notin M$ and $a' \notin M'$ such that $aAa' = 0$ [2]). However, we shall show that each ring homomorphism $f : R_1 \to R_2$, where R_1 and R_2 are strongly harmonic rings such that each maximal 2-sided ideal contains a symmetric ideal (an ideal I of R is called *symmetric* if $abc \in I$ implies $acb \in I$) induces naturally a continuous mapping from $\operatorname{Max} R_2$ to $\operatorname{Max} R_1$.

2. MAIN RESULTS

Throughout the paper, all the rings are assumed to have identity and are not necessarily commutative. First we discuss a kind of ring which includes the class of strongly harmonic rings without using the Axiom of Choice (AC), and is the same using AC.

DEFINITION 1: Let R be a ring and $\operatorname{Id} R$ the lattice of all 2-sided ideals of R. Then we say that $\operatorname{Id} R$ is normal if for each pair $I_1, I_2 \in \operatorname{Id} R$ with $I_1 + I_2 = R$, there exist $J_1, J_2 \in \operatorname{Id} R$ such that

$$I_1 + J_1 = R = I_2 + J_2 \quad \text{and} \quad J_1 J_2 = 0.$$

As usual, if R is a ring, then $\operatorname{Spec} R$ denotes the space of all prime 2-sided ideals of R endowed with the hull-kernel topology. Define $S : \operatorname{Id} R \to \operatorname{Id} R$ by

$$S(I) = \cap\{P \in \operatorname{Spec} R \mid I \subseteq P\}$$

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for each $I \in \operatorname{Id} R$. Then it is easy to see that the image $S(\operatorname{Id} R)$ is isomorphic to the lattice of open sets of $\operatorname{Spec} R$. Thus the topological space $\operatorname{Spec} R$ is normal if and only if for each pair $I_1, I_2 \in \operatorname{Id} R$ with $S(I_1 + I_2) = R$, there exist $J_1, J_2 \in \operatorname{Id} R$ such that

$$S(J_1 + I_1) = R = S(J_2 + I_2) \quad \text{and} \quad S(J_1 J_2) = S(0).$$

Next, we note that if $S(I + J) = R$ and PIT holds, then $I + J = R$. In fact, if $1 \notin (I + J)$, then there is a prime ideal P containing $(I + J)$ which implies that $1 \notin S(I + J)$. Hence we have:

Lemma 1. *Let R be a ring and suppose PIT holds. Then $\operatorname{Spec} R$ is normal if and only if for each pair $I_1, I_2 \in \operatorname{Id} R$ with $I_1 + I_2 = R$, there exist $J_1, J_2 \in \operatorname{Id} R$ such that $I_1 + J_1 = R = I_2 + J_2$ and $S(J_1)S(J_2) \subseteq S(0)$.*

Since $S(IJ) = S(I) \cap S(J) \supseteq S(I)S(J)$, we immediately have:

Corollary. *Let R be a ring. If $\operatorname{Id} R$ is normal, then $\operatorname{Spec} R$ is normal.*

Theorem 1. *Let R be a ring and suppose PIT holds. If $\operatorname{Spec} R$ is normal, then each prime 2-sided ideal is contained in a unique maximal 2-sided ideal. Moreover, the maximal 2-sided ideal space $\operatorname{Max} R$ is a non-empty retract of $\operatorname{Spec} R$.*

Proof: First we show that for each $P \in \operatorname{Spec} R$, there exists a unique maximal ideal containing it. Define

$$F_P = \{I \in \operatorname{Id} R \mid I + P = R\}.$$

Then the family F_P satisfies

(i) If $I_1 + I_2 \in F_P$, then either $I_1 \in F_P$ or $I_2 \in F_P$.
(ii) If $I \in F_P$ and $I \subseteq J$, then $J \in F_P$.

In fact, if $I_1 + I_2 \in F_P$, then $I_1 + (I_2 + P) = R$ by definition. Then by Lemma 1, there are $J_1, J_2 \in \operatorname{Id} R$ such that $J_1 + I_1 = R = J_2 + I_2 + P$ and $S(J_1)S(J_2) \subseteq S(0)$. Moreover, we have either $J_1 \subseteq P$ or $J_2 \subseteq P$ since P is prime and hence either $I_1 + P = R$ or $I_2 + P = R$, which implies either $I_1 \in F_P$ or $I_2 \in F_P$ by the definition of F_P. (ii) is clear.

Now let

$$M_P = \sum \{I \in \operatorname{Id} R \mid I \notin F_P\}.$$

We see that $1 \notin M_P$ by Property (i); that is, M_P is proper. Since $P \notin F_P$, it follows that $P \subseteq M_P$. Next we show that M_P is a maximal ideal: if $I \not\subseteq M_P$, then $I \in F_P$ and so $I + P = R$ which imlpies that $I + M_P = R$. Hence we have shown that for each prime 2-sided ideal of R, there is a maximal 2-sided ideal containing it. Note that M_P is the unique maximal 2-sided ideal containing P since $M' = M_{M'} = M_P$ for each $M' \in \operatorname{Max} R$ with $M' \supseteq P$.

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Now we define a mapping $m : \operatorname{Spec} R \to \operatorname{Max} R$ by sending each $P \in \operatorname{Spec} R$ to $M_P \in \operatorname{Max} R$. It remains to show that m is continuous. Let O_I be the set of all prime 2-sided ideals which do not contain I, where $I \in \operatorname{Id} R$. Then O_I is a basic open subset of $\operatorname{Spec} R$ and it is easy to check that $O_I = O_{S(I)}$. Choose an O_I with $m(P) \in O_I \cap \operatorname{Max} R$. Then $m(P) \not\supseteq I$ so that $I + P = R$, and hence there are $J_1, J_2 \in \operatorname{Id} R$ such that $S(J_1)S(J_2) \subseteq S(0)$ and $J_1 + I = R = J_2 + P$; this means $J_2 \not\subseteq P$. We claim that

$$O_{J_2} \subseteq m^{-1}(O_I \cap \operatorname{Max} R).$$

In fact, if $P' \in O_{J_2}$, that is, $P' \not\supseteq S(J_2)$, then $P' \supseteq S(J_1)$ since $S(J_1)S(J_2) \subseteq S(0)$ and P' is prime. Thus $P' + I = R$ so that $I \in F_{P'}$ and $I \not\subseteq m(P')$. Hence m is continuous and this completes the proof. ∎

Corollary 1. *Let R be a ring such that* Id R *is normal and suppose that PIT holds. Then each prime 2-sided ideal of R is contained in a unique maximal 2-sided ideal.*

As immediate consequences, each non-empty closed subset of $\operatorname{Spec} R$ contains a maximal ideal and $\operatorname{Max} R$ is a non-empty compact T_2 space.

Lemma 2. *Let R be a ring such that* Id R *is normal. If F is a closed subset of* Spec R *and O is an open subset of* Spec R *satisfying $O \supseteq (F \cap \operatorname{Max} R)$, then PIT implies $O \supseteq F$.*

Proof: Suppose that $P \in F$ but $P \notin O$. Then the closure $\overline{\{P\}}$ of P is disjoint from O. On the other hand, $\overline{\{P\}} \cap F \cap \operatorname{Max} R \neq \emptyset$ by Corollary 1, a contradiction. ∎

Theorem 2. *If PIT holds, then* Id R *is normal if and only if R is strongly harmonic and satisfies MIT.*

Proof: Suppose that R is strongly harmonic and satisfies MIT. First we shall show that for any two disjoint non-empty closed subsets F_1 and F_2 of $\operatorname{Spec} R$, there exist J_1 and J_2 in $\operatorname{Id} R$ such that $F_1 \subseteq O_{J_1}$, $F_2 \subseteq O_{J_2}$ and $J_1 J_2 = 0$. By Corollary 1, we see that $\operatorname{Max} R$ is compact T_2 and that $F_1 \cap \operatorname{Max} R$ and $F_2 \cap \operatorname{Max} R$ are two non-empty disjoint closed subsets of $\operatorname{Max} R$ and hence are non-empty compact subsets of $\operatorname{Max} R$. Fix an $M \in F_1 \cap \operatorname{Max} R$. Then for each $M' \in F_2 \cap \operatorname{Max} R$, we have $I_{M'} \not\subseteq M'$ and $J_M \not\subseteq M$ with $I_{M'} J_M = 0$. By the compactness of $F_2 \cap \operatorname{Max} R$, we can find a finite number of 2-sided ideals, say $I_1, I_2, \dots, I_n, J_1, J_2, \dots, J_n$, such that $M \in O_{J_1 J_2 \dots J_n}$ and $F_2 \cap \operatorname{Max} R \subseteq O_{(I_1 + I_2 + \dots + I_n)}$ and $I_i J_i = 0, (i = 1, 2, \dots, n)$. Hence

$$(I_1 + I_2 + \dots + I_n)(J_1 J_2 \dots J_n) = 0.$$

Furthermore, by Lemma 2, we have $F_2 \subseteq O_{(I_1 + I_2 + \dots + I_n)}$. Repeating the above procedure, we finally find $I, J \in \operatorname{Id} R$ such that $IJ = 0$, $F_1 \subseteq O_I$ and $F_2 \subseteq O_J$.

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Now we are going to show that $\operatorname{Id} R$ is normal. For this, let $I_1, I_2 \in \operatorname{Id} R$ with $I_1 + I_2 = R$ and for $i = 1, 2$, let

$$D_{I_i} = \{P \in \operatorname{Spec} R \mid P \supseteq I_i\}.$$

Then D_{I_1} and D_{I_2} are disjoint closed subsets of $\operatorname{Spec} R$ and so there are $J_1, J_2 \in \operatorname{Id} R$ such that $O_{J_1} \supseteq D_{I_1}$, $O_{J_2} \supseteq D_{I_2}$ and $J_1 J_2 = 0$. We claim that $I_1 + J_1 = R = I_2 + J_2$. If $1 \notin I_1 + J_1$, then there is a prime 2-sided ideal P with $P \supseteq I_1 + J_1$, which implies that $P \in D_{I_1}$ since $P \supseteq I_1$. Since $O_{J_1} \supseteq D_{I_1}$, thus $P \in O_{J_1}$ which means $P \not\supseteq J_1$, a contradiction. Hence $I_1 + J_1 = R$. Similarly $I_2 + J_2 = R$.

The other implication follows from Theorem 1. This completes the proof. □

In a similar way, we obtain:

Theorem 3. (see [**9**, Theorem 2.3]) *If PIT holds, then the lattice* $\operatorname{Id}_r R$ *of right ideals of* R *is normal if and only if* R *is Gelfand and satisfies MIT.*

As an immediate consequence, the maximal right ideal space $\operatorname{Max}_r R$ of R is a non-empty retract of the spectrum $\operatorname{Spec}_r R$ of right ideals of R.

Now we study the functorial property of the assignment to each strongly harmonic ring of its maximal ideal space. Recall that an ideal I of a ring R is symmetric if $abc \in I$ implies $acb \in I$ for any $a, b, c \in R$. Then a ring R is called *feebly symmetric* if each maximal ideal (if it exists) of R contains a symmetric ideal.

Clearly a Gelfand ring is strongly harmonic, and since its maximal right ideal is symmetric, it is also feebly symmetric.

The existence of the sheaf representation of a strongly harmonic ring was established first by Koh [**2**] and then by Simmons [**7**]. We shall use the description provided by Simmons. Let R be a ring. Then $I \in \operatorname{Id} R$ is called *uniformly virginal* if for each $a \in I$

$$I + \operatorname{Ann}(aR) = R,$$

where $\operatorname{Ann}(X)$ denotes the right annihilator of X. We denote by ΨR the set of all such ideals. It was shown by Simmons [**7**, Theorem 2.4] that ΨR is a subframe of $\operatorname{Id} R$. Now for each $I \in \operatorname{Id} R$, we write $\operatorname{Wir}(I)$ the greatest uniformly virginal ideal contained in I. Then

Lemma 3. [**7**, Theorem 5.3] *Let* R *be a ring. Then the following conditions are equivalent:*

(i) *R is strongly harmonic.*
(ii) *For each $I \in \operatorname{Id} R$ and $M \in \operatorname{Max} R$, $\operatorname{Wir}(I) \subseteq M$ implies $I \subseteq M$.*
(iii) *If $I, J \in \operatorname{Id} R$ with $I + J = R$, then $\operatorname{Wir}(I) + \operatorname{Wir}(J) = R$.*

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REMARK 1. We note that if R is a ring with $\mathrm{Id}\,R$ normal, then properties (ii) and (iii) in Lemma 3 remain true, by using only PIT.

REMARK 2. These results can be generalised to a more general setting called right unital quantales (for the details, see [**10**]). Also we can extend the result in Theorem 2 as follows: A ring R is strongly harmonic if and only if for any two distinct maximal 2-sided ideals M and M' of R, there exist $I_1, I_2 \in \mathrm{Id}\,R$ such that $I_1 \not\subseteq M_1$, $I_2 \not\subseteq M_2$ and $I_1 \cap I_2 = 0$ if and only if for any $I_1, I_2 \in \mathrm{Id}\,R$ with $I_1 + I_2 = R$, there exist $J_1, J_2 \in \mathrm{Id}\,R$ such that $I_1 + J_1 = R = I_2 + J_2$ and $J_1 \cap J_2 = 0$. Similar results hold for Gelfand rings.

The sheaf representation $O_{\mathrm{Max}\,R}$ obtained is that of which the stalk of each $M \in \mathrm{Max}\,A$ is the factor ring $R/\mathrm{Wir}\,(M)$: the canonical isomorphism from R to the ring of sections is that which assigns to each $a \in R$ the section obtained by taking the canonical image $a_M \in R/\mathrm{Wir}\,(M)$ of $a \in R$, for each $M \in \mathrm{Max}\,R$.

DEFINITION 2: A ringed space (X, O_X) is called *quasi-local* if each stalk of O_X has a unique maximal 2-sided ideal and is called *feebly symmetric* if each stalk is feebly symmetric.

Following Mulvey [**4**] and [**6**], we have:

THEOREM 4. *Let (X, O_X) be any ringed space and let R be the ring of the global sections of (X, O_X). If (X, O_X) is compact and quasi-local, then $\mathrm{Id}\,R$ is normal. In addition, if each stalk of (X, O_X) is feebly symmetric, then R is feebly symmetric.*

PROOF: The existence of maximal 2-sided ideals of R follows from the fact that the ringed space (X, O_X) is compact and quasi-local. Then suppose that M and M' are distinct maximal 2-sided ideals of R. By the compactness theorem [**5**, Theorem 2.3] for ringed space, there exist $x, x' \in X$ such that

$$M \supseteq \{a \in R \mid a_x = 0\} \quad \text{and} \quad M' \supseteq \{a \in R \mid a_{x'} = 0\}.$$

Since the ringed space (X, O_X) is quasi-local, the stalk of O_X at each $x \in X$ has a unique maximal 2-sided ideal. Moreover, at any $x \in X$, the stalk of O_X is isomorphic to the ring R of global sections factored by the ideal $\{a \in R \mid a_x = 0\}$. The elements x and x' in X corresponding to two distinct maximal ideals are therefore distinct. Since X is compact and T_2, there exist two disjoint open neighbourhoods U and U' of x and x' in X, respectively. By the compactness of the ringed space (X, O_X), there exist $a, a' \in R$ having supports in U and U' respectively, and such that $a_x = 1$ and $a_{x'} = 1$ [**5**, Theorem 1.2]. Then $a \notin M$ and $a' \notin M'$. Furthermore, any product of elements of R which contains both a and a' in its expression must be zero, and so $aRa' = 0$. Hence R is strongly harmonic and satisfies MIT and so $\mathrm{Id}\,R$ is normal, using Theorem 2. Now, suppose in addition that (X, O_X) is feebly symmetric and

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$M \in \operatorname{Max} R$. Then there is $x \in X$ such that $M \supseteq \{a \in R \mid a_x = 0\}$ and hence the stalk $O_{X,x}$ is isomorphic to the ring R factored by the ideal $\{a \in R \mid a_x = 0\}$. Since each stalk is feebly symmetric, it follows that M contains a symmetric ideal of R so that R is feebly symmetric. This completes the proof. ∎

The converse is also true. We need the following results where we need to use Zorn's Lemma:

LEMMA 4. *Let R be a ring with* $\operatorname{Id} R$ *normal. If I is a symmetric ideal of R, then the minimal prime 2-sided ideal containing I is completely prime and every maximal 2-sided ideal containing I contains a completely prime ideal which contains I.*

PROOF: Since R/I is a symmetric ring, each minimal prime 2-sided ideal of R/I is completely prime (see [**8**, Lemma 3.2], [**1**] or [**3**]). The existence of minimal prime ideals is guaranteed by Zorn's Lemma. ∎

THEOREM 5. *If R is a ring satisfying* $\operatorname{Id} R$ *is normal, then the ringed space* $(\operatorname{Max} R, O_{\operatorname{Max} R})$ *is compact and quasi-local. In addition, if R is feebly symmetric, then Zorn's lemma implies that each stalk of* $(\operatorname{Max} R, O_{\operatorname{Max} R})$ *is feebly symmetric.*

PROOF: First we note that $\operatorname{Max} R$ is a non-empty compact T_2 space. Next, for any two distinct M_1 and M_2 in $\operatorname{Max} R$, we have, by Lemma 3(iii),

$$\text{Wir }(M_1) + \text{Wir }(M_2) = R.$$

Hence there are $a_1 \in \text{Wir }(M_1)$ and $a_2 \in \text{Wir }(M_2)$ with $a_1 + a_2 = 1$ and so $a_2 = 1 - a_1$ is a unit in $R/\text{Wir }(M_1)$, whence $(a_2)_{M_1} = 1$. On the other hand, $(a_2)_{M_2} = 0$ since $a_2 \in \text{Wir }(M_2)$. Hence $(\operatorname{Max} R, O_{\operatorname{Max} R})$ is a compact ringed space. The fact that $(MaxR, O_{MaxR})$ is a quasi-local follows from the fact that each maximal ideal of R contains the ideal $\text{Wir }(M)$ for a unique $M \in \operatorname{Max} R$ by Lemma 3(ii).

Finally, if, in addition, R is feebly symmetric, then for the last assertion, it suffices to show that each maximal 2-sided ideal of R contains a completely prime ideal, the fact of which follows from Lemma 4. This completes the proof. ∎

REMARK. If R is a ring satisfying the lattice $\operatorname{Id}_r R$ of right ideals of R is normal, then PIT suffices to imply that $(\operatorname{Max} R, O_{\operatorname{Max} R})$ is a compact local ringed space, in the sense of Mulvey [**4**].

Thus, using AC, we then can show that the assignment to each ring R of its compact quasi-local ringed space $(\operatorname{Max} R, O_{\operatorname{Max} R})$ determines a functor from the dual of the category of strongly harmonic rings to the catogory of compact quasi-local ringed spaces. However, to show that the restriction of the above assignment to the subcategory consisting of those rings R such that $\operatorname{Id}_r R$ is normal, determines a functor, we need to use only PIT. To do this, we need one more notion. For each maximal 2-sided ideal

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M (if it exists) of R, we write Q_M for the union of all completely prime ideals (if any exists) of R contained in M. Using Lemma 4, we immediately have:

Lemma 5. *Let R be a ring and M a maximal 2-sided ideal. Then each union of symmetric ideals contained in M is contained in Q_M.*

Now for a given ring R, let $\mathrm{Qcp}\, R$ be the collection of all non-empty unions of completely prime ideals which is contained in some proper ideals of R. For $I \in \mathrm{Id}\, R$, we define

$$O_I = \{P \in \mathrm{Qcp}\ R \mid I \not\subseteq P\},$$

and consider the collection

$$\{O_I : I \in \mathrm{Id}\, R\}$$

to be a subbase of a topology on $\mathrm{Qcp}\, R$. Then the subspace $\mathrm{Cspec}\, R$ of $\mathrm{Qcp}\, R$, consisting of all completely prime ideals of R is precisely the subspace of $\mathrm{Spec}\, R$. Moreover, we have:

Lemma 6. *Let R be a ring and suppose that* $\mathrm{Id}\, R$ *is normal. Then* $\mathrm{Max}\, R$ *is a retract of $\mathrm{Qcp}\ R$.*

Proof: Define $\mu : \mathrm{Qcp}\ R \to \mathrm{Max}\, R$ by assigning each element Q of $\mathrm{Qcp}\ R$, the unique maximal 2-sided ideal containing Q. We need to prove that μ is continuous. Let $\mathcal{F}$ be a closed subset of $\mathrm{Max}\, R$. Then we have to show that $\mu^{-1}(\mathcal{F})$ is closed in $\mathrm{Qcp}\ R$. For this purpose, let

$$F = \bigcap\{M \in \mathcal{F} \mid M \in \mathrm{Max}\, R\} \quad \text{and} \quad I = \bigcap\{Q \in \mathrm{Qcp}\ R \mid \mu Q \in \mathcal{F}\}.$$

Then we observe that

$$I = \bigcap\{P \in \mathrm{Cspec}\ R \mid \mu(P) \in \mathcal{F}\}.$$

Let $Q \supseteq I$ where $Q \in \mathrm{Qcp}\ R$; we have to show that $\mu(Q) \in \mathcal{F}$. We first observe that if Q is a prime 2-sided ideal and

$$Q \subseteq B = \bigcup\{M \mid M \in \mathcal{F}\},$$

then the unique maximal 2-sided ideal containing Q is in $\mathcal{F}$. In fact, we see that $Q + F$ is an ideal contained in B; hence there is a maximal 2-sided ideal M which contains $Q + F$. Since $M \supseteq F$ and $\mathcal{F}$ is closed, thus $M \in \mathcal{F}$. Moreover, M is the unique maximal ideal containing Q.

Now we want to find a prime 2-sided ideal P which is contained in $Q \cap B$ (and hence $\mu P = \mu Q$ is in $\mathcal{F}$, as required). Consider the multiplicative system

$$S = \{s_1 t_1 s_2 t_2 \cdots s_n t_n \mid s_i \notin B, t_i \notin Q, i = 1, 2, \ldots, n,\ n \in N\}.$$

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We claim that $\mathcal{S}$ does not contain 0. Suppose that $s_1t_1s_2t_2\cdots s_nt_n = 0 \in \mathcal{S}$. Put $t = t_1t_2\cdots t_n$. Then $t \notin Q$, since $Q \in \text{Qcp } R$. Hence there exists $P' \in \text{Cspec } R \cap \mu^{-1}(\mathcal{F})$ such that $t \notin P'$. Since $P' \subseteq B$, it follows that each $s_i \notin P'$, and since P' is a completely prime ideal, we conclude that $s_1s_2\cdots s_n \notin P'$, a contradiction. Hence there exists a prime ideal P disjoint from the multiplicative system, and so, in particular, $P \subseteq Q \cap B$. The continuity of μ now follows. ∎

Lemma 7. *Let $f : R_1 \to R_2$ be a ring homomorphism. Then*

$$\phi = f^{-1} : \text{Qcp } R_2 \to \text{Qcp } R_1$$

is a continuous mapping.

Proof: First we note that ϕ sends each completely prime ideal of R_2 to a completely prime ideal of R_1, and thus sends each union of completely prime ideals of R_2 to a union of completely prime ideals of R_1. Hence ϕ is well-defined.

Now for each non-empty open set O_I of $\text{Qcp } R_1$, we shall show that $\phi^{-1}(O_I)$ is open in $\text{Qcp } R_2$. In fact, this follows from

$$\cup P_i \in \phi^{-1}(O_I) \iff \phi(\cup P_i) \in O_I$$

$$\iff \phi(\cup P_i) \not\supseteq I \iff f^{-1}(\cup P_i) \not\supseteq I$$

$$\iff \cup P_i \not\supseteq f(I) \iff \cup P_i \in O_{f(I)}.$$

∎

Lemma 8. *The assignment ψ to each maximal ideal M of the union Q_M of all completely prime ideals contained in M, is a continuous mapping from* $\text{Max } R$ *to* $\text{Qcp } R$.

Proof: Let $\mathcal{F}$ be a closed subset of $\text{Qcp } R$, and put

$$F = \bigcap\{P \mid P \in \mathcal{F}\} \quad \text{and} \quad I = \bigcap\{M \mid Q_M \in \mathcal{F}\}.$$

We have to show that if $M \in \text{Max } R$ with $M \supseteq I$, then $Q_M \in \mathcal{F}$. First we see that $F \subseteq I$ and F is a union of symmetric ideals since it is the intersection of unions of prime symmetric ideals. Now let M be a maximal ideal containing I. Since $F \subseteq I$, M contains F so that $P_M \supseteq F$ by Lemma 5 and hence $P_M \in \mathcal{F}$ since $\mathcal{F}$ is closed. Thus the proof is completed. ∎

Theorem 6. *Let R be a feebly symmetric ring satisfying* $\text{Id } R$ *is normal. Then the mapping ψ defined in Lemma 8 is a homeomorphism from* $\text{Max } R$ *to a subspace*

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of Qcp R, consisting of those completely prime ideals with the form Q_M for some $M \in \operatorname{Max} R$.

PROOF: It is clear that ψ is injective since each prime ideal is contained in a unique maximal ideal. The restriction of the mapping μ, defined in Lemma 6, is the continuous inverse of ψ. This completes the proof. ∎

By Theorem 6, we can then identify $\operatorname{Max} R$ as a subspace of Qcp R.

Now for each ring homomorphism $f : R_1 \to R_2$, define

$$\rho_f = \mu f^{-1} \psi,$$

where μ and ψ are as given in Lemmas 6 and 8, respectively. The induced mapping from $\operatorname{Max} R_2$ to $\operatorname{Max} R_1$ is continuous, by Lemmas 6, 7 and 8. Therefore this determines a functor from the dual of the category of feebly symmetric rings satisfying $\operatorname{Id} R$ are normal and ring homomorphisms to the category of compact Hausdorff spaces and continuous mappings.

THEOREM 7. *The assignment to each ring of the maximal ideal space $\operatorname{Max} R$ is functorial on the category of feebly symmetric rings, with $\operatorname{Id} R$ normal, and ring homomorphisms.*

PROOF: Let

$$R_1 \xrightarrow{f} R_2 \xrightarrow{g} R_3$$

be ring homomorphisms. We have to show that

$$\rho_{gf} = \rho_f \rho_g : \operatorname{Max} R_3 \to \operatorname{Max} R_1,$$

or equivalently;

$$\mu_1 f^{-1} \psi_2 \mu_2 g^{-1} \psi_3 = \mu_1 f^{-1} g^{-1} \psi_3.$$

In fact, for each $M \in \operatorname{Max} R_3$, $\mu_2 g^{-1} \psi_3(M)$ is the unique maximal ideal containing $g^{-1}\psi_3(M)$, which is a union of completely prime ideals; and $\psi_2 \mu_2 g^{-1} \psi_3(M)$ is the union of all completely prime ideals contained in $\mu_2 g^{-1} \psi_3(M)$, so that

$$\psi_2 \mu_2 g^{-1} \psi_3(M) \supseteq g^{-1} \psi_3(M).$$

Hence

$$f^{-1}\left(\psi_2 \mu_2 g^{-1} \psi_3(M)\right) \supseteq f^{-1}\left(g^{-1} \psi_3(M)\right).$$

Now the conclusion follows from the fact that each prime 2-sided ideals is contained in a unique maximal ideal. ∎

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DEFINITION 3: A ring homomorphism $f : R_1 \to R_2$ is called fibered if for any $M_1 \in \operatorname{Max} R_1$ and for any $a \in \operatorname{Wir}(M_1)$, we have $f(a) \in \operatorname{Wir}(M_2)$, for each $M_2 \in \operatorname{Max} R_2$ with $\rho_f(M_2) = M_1$.

In [5], it was shown that if R_1 and R_2 are Gelfand then each ring homomorphism is fibered. Hence we have the following generalisations of Gelfand-Mulvey duality.

THEOREM 8. (i) *The functor from the category of compact quasi-local ringed spaces to the category of feebly symmetric strongly harmonic rings and fibred ring homomorphisms, obtained by assigning to each ringed space* (X, O_X) *the ring of sections, determines a duality between them.*

(ii) *The restriction of the above functor to the subcategory consisting of those compact local ringed spaces to the category of those rings* R *such that* $\operatorname{Id}_r R$ *is normal and ring homomorphisms, determines a duality between them and in this case we need to use only PIT.*

REMARK. We still do not know whether a feebly symmetric strongly harmonic ring is Gelfand.

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Department of Pure Mathematics
University of Sydney
New South Wales 2006
Australia

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