On the Direction-Cosines of the Axes of the Conicoid $f(x\,y\,z) \equiv a\,x^2+b\,y^2+c\,z^2+2\,f\,y\,z+2\,g\,z\,x+2\,h\,x\,y=1$.

Some time ago I received from Dr Muirhead the following theorem:—"If l_r, m_r, n_r $(r=1, 2, \dots 3)$ are the direction-cosines of the axes of the conicoid

$$f(x, y, z)=1, \quad f\,l_1\,l_2\,l_3+g\,m_1\,m_2\,m_3+h\,n_1\,n_2\,n_3=0."$$

In this note a proof and extension of the theorem are given.

The equations for the direction-cosines are

$$\frac{a\,l+h\,m+g\,n}{l}=\frac{h\,l+b\,m+f\,n}{m}=\frac{g\,l+f\,m+c\,n}{n}. \quad \dots\dots(1)$$

Therefore $$a+\frac{h\,m_2}{l_2}+\frac{g\,n_2}{l_2}=\frac{h\,l_2}{m_2}+b+\frac{f\,n_2}{m_2},$$

and $$a+\frac{h\,m_3}{l_3}+\frac{g\,n_3}{l_3}=\frac{h\,l_3}{m_3}+b+\frac{f\,n_3}{m_3}.$$

Subtracting, and remembering that $m_2\,n_3-m_3\,n_2=l_1$, etc., we obtain

$$\frac{g\,m_1-h\,n_1}{l_2\,l_3}=\frac{h\,n_1-f\,l_1}{m_2\,m_3}=(\text{similarly})\ \frac{f\,l_1-g\,m_1}{n_2\,n_3}.$$

Multiplying numerators and denominators by $f\,l_1$, $g\,m_1$, and $h\,n_1$ respectively, and adding, we get

$$f\,l_1\,l_2\,l_3+g\,m_1\,m_2\,m_3+h\,n_1\,n_2\,n_3=0. \quad \dots\dots\dots\dots\dots\dots (2)$$

If $D\equiv a\,b\,c+2fgh-af^2-bg^2-ch^2$ and $A\equiv b\,c-f^2$, etc., we find that each ratio in (1)

$$=\frac{D\,l}{A\,l+H\,m+G\,n}=\frac{D\,m}{H\,l+B\,m+F\,n}=\frac{D\,n}{G\,l+F\,m+C\,n}$$

(Geometrically, these follow from the fact that a cone and its reciprocal are coaxal, and (1) gives the direction-cosines of the axes of the cone $f(x, y, z)=0$).

Therefore as above, we prove

$$F\,l_1\,l_2\,l_3+G\,m_1\,m_2\,m_3+H\,n_1\,n_2\,n_3=0. \quad \dots\dots\dots\dots\dots (3)$$

From (2) and (3)

$$\frac{l_1\,l_2\,l_3}{gH-h\,G}=\frac{m_1\,m_2\,m_3}{h\,F-f\,H}=\frac{n_1\,n_2\,n_3}{f\,G-g\,F}.$$

Now if the axes are OA, OB, OC, the cone through the coordinate axes and OA, OB, OC is easily seen to be

$$\frac{l_1\,l_2\,l_3}{x}+\frac{m_1\,m_2\,m_3}{y}+\frac{n_1\,n_2\,n_3}{z}=0,$$

and the cone which touches the planes BOC, COA, AOB and the coordinate planes is

$$\sqrt{l_1 l_2 l_3 x} + \sqrt{m_1 m_2 m_3 y} + \sqrt{n_1 n_2 n_3 z} = 0.$$

Substituting for $l_1 l_2 l_3$, $m_1 m_2 m_3$, and $n_1 n_2 n_3$, we obtain the equations of these cones.

R. J. T. BELL.

A Method of obtaining Examples on the Multiplication of Determinants.

In the ordinary text-books on Algebra there is a lack of suitable examples on Multiplication of Determinants. Most of the examples that are given are particular cases of the theorem

$$D\,\Delta = D^n,$$

in which

$$D = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}, \qquad \Delta = \begin{vmatrix} A_1 & A_2 & \dots & A_n \\ B_1 & B_2 & \dots & B_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

where A_1, A_2, ..., B_1, ..., are the co-factors of a_1, a_2, ..., b_1..., in D.

If the determinant D is chosen at random, in most cases the second determinant Δ will be too complicated. It is easy, however, to choose D so that factors can be taken out of Δ; and thus a sufficiently simple second determinant is obtained.

For example, let

$$D = \begin{vmatrix} b & a & a \\ a & b & a \\ a & a & b \end{vmatrix} = (2a+b)(a-b)^2.$$

Then

$$\Delta = \begin{vmatrix} b^2-a^2 & a^2-ab & a^2-ab \\ a^2-ab & b^2-a^2 & a^2-ab \\ a^2-ab & a^2-ab & b^2-a^2 \end{vmatrix}.$$

Let the factor $b-a$ be taken out of each row of Δ. Then, multiplying the determinant so obtained by D, we have

$$\begin{vmatrix} b & a & a \\ a & b & a \\ a & a & b \end{vmatrix} \begin{vmatrix} a+b & -a & -a \\ -a & a+b & -a \\ -a & -a & a+b \end{vmatrix} = \begin{vmatrix} b^2+ba-2a^2 & 0 & 0 \\ 0 & b^2+ba-2a^2 & 0 \\ 0 & 0 & b^2+ba-2a^2 \end{vmatrix}$$

$$= (b-a)^3 (b+2a)^3.$$