Canad. Math. Bull.Vol. **36** (1), 1993 pp. 3–7

SOME CHARACTERIZATIONS OF DEDEKIND α-COMPLETENESS OF A RIESZ SPACE

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ABSTRACT. A vector lattice F is said to be Dedekind α-complete, where α is a cardinal number, provided that each non-empty order bounded subset D of F satisfying $\text{card}(D) \le \alpha$ has a supremum. Several characterizations of this property are presented here.

1. **Introduction.** In [AG] it was shown that, for E and F Archimedean Riesz spaces, the space of all regular operators from E into F forms a Riesz space for all choices of E precisely when F is Dedekind complete. In the course of that proof it is shown that if the regular operators from $\ell_0^\infty(\mathbb{N})$ into F forms a Riesz space, then F is Dedekind σ-complete (see §2 for definitions). The converse to the last statement is also true and it is proved in [W], Theorem 5.2. Our aim in this paper is to show that we can characterize Dedekind α-complete Riesz spaces, F, as those for which the regular operators from $\ell_0^\infty(I)$, where I is a set of cardinality α, into F form a Riesz space.

The proof that we offer needs a transfinite induction argument. Whilst it is obvious that a Riesz space is Dedekind σ-complete if and only if every *increasing* sequence has a supremum, the corresponding result for α-completeness apparently does not seem to be known. This is surprising in view of the fact that the use of transfinite sequences, *i.e.*, the families which are order isomorphic to ordinals, has been considered rather important (for example, even the original definition of order continuous functionals given in [KVP, page 406], was given in terms of transfinite sequences) and was the subject of some investigation, especially by the school of L. V. Kantorovich [AV], [A], [VG]. What is even more surprising is that, though, as was shown in [AV], the transfinite sequences are insufficient to characterize some "classical" properties in Dedekind complete Banach lattices*, nevertheless, as we will show, they are sufficient to characterize Dedekind α-completeness.

This work was made possible by a NATO Collaborative Research Grant CRG-890909 which allowed Dr. Wickstead to visit IUPUI during the Summer of 1990.

The second author would like to thank all those who made his stay at IUPUI so enjoyable, especially Professors Abramovich, Aliprantis, Burkinshaw and Ng.

Received by the editors July 3, 1991 .

AMS subject classification: Primary: 47B55; secondary: 46A40.

* For example, it is shown in [AV] that transfinite sequences are not enough to characterize the Levi property, *i.e.*, there exists a Dedekind complete Banach lattice without the Levi property, but in which every norm bounded transfinite sequence has a supremum.

https://doi.org/10.4153/CMB-1993-001-3 Published online by Cambridge University Press

2. **Preliminaries.** Recall that a Riesz space F is *Dedekind complete* if every non-empty subset of F, which is bounded above, has a supremum. If α is a cardinal, then F is said to be *Dedekind α-complete* if every non-empty subset of cardinality at most α, which is bounded above, has a supremum. If $\alpha = \aleph_0$, the first infinite cardinal, then this property is usually called *Dedekind σ-completeness*.

If E and F are Riesz spaces, then a linear operator $T: E \longrightarrow F$ is *positive* provided $x \in E_+ \Rightarrow Tx \in F_+$. The positive operators from E into F form a cone, which induces an order on the linear space of differences of all positive operators, $L^r(E,F)$, the so-called *regular* operators from E into F. In general this ordered linear space will not be a Riesz space. It is if F is Dedekind complete. The Theorem of [AG] asserted precisely that it is only the Dedekind complete F for which $L^r(E,F)$ is a Riesz space for all choices of E. The Dedekind complete Riesz spaces F have an even stronger property. An operator from E into F is termed *order bounded* if it maps order bounded sets in E to order bounded sets in F. We denote the space of all order bounded operators from E into F by $L^b(E,F)$. If F is Dedekind complete, then $L^r(E,F) = L^b(E,F)$, *i.e.*, all order bounded operators from E into F are regular. This last property does *not* characterize Dedekind complete Riesz spaces F ([AG], Proposition 2), but the assumption that $L^b(E,F)$ is a Riesz space does ([AG], Theorem).

By $\ell_0^\infty(I)$ we will denote the space of all real-valued functions on the set I which are constant except on a finite set. When given the usual linear operations and the pointwise partial order this is a Riesz space. We will denote the constantly one function in $\ell_0^\infty(I)$ by $\mathbf{1}$ and use $\mathbf{e}_i$ to denote the characteristic function of $\{i\}$. Note that the set $\{\mathbf{e}_i : i \in I\} \cup \{\mathbf{1}\}$ is a Hamel basis for $\ell_0^\infty(I)$. If α is a cardinal and the cardinality of a set I is α, then we write simply $\ell_0^\infty(\alpha)$ instead of $\ell_0^\infty(I)$. An extensive study of regular operators from or into space $\ell_0^\infty(\mathbb{N})$ is presented in [AW].

We refer the reader to [AB], [LZ] or [V] for any unexplained terms from the theory of Riesz spaces.

3. **The characterization.**

THEOREM. *For a fixed cardinal number α the following conditions on a Riesz space F are equivalent:*

(1) Any subset of F of cardinality at most α, which has an upper bound, has a supremum.

(2) If η is an ordinal of cardinality at most α, $f: \eta \longrightarrow F$ is an increasing function and $f(\eta)$ has an upper bound, then $f(\eta)$ has a supremum.

(3) If η is an initial ordinal of cardinality at most α, $f: \eta \longrightarrow F$ is an increasing function and $f(\eta)$ has an upper bound, then $f(\eta)$ has a supremum.

(4) $L^b\big(\ell_0^\infty(\alpha), F\big)$ is a Riesz space.

(5) $L^r\big(\ell_0^\infty(\alpha), F\big)$ is a Riesz space.

PROOF. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and that (4) $\Rightarrow$ (5).

In order to establish that (3) $\Rightarrow$ (1), we may suppose that there is some subset of F with an upper bound but no supremum (else (1) is certainly true). Let β be the smallest

https://doi.org/10.4153/CMB-1993-001-3 Published online by Cambridge University Press

cardinal of a set $B \subset F$ which is bounded above but has no supremum. Let η be the first ordinal of cardinality equal to β. Index B by the elements of η, so that $B = \{b_i : i < \eta\}$. For each $i < \eta$, card$(i) < \beta$, so that $f(i) = \sup\{b_j : j < i\}$ exists in F, by the definition of β. Now $f\colon \eta \longrightarrow F$ is an increasing function which is bounded above. To establish (1) we need to prove that $\beta > \alpha$. If not, then $\beta \leq \alpha$ so that (3) guarantees that $\sup f(\eta)$ exists. Clearly $\sup f(\eta) \geq f(i+1) \geq b_i$ for each $i \in \eta$, so that $\sup f(\eta)$ is an upper bound for B. On the other hand any upper bound c for B will also be an upper bound for each set $\{b_j : j < i\}$, so $c \geq f(i)$ and hence $c \geq \sup f(\eta)$. Thus $\sup f(\eta)$ is the supremum of B, contrary to hypothesis. Thus $\beta > \alpha$ and hence (1) holds.

In order to prove that (1) $\Rightarrow$ (4), notice first that for every $x \in \ell_0^\infty(\alpha)_+$ we may find a subset A of the order interval $[0,x]$ of cardinality at most α, which is dense for the supremum norm. If $T\colon \ell_0^\infty(\alpha) \longrightarrow F$ is order bounded, let $T([-\mathbf{1},\mathbf{1}]) \subseteq [-y,y]$ then $T(A)$ is relatively uniformly dense in $T([0,x])$ with respect to y. By (1) $T(A)$ has a supremum in F which must also be the supremum of $T([0,x])$. It is now routine to define T^+ on $\ell_0^\infty(\alpha)_+$ by $T^+x = \sup T([0,x])$ and extend it to a linear operator on the whole of $\ell_0^\infty(\alpha)$. The operator T^+ is the supremum of T and the zero operator showing that $L^b\big(\ell_0^\infty(\alpha), F\big)$ is a Riesz space.

Finally we will prove that (5) $\Rightarrow$ (3). Before doing this we show that if (5) holds for α then it also holds for any infinite cardinal $\beta < \alpha$. We may suppose that $\beta \subset \alpha$ and define $J\colon \ell_0^\infty(\beta) \longrightarrow \ell_0^\infty(\alpha)$ by extending elements of $\ell_0^\infty(\beta)$ to have a constant value on $\alpha \setminus \beta$ (the same value that they take on all but a finite number of points of β). We also have the restriction map $R\colon \ell_0^\infty(\alpha) \longrightarrow \ell_0^\infty(\beta)$ and clearly $R \circ J$ is the identity on $\ell_0^\infty(\beta)$. Note that both R and J are positive. If $T \in L^r\big(\ell_0^\infty(\beta), F\big)$, then $T \circ R \in L^r\big(\ell_0^\infty(\alpha), F\big)$, so has a positive part $(T \circ R)^+$. Consider $(T \circ R)^+ \circ J \in L^r\big(\ell_0^\infty(\beta), F\big)$. Obviously this operator is positive. If $x \in \ell_0^\infty(\beta)_+$, then $Jx \geq 0$ so that $(T \circ R)^+(Jx) \geq (T \circ R)(Jx) = Tx$. Thus $(T \circ R)^+ \circ J$ is a positive majorant for T. If S is any other positive majorant for T, then $S \circ R \geq T \circ R, 0$ so that $S \circ R \geq (T \circ R)^+$, and hence $S = S \circ R \circ J \geq (T \circ R)^+ \circ J$, showing that $(T \circ R)^+ \circ J$ is actually the positive part of T and $L^r\big(\ell_0^\infty(\beta), F\big)$ is indeed a Riesz space.

Now suppose that (3) fails. Let η be the initial ordinal of lowest cardinality for which it fails. Then $\beta = \text{card}(\eta) \leq \alpha$. In view of the preceding paragraph we know that $L^r\big(\ell_0^\infty(\eta), F\big)$ is a Riesz space. Let $f\colon \eta \longrightarrow F$ be any increasing function for which $f(\eta)$ has an upper bound but no supremum. Without loss of generality we may suppose that $f(0) = 0$, the zero element in F. Define $T\colon \ell_0^\infty(\eta) \longrightarrow F$ as follows

$$T\mathbf{1} = 0$$
$$T(\mathbf{e}_0) = f(0)$$
$$T(\mathbf{e}_i) = f(i) - \bigvee_{j<i} f(j), \quad i \in \eta$$

This supremum exists as card$(i) < \beta$ and because if (3) holds with α replaced by card(i) then so does (2). If c is any upper bound for $f(\eta)$ in F, then we may define $T_c\colon \ell_0^\infty(\eta) \longrightarrow F$

https://doi.org/10.4153/CMB-1993-001-3 Published online by Cambridge University Press

by

$$T_c\mathbf{1} = c$$
$$T_c(\mathbf{e}_i) = T(\mathbf{e}_i).$$

We claim that T_c is a positive majorant for T, thus showing that T is regular. If $x = \mathbf{1} + \sum_{k\in K} x_k\mathbf{e}_k \in \ell_0^\infty(\eta)_+$, where $K = \{k_1, k_2, \ldots, k_n\}$ is a finite subset of η with $k_1 > k_2 > \cdots > k_n$, then (noting that each $x_k \geq -1$ and that $T(\mathbf{e}_k) \geq 0$) we have

$$\begin{aligned} T_c(x) &= T_c\mathbf{1} + \sum_{k\in K} x_k T(\mathbf{e}_k) \\ &\geq T_c\mathbf{1} - \sum_{k\in K} T(\mathbf{e}_k) \\ &= c - \Big[f(k_1) - \bigvee_{j<k_1} f(j) + f(k_2) - \bigvee_{j<k_2} f(j) + \cdots + f(k_n) - \bigvee_{j<k_n} f(j)\Big] \\ &\geq c - f(k_1) + \bigvee_{j<k_n} f(j) \\ &\geq c - f(k_1) \geq 0. \end{aligned}$$

Also, for the same x as above, we have

$$\begin{aligned} (T_c - T)(x) &= \big(T_c(\mathbf{1}) - T(\mathbf{1})\big) + \sum_{k\in K}\big(T_c(\mathbf{e}_k) - T(\mathbf{e}_k)\big) \\ &= c \geq f(0) = 0. \end{aligned}$$

Since any positive element of $\ell_0^\infty(\eta)$ is a positive multiple of such an x, it follows that $T_c \geq T, 0$ as claimed.

By hypothesis, T^+ exists in $L^r\big(\ell_0^\infty(\eta), F\big)$. If K is any finite subset of η then we have

$$\mathbf{1} \geq \sum_{k\in K} \mathbf{e}_k \geq 0$$

so that

$$T^+(\mathbf{1}) \geq T^+\Big(\sum_{k\in K} \mathbf{e}_k\Big) \geq \sum_{k\in K} T(\mathbf{e}_k).$$

We claim that if $i < \eta$, then any upper bound for all the sums $\sum_{k\in K} T(\mathbf{e}_k)$, where K is a finite set for which each $k \in K$ is at most i, must be at least $f(i)$. If not, let i_0 be the first ordinal for which this fails. Then $i_0 \neq 0$ as

$$\bigvee_{k\leq 0} T(\mathbf{e}_k) = T(\mathbf{e}_0) = f(0)$$

by definition. Otherwise, if u is an upper bound for all such sums $\sum_{k\in K} T(\mathbf{e}_k)$, with each $k \leq i_0$, then $u - T(\mathbf{e}_{i_0})$ will be an upper bound for all sums $\sum_{k\in K} T(\mathbf{e}_k)$, where each $k < i_0$. In particular, for each $j < i_0$, it will be an upper bound for the sums $\sum_{k\in K} T(\mathbf{e}_k)$ where each $k \leq j$. By definition of i_0, any such upper bound is at least $f(j)$. Thus $u - T(\mathbf{e}_{i_0}) \geq f(j)$ for all $j < i_0$. That is, we must have

$$u \geq T(\mathbf{e}_{i_0}) + \bigvee_{j<i_0} f(j) = f(i_0).$$

In other words we have proved that $T^+(\mathbf{1}) \geq f(i)$ for all $i \in \eta$. For any upper bound c of $f(\eta)$, T_c is a positive majorant of T so that $T_c \geq T^+$. Thus $c = T_c(\mathbf{1}) \geq T^+(\mathbf{1})$ and $T^+(\mathbf{1})$ is the supremum of $f(\eta)$. This contradicts the choice of η and f, so that we indeed have (3) holding.

ADDED IN PROOF. The following result of D. Fremlin and M. Laszkovich has been given recently in [SW]. We are mentioning it here as it also describes a situation (similar to those discussed in the introduction) where the transfinite sequences suffice.

THEOREM (FREMLIN-LASZKOVICH). *Let P be a partially ordered set such that each upper bounded, well-ordered subset of P has a supremum. Then each upper bounded, directed subset of P has a supremum.*

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