Canadian Mathematical Bulletin
doi:10.4153/CMB-2011-030-1

Yet Another Solution to the Burnside Problem for Matrix Semigroups

Benjamin Steinberg

Abstract. We use the kernel category to give a finiteness condition for semigroups. As a consequence we provide yet another proof that finitely generated periodic semigroups of matrices are finite.

1 Introduction

Schur proved that every finitely generated periodic group of matrices over a field is finite [3]. McNaughton and Zalcstein established the corresponding result for semigroups [7]. The case of the rational field was essentially handled earlier on by Schützenberger [2, 11]. Since then, a number of proofs of this result have appeared, see [2, 4, 5, 8, 12]. Here we use "global semigroup theory."

2 A Finiteness Condition for Monoids

Let $\varphi\colon M \to N$ be a homomorphism of monoids. Define the *trace* of φ at $(n_1, n_2) \in N \times N$ to be the quotient $\operatorname{tr}_\varphi(n_1, n_2)$ of the submonoid

$$\{m \in M \mid n_1\varphi(m) = n_1, \varphi(m)n_2 = n_2\}$$

by the congruence $\equiv$ given by $m \equiv m'$ if, for all $m_i \in \varphi^{-1}(n_i)$ $(i = 1, 2)$, one has $m_1 m m_2 = m_1 m' m_2$. Recall that a monoid is called *locally finite* if all its finitely generated submonoids are finite. In this section, we prove the following finiteness result.

Theorem 2.1 *Let $\varphi\colon M \to N$ be a homomorphism of monoids such that N is locally finite and $\operatorname{tr}_\varphi(n_1, n_2)$ is locally finite for all $n_1, n_2 \in N$. Then M is locally finite.*

To prove this theorem, we use the kernel category from [10] (see also [9, § 2.6]). In this paper, we perform composition in small categories diagramatically: fg means do f first and then g. If C is a category, then $C(c_1, c_2)$ will denote the hom set of arrows from c_1 to c_2. We use $C(c)$ as shorthand for the endomorphism monoid $C(c, c)$. A category is said to be *locally finite* if all its finitely generated subcategories are finite. First we need a lemma from [6]; the proof is essentially that of Kleene's Theorem.

Lemma 2.2 ([6]) *Let C be a category, all of whose endomorphism monoids are locally finite. Then C is locally finite.*

Received by the editors November 17, 2008.
Published electronically March 7, 2011.
The author was supported in part by NSERC
AMS subject classification: **20M30**, 20M32.
Keywords: Burnside problem, kernel category.

Proof Let Γ be a finite graph with vertex set V and let $\varphi\colon \Gamma^* \to C$ be a functor that is injective on vertices (where Γ^* denotes the free category on Γ). For each $X \subseteq V$ and $v, v' \in V$, define $\Gamma^X_{v,v'}$ to be the set of all paths in Γ from v to v' visiting only vertices from X outside of its initial and terminal vertices. Then $\varphi(\Gamma^*) = \bigcup_{v,v'} \varphi(\Gamma^V_{v,v'})$ and so it suffices to show each $\varphi(\Gamma^V_{v,v'})$ is finite. We proceed by establishing that each $\varphi(\Gamma^X_{v,v'})$ is finite by induction on $|X|$. Since $\Gamma^{\varnothing}_{v,v'}$ is just the set of edges from v to v' and Γ is finite, the case $|X| = 0$ is handled. Suppose $X \neq \varnothing$ and choose $x \in X$. Plainly,

$$\varphi(\Gamma^X_{v,v'}) = \varphi(\Gamma^{X\setminus\{x\}}_{v,v'}) \cup \varphi(\Gamma^{X\setminus\{x\}}_{v,x})\big(\varphi(\Gamma^{X\setminus\{x\}}_{x,x})\big)^*\varphi(\Gamma^{X\setminus\{x\}}_{x,v'}) \tag{2.1}$$

and by induction all sets $\varphi(\Gamma^{X\setminus\{x\}}_{q,q'})$ are finite. Since the endomorphism monoid $C(\varphi(x))$ is locally finite, we conclude $(\varphi(\Gamma^{X\setminus\{x\}}_{x,x}))^*$ is also finite. Finiteness of $\varphi(\Gamma^X_{v,v'})$ now follows from (2.1). ■

Let $\varphi\colon M \to N$ be a homomorphism of monoids. Following [10], we define a small category K_φ, called the *kernel category* of φ. The object set of K_φ is $N \times N$. The arrows of K_φ are equivalence classes $[n_1, m, n_2]$ of triples $(n_1, m, n_2) \in N \times M \times N$, where

$$[n_1, m, n_2]\colon (n_1, \varphi(m)n_2) \to (n_1\varphi(m), n_2)$$

and two triples $(n_1, m, n_2), (n_1, m', n_2)$ are identified if

$$n_1\varphi(m) = n_1\varphi(m'), \quad \varphi(m)n_2 = \varphi(m')n_2, \quad m_1mm_2 = m_1m'm_2$$

for all $m_i \in \varphi^{-1}(n_i)$ with $i = 1, 2$. Composition is given by

$$[n_1, m, \varphi(m')n_2][n_1\varphi(m), m', n_2] = [n_1, mm', n_2];$$

the identity at (n_1, n_2) is $[n_1, 1, n_2]$. One easily verifies that K_φ is a category [10] (see also [9, § 2.6]). Moreover, the endomorphism monoid $K_\varphi(n_1, n_2)$ is isomorphic to $\mathrm{tr}_\varphi(n_1, n_2)$.

It turns out that K_φ is finitely generated whenever the domain is finitely generated and the codomain is finite.

Proposition 2.3 *Suppose $\varphi\colon M \to N$ is a homomorphism of monoids with M finitely generated and N finite. Then K_φ is finitely generated.*

Proof Let X be a finite generating set for M. Then K_φ is generated by all arrows of the form $[n_1, x, n_2]$ with $x \in X$. Indeed, if $m = x_1 \cdots x_r$, then

$$[n_1, m, n_2] = [n_1, x_1, \varphi(x_2 \cdots x_r)n_2][n_1\varphi(x_1), x_2, \varphi(x_3 \cdots x_r)n_2] \cdots [n_1\varphi(x_1 \cdots x_{r-1}), x_r, n_2],$$

as required. ■

We can now prove Theorem 2.1

https://doi.org/10.4153/CMB-2011-030-1 Published online by Cambridge University Press

Proof of Theorem 2.1 Let X be a finite subset of M; put $M' = \langle X \rangle$ and $N' = \varphi(M')$. Then N' is finite. On the other hand, for $n_1, n_2 \in N'$, clearly $\mathrm{tr}_{\varphi|_{M'}}(n_1, n_2)$ is a quotient of a submonoid of $\mathrm{tr}_\varphi(n_1, n_2)$ and hence locally finite. Thus without loss of generality we may assume that M is finitely generated and N is finite.

From Proposition 2.3 we conclude K_φ is finitely generated. Because each endomorphism monoid of K_φ is of the form $\mathrm{tr}_\varphi(n_1, n_2)$, whence locally finite by hypothesis, Lemma 2.2 yields K_φ finite. Define a map $\psi\colon M \to K_\varphi$ by $\psi(m) = [1, m, 1]$. As $1 \in \varphi^{-1}(1)$, the equality $\psi(m) = \psi(m')$ implies that $m = 1m1 = 1m'1 = m'$ and hence ψ is injective. Thus M is finite. ■

3 The Burnside Problem for Matrix Semigroups

By a periodic semigroup, we mean a semigroup so that each cyclic subsemigroup is finite. Like many proofs of the McNaughton–Zalcstein Theorem, we begin with the case of an irreducible representation. We are not really doing anything new here; our proof roughly follows [5].

Proposition 3.1 *Let S be a finitely generated irreducible periodic subsemigroup of $M_n(K)$, where K is an algebraically closed field. Then M is finite.*

Proof By a well-known theorem of Burnside, there are no proper irreducible subalgebras of $M_n(K)$ and hence we can find elements $s_1, \dots, s_{n^2}$ of S forming a basis for $M_n(K)$. The trace form $(A, B) \mapsto \mathrm{tr}(AB)$ is a non-degenerate bilinear form on $M_n(K)$; let $s_1^*, \dots, s_{n^2}^*$ be the corresponding dual basis. Denote by F the prime field of K. Suppose X is a finite generating set for S and consider the finite set A of elements $\mathrm{tr}(xs_is_j)$ and $\mathrm{tr}(xs_i)$ with $x \in X \cup \{1\}$, $1 \le i \le n^2$. Put $E = F(A)$. Since S is periodic, $\mathrm{tr}(s)$ is either 0 or a sum of roots of unity for $s \in S$. Consequently, E is a finite extension of F.

First we show that each s_i^*, for $i = 1, \dots, n^2$, can be written as a linear combination over E of $s_1, \dots, s_{n^2}$. Indeed, let $C = (c_{ij})$ be the matrix given by $s_i^* = \sum c_{ij}s_j$ and $D = (d_{ij})$ where $d_{ij} = \mathrm{tr}(s_is_j)$. Then $D \in M_n(E)$ and $CD = I$, since

$$\sum c_{ik}d_{kj} = \sum c_{ik}\,\mathrm{tr}(s_ks_j) = \mathrm{tr}\Bigl(\sum c_{ik}s_ks_j\Bigr) = \mathrm{tr}(s_i^*s_j) = \delta_{ij}.$$

Thus $C = D^{-1} \in M_n(E)$, as required.

Now for all $s \in S$, we can write

$$s = \sum_i \mathrm{tr}(ss_i^*)s_i = \sum_{i,j} c_{ij}\,\mathrm{tr}(ss_j)s_i. \tag{3.1}$$

We claim that each element of S can be written as a linear combination over E of $s_1, \dots, s_{n^2}$. The proof is by induction on length. For $x \in X$, the claim is immediate from the definition of E and (3.1). Suppose $s = xs'$ with $x \in X$ and that $s' = \sum a_is_i$ with the $a_i \in E$. Then $s = xs' = x\sum a_is_i$ and so $\mathrm{tr}(ss_j) = \sum a_i\,\mathrm{tr}(xs_is_j) \in E$. An application of (3.1) proves the claim. As a consequence of the claim, it follows $\mathrm{tr}(s) \in E$ for all $s \in S$.

https://doi.org/10.4153/CMB-2011-030-1 Published online by Cambridge University Press

Let $T = \{\mathrm{tr}(s) \mid s \in S\}$. We aim to prove that T is finite. Assuming this is true, it follows from (3.1) that S has at most $|T|^{n^2}$ elements. We have two cases. Suppose first that K has characteristic $p > 0$. Then E is a finite field and so trivially $T \subseteq E$ is finite. Next assume that K has characteristic 0. Let k be the subfield obtained by adjoining to $\mathbb{Q}$ all entries of the elements of X. Then k is finitely generated over $\mathbb{Q}$ and $S \subseteq M_n(k)$. The possible eigenvalues of an element $s \in S$ are zero and roots of unity satisfying a degree n polynomial over k (namely, the characteristic polynomial of s). But a finitely generated extension field of $\mathbb{Q}$ has only finitely many such roots of unity [3, Proof of Theorem (36.2)]. Since the trace is the sum of the eigenvalues, it follows that also in this case T is finite. ■

The novel part of our proof is how we handle the reduction to the irreducible case. The following lemma is a variant of a result from [1].

Lemma 3.2 *Let K be a ring with unit and $M \subseteq M_n(K)$ be a monoid of block upper triangular matrices*

$$\begin{pmatrix} S & * \\ 0 & T \end{pmatrix} \text{ with } S \subseteq M_m(K) \text{ and } T \subseteq M_r(K).$$

Let φ be the projection to the diagonal block and set $N = \varphi(M)$. Then for all $n_1, n_2 \in N$ the monoid $\mathrm{tr}_\varphi(n_1, n_2)$ embeds in the additive group $M_{m,r}(K)$.

Proof Fix $n_1, n_2 \in N$ and put $R = \{m \in M \mid n_1 m = n_1, m n_2 = n_2\}$. Suppose that $n_1 = (X, Y)$ and $n_2 = (U, V)$. We define a homomorphism $\psi\colon R \to M_{m,r}(K)$ as follows. Given

$$m = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in R,$$

define $\psi(m) = XBV$. Note that

$$XA = X, \quad YC = Y, \quad AU = U, \quad CV = V \tag{3.2}$$

by definition of R. Using this we compute

$$\begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} U & W \\ 0 & V \end{pmatrix} = \begin{pmatrix} XU & XW + XBV + ZV \\ 0 & YV \end{pmatrix}. \tag{3.3}$$

Thus $\psi(m) = XBV$ determines and is determined by the right-hand side of (3.3) for any given Z, W. Therefore, $\psi(m) = \psi(m')$ if and only if m and m' represent the same element of $\mathrm{tr}_\varphi(n_1, n_2)$.

It remains to verify that ψ is a homomorphism to the additive group $M_{m,r}(K)$. It clearly sends the identity matrix to 0. Also if

$$a = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \quad b = \begin{pmatrix} A' & B' \\ 0 & C' \end{pmatrix} \in R,$$

then $\psi(a) + \psi(b) = XBV + XB'V$, whereas

$$ab = \begin{pmatrix} AA' & AB' + BC' \\ 0 & CC' \end{pmatrix}.$$

So $\psi(ab) = X(AB' + BC')V = XAB'V + XBC'V = XB'V + XBV$ since $a, b \in R$ (3.2). ■

We are now in a position to complete our proof of the theorem of McNaughton and Zalcstein [7].

Theorem 3.3 (McNaughton and Zalcstein) *Let S be a finitely generated periodic semigroup of $n \times n$ matrices over a field K. Then S is finite.*

Proof Without loss of generality we may assume that K is algebraically closed. Also assume that S contains the identity matrix, since if it does not, we may adjoin it. We proceed by induction on n. If S is irreducible, we are done by Proposition 3.1. Otherwise, we can write

$$S = \begin{pmatrix} M_1 & * \\ 0 & M_2 \end{pmatrix},$$

where $M_1 \subseteq M_m(K)$ and $M_2 \subseteq M_r(K)$ are finitely generated periodic semigroups of matrices of strictly smaller degrees $m, r < n$. By induction $M_1 \times M_2$ is finite. Consider the projection $\varphi\colon S \to M_1 \times M_2$. Lemma 3.2 yields that $\mathrm{tr}_\varphi(x, y)$ is a periodic subsemigroup of the additive group $M_{m,r}(K)$, for any $x, y \in M_1 \times M_2$. In the case K has characteristic 0, this implies each such trace is trivial; if the characteristic of K is $p > 0$, then each such trace is an elementary abelian p-group. In either case, it follows that each trace is locally finite and so Theorem 2.1 implies that S is finite. ■

References

[1] J. Almeida, S. W. Margolis, B. Steinberg, and M. V. Volkov, *Representation theory of finite semigroups, semigroup radicals and formal language theory.* Trans. Amer. Math. Soc. **361**(2009), no. 3, 1429–1461. doi:10.1090/S0002-9947-08-04712-0

[2] J. Berstel and C. Reutenauer, *Rational Series and Their Languages.* EATCS Monographs on Theoretical Computer Science 12, Springer-Verlag, Berlin, 1988.

[3] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*. John Wiley & Sons, New York, 1988. Reprint of the 1962 original.

[4] A. de Luca and S. Varricchio, *Finiteness and regularity in semigroups and formal languages*. EATCS Monographs in Theoretical Computer Science, Springer-Verlag, Berlin, 1999.

[5] A. Freedman, R. N. Gupta, and R. M. Guralnick, *Shirshov's theorem and representations of semigroups. Olga Taussky-Todd: in memoriam.* Pacific J. Math. (1997) Special Issue, 159–176.

[6] B. Le Saëc, J.-E. Pin, and P. Weil, *Semigroups with idempotent stabilizers and applications to automata theory.* Internat. J. Algebra Comput. **1**(1991), no. 3, 291–314. doi:10.1142/S0218196791000195

[7] R. McNaughton and Y. Zalcstein, *The Burnside problem for semigroups.* J. Algebra **34**(1975), 292–299. doi:10.1016/0021-8693(75)90184-2

[8] J. Okniński, *Semigroups of Matrices.* Series in Algebra 6, World Scientific Publishing, River Edge, NJ, 1998.

[9] J. Rhodes and B. Steinberg, *The q-Theory of Finite Semigroups*. Springer Monographs in Mathematics, Springer, New York, 2009.

[10] J. Rhodes and B. Tilson, *The kernel of monoid morphisms.* J. Pure Appl. Algebra **62**(1989), no. 3, 227–268. doi:10.1016/0022-4049(89)90137-0

[11] M. P. Schützenberger, *Finite counting automata.* Information and Control **5**(1962), 91–107. doi:10.1016/S0019-9958(62)90244-9

[12] H. Straubing, *The Burnside problem for semigroups of matrices.* In: Combinatorics on Words. Academic Press, Toronto, 1983, pp. 279–295.

School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6
e-mail: bsteinbg@math.carleton.ca