THE BAIRE CLOSURE AND ITS LOGIC

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Abstract. The Baire algebra of a topological space X is the quotient of the algebra of all subsets of X modulo the meager sets. We show that this Boolean algebra can be endowed with a natural closure operator, resulting in a closure algebra which we denote $\mathbf{Baire}(X)$. We identify the modal logic of such algebras to be the well-known system S5, and prove soundness and strong completeness for the cases where X is crowded and either completely metrizable and continuum-sized or locally compact Hausdorff. We also show that every extension of S5 is the modal logic of a subalgebra of $\mathbf{Baire}(X)$, and that soundness and strong completeness also holds in the language with the universal modality.

§1. Introduction. Canonical examples of Boolean algebras include the powerset $\wp(X)$ of a set X, as well as its subalgebras; indeed, every Boolean algebra is isomorphic to a subalgebra of a powerset algebra (see, e.g., [19, p. 28]). Alternately, one can consider quotients of the form $\wp(X)/\mathbf{I}$, where $\mathbf{I}$ is a suitable ideal of $\wp(X)$. Some of the most familiar such ideals are the ideal $\mathbf{N}$ of null sets when X is a measure space (see, e.g., [19, p. 233]), or the ideal $\mathbf{M}$ of meager sets when X is a topological space (see, e.g., [19, p. 182]); recall that a set is *nowhere dense* if the interior of its closure is empty, and that it is *meager* if it is a countable union of nowhere dense sets. The quotient $\wp(X)/\mathbf{M}$ gives rise to the *Baire algebra* of X.

When X is a topological space, the powerset algebra of X comes equipped with the usual closure operator $\mathbf{c}\colon \wp(X) \to \wp(X)$. This operator satisfies some familiar properties also known as the *Kuratowski axioms,* including, e.g., $A \subseteq \mathbf{c}A$ (see Section 2 for the full list), and any operator satisfying these axioms uniquely determines a topology on X. One can more generally consider a Boolean algebra B with an operator $\mathbf{c}\colon B \to B$ satisfying the same axioms; such algebras are the *closure algebras* of McKinsey and Tarski [24]. It is then a natural question to ask whether a closure operator on $\wp(X)$ carries over to quotients $\wp(X)/\mathbf{I}$ in a meaningful way. This question has already been answered in the affirmative by Fernández-Duque [12] and Lando [22] in the setting of the *Lebesgue measure algebra,* defined as the quotient of the Borel sets of reals modulo the null sets. In an unpublished work, Bjorndahl has also considered validity modulo the nowhere dense sets, although he does not work directly with the algebraic quotient. In this article, we will explore this question in the setting of the Baire algebra of a topological space X.

One subtlety when defining a closure operator for such quotients is that, denoting the equivalence class of Y by $[Y]$, the definition $\mathbf{c}[Y] := [\mathbf{c}Y]$ does not yield a

well-defined operation: for example, in the above-mentioned quotients, $[\mathbb{Q}] = [\varnothing]$ (as $\mathbb{Q}$ is both meager and of measure zero), yet $[\mathbf{c}\mathbb{Q}] = [\mathbb{R}] \neq [\varnothing] = [\mathbf{c}\varnothing]$. Instead, we must compute the closure of a set directly within the quotient, by the expression

$$\mathbf{c}a = \inf\{[C] : a \sqsubseteq [C] \text{ and } C \text{ is closed}\},$$

where $\sqsubseteq$ denotes the partial order on the quotient algebra. We refer to the element $\mathbf{c}a$ as the *Baire closure* of a. We utilize a nontrivial fact in topology that the Boolean algebra of Borel sets of an arbitrary topological space modulo the ideal of meager sets is complete (see [28, p. 75]) to show that this produces a closure operator on the Baire algebra of *any* topological space X, and we denote the resulting closure algebra by **Baire**(X).

Much as Boolean algebras provide semantics for propositional logic, closure algebras provide semantics for *modal logic,* which extends propositional logic with an operator $\diamond$ that we will interpret as a closure operator, along with its dual $\square$, interpreted as interior. Such topological semantics of modal logic, as well as the closely related intuitionistic logic, predates their now-widespread relational semantics. For intuitionistic logic it was first developed by Stone [29] and Tarski [30], and for modal logic by Tsao-Chen [31], McKinsey [23], and McKinsey and Tarski [24]. Under this interpretation, the modal logic of all topological spaces turns out to be the well-known modal system of Lewis, S4 (see Section 2 for the definition). Other well-known extensions of S4 also turn out to be the modal logics of interesting topological spaces. To give a couple of examples:

- S4.2 := S4 + $\diamond\square p \to \square\diamond p$ is the modal logic of all extremally disconnected spaces [13].
- S4.3 := S4 + $\square(\square p \to q) \vee \square(\square q \to p)$ is the modal logic of all hereditarily extremally disconnected spaces [1].

Here we recall that a space X is extremally disconnected if the closure of each open set is open, and X is hereditarily extremally disconnected if each subspace of X is extremally disconnected.

These topological completeness results can be strengthened as follows. By the celebrated McKinsey–Tarski theorem [24], S4 is the modal logic of any crowded metrizable space.[1] In fact, S4 is strongly sound and complete with respect to any crowded metrizable space [20, 14]. In particular, S4 is the modal logic of the real unit interval $[0, 1]$. On the other hand, S4.2 is the modal logic of the Gleason cover of $[0, 1]$ (see [4]), while S4.3 is the modal logic of a countable (hereditarily) extremally disconnected subspace of the Gleason cover of $[0, 1]$ (see [1]). In contrast, S5 := S4 + $\diamond p \to \square\diamond p$, which is one of the best known modal logics, is not complete with respect to any class of spaces that satisfy even weak separation axioms. Indeed, $\diamond\square p \to \square p$ is valid in a topological space iff each open set is also closed. Thus, a T_0-space validates S5 iff it is discrete.

One can also characterize the modal logics of closure algebras that are not based on a powerset. Fernández-Duque [12] and Lando [22] have shown that S4 is the logic

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[1] We recall that a space X is *crowded* or *dense-in-itself* if it has no isolated points. It is worth pointing out that the original McKinsey–Tarski theorem also had the separability (equivalently the second countability) assumption on X, which was later removed by Rasiowa and Sikorski [25] by an elaborate use of the Axiom of Choice. For a modern proof of this result, see [3].

of the Lebesgue measure algebra, and in this article we will characterize the logic of Baire algebras. An element $c \in \mathbf{Baire}(X)$ is closed (resp. open) if $c = [C]$ for some closed (resp. open) C. A distinguished feature of $\mathbf{Baire}(X)$ is that $c \in \mathbf{Baire}(X)$ is open iff it is closed. This yields that S5 is sound with respect to $\mathbf{Baire}(X)$ for any topological space X. For completeness, we refine Hewitt's [18] well-known concept of *resolvability* to that of *Baire resolvability*. A space is resolvable if it can be partitioned into two dense sets. Similarly, a closure algebra B is resolvable if there are a, b that are *orthogonal* ($a \wedge b = 0$) and *dense* ($\mathbf{c}a = \mathbf{c}b = 1$). If we denote $|\mathbb{R}|$ by $\mathfrak{c}$, we show that if X is crowded and either a complete metric space of cardinality $\mathfrak{c}$ or a locally compact Hausdorff space, then $\mathbf{Baire}(X)$ is resolvable. In fact, such algebras are $\mathfrak{c}$-resolvable, meaning that we can find $\mathfrak{c}$-many dense and pairwise orthogonal elements of $\mathbf{Baire}(X)$. Our main tool in proving these results is the Disjoint Refinement Lemma (see, e.g., [10, Lemma 7.5]).

Using these resolvability results we show that if X is a crowded, continuum-sized, complete metrizable space, then S5 is strongly complete for $\mathbf{Baire}(X)$, yielding a variant of the McKinsey–Tarski theorem for S5. In view of $\mathfrak{c}$-resolvability, strong completeness holds even if we extend the propositional language with continuum-many propositional variables. This yields a sharper version of strong completeness than that given in the literature, where only countable languages are typically considered. Some of our other main results include that every extension of S5 is the modal logic of some subalgebra of $\mathbf{Baire}(X)$ for any crowded second-countable completely metrizable space X. These results also hold if we replace completely metrizable by locally compact Hausdorff. Finally, we show how to extend our results to the setting of the universal modality.

§2. Preliminaries. We assume some basic familiarity with Boolean algebras, topological spaces, and ordinal and cardinal arithmetic (see, e.g., [19, 11, 10]). In this section we will review closure algebras and their relation to modal logic.

2.1. Closure algebras. For each topological space X, the powerset $\wp(X)$ is a Boolean algebra and the closure operator $\mathbf{c} : \wp(X) \to \wp(X)$ satisfies the Kuratowski axioms:

$$A \subseteq \mathbf{c}A, \; \mathbf{cc}A \subseteq \mathbf{c}A, \; \mathbf{c}(A \cup B) = \mathbf{c}A \cup \mathbf{c}B, \text{ and } \mathbf{c}\varnothing = \varnothing.$$

We call the pair $(\wp(X), \mathbf{c})$ the *Kuratowski algebra* of X and denote it by $\mathbf{Kur}(X)$.

McKinsey and Tarski [24] generalized the concept of a closure on $\wp(X)$ to that of a closure on an arbitrary Boolean algebra.

DEFINITION 2.1. A *closure algebra* is a pair $(B, \mathbf{c})$, where B is a Boolean algebra and $\mathbf{c} \colon B \to B$ satisfies, for every $a, b \in B$,

1. $a \leq \mathbf{c}a$,
2. $\mathbf{cc}a \leq \mathbf{c}a$,
3. $\mathbf{c}(a \vee b) = \mathbf{c}a \vee \mathbf{c}b$, and
4. $\mathbf{c}0 = 0$.

Let $\mathbf{i}a := -\mathbf{c}-a$. If $\mathbf{c}$ moreover satisfies

$$\mathbf{c}a = \mathbf{ic}a,$$

then $(B, \mathbf{c})$ is a *monadic algebra*.

REMARK 2.2. The notion of a monadic algebra is due to Halmos [16].

As is the case for Boolean algebras, closure algebras can be represented as subalgebras of algebras based on a powerset.

THEOREM 2.3 [24, Theorem 2.4]. *Each closure algebra is isomorphic to a subalgebra of the Kuratowski algebra $\mathbf{Kur}(X)$ of some topological space X.*

Recall that Boolean algebras are partial orders $(B, \leq)$ with a least element and a greatest element, usually denoted by 0 and 1, such that for $a, b \in B$ their meet (greatest lower bound) $a \wedge b$ and complement $-a$ are defined, and satisfy the usual axioms (see, e.g., [28, 17, 19]). If $S \subseteq B$, then $\bigwedge S$ denotes its meet and $\bigvee S$ its join when they exist; the algebra B is *complete* provided these always exist. The next definition goes back to Halmos [16].

DEFINITION 2.4. We say that $A \subseteq B$ is *relatively complete* in B if for each $b \in B$, the set $\{a \in A \mid b \leq a\}$ has a least element (which then is $\bigwedge\{a \in A \mid b \leq a\}$).

Of particular importance are countable joins and meets, also called σ-*joins* and σ-*meets;* the algebra B is σ-*complete* if such joins and meets always exist, in which case B is a σ-*algebra*. An *ideal* is a set $\mathbf{I} \subseteq B$ which is closed under finite joins and with each element $a \in \mathbf{I}$ also contains all elements underneath a; it is *proper* if $1 \notin \mathbf{I}$; and it is a σ-*ideal* if it is closed under σ-joins. For an ideal $\mathbf{I}$, recall that $B/\mathbf{I}$ is the set of equivalence classes of B under the equivalence relation given by $a \sim b$ if $a - b, b - a \in \mathbf{I}$. We order $B/\mathbf{I}$ by $[a] \leq [b]$ if $a - b \in \mathbf{I}$. The quotient $B/\mathbf{I}$ is then a Boolean algebra; moreover, if B is σ-complete and $\mathbf{I}$ is a σ-ideal, then $B/\mathbf{I}$ is σ-complete (see, e.g., [17, p. 56]).

2.2. Modal logic. We will work with the basic unimodal language as well as its extension with the universal modality. However, as we are interested in strong completeness for possibly uncountable sets of formulas, we want to allow for an arbitrary number of propositional variables. Given a cardinal λ, let $\mathbb{P}^\lambda = \{p_\iota \mid \iota < \lambda\}$ be a set of λ-many propositional variables. The *modal language* $\mathcal{L}^\lambda_{\Diamond\forall}$ is defined by the grammar (in Backus–Naur form)

$$\varphi, \psi := \quad p \quad | \quad \varphi \wedge \psi \quad | \quad \neg\varphi \quad | \quad \Diamond\varphi \quad | \quad \forall\varphi,$$

where $p \in \mathbb{P}^\lambda$. We also use standard shorthands, for example, defining $\bot$ as $p \wedge \neg p$ (where p is any fixed variable), $\varphi \vee \psi$ as $\neg(\neg\varphi \wedge \neg\psi)$, and $\Box\varphi$ as $\neg\Diamond\neg\varphi$. We denote the $\forall$-free fragment by $\mathcal{L}^\lambda_{\Diamond}$, and we omit the superscript λ when $\lambda = \omega$.

We use $\Diamond$ as primitive rather than $\Box$ since, historically, algebraic semantics was presented in terms of closure-like operators on Boolean algebras. For our purposes, a *logic* is understood as a set of formulas closed under modus ponens, substitution, and necessitation. If Λ is any logic over $\mathcal{L}$, then Λ^λ denotes the least logic over $\mathcal{L}^\lambda$ containing $\Lambda \cap \mathcal{L}^\lambda$.

We restrict our attention to logics above S4, and especially to S5 and its extensions. A standard axiomatization of S4 in $\mathcal{L}^\lambda_{\Diamond}$ (i.e., of $\mathsf{S4}^\lambda$) is given by all (classical)

propositional tautologies and the axioms and rules:

$$\mathbf{M} : \Diamond(p \vee q) \to \Diamond p \vee \Diamond q \qquad\qquad \mathbf{Sub} : \frac{\varphi(p_1, \ldots, p_n)}{\varphi(\psi_1, \ldots, \psi_n)}$$

$$\mathbf{T} : p \to \Diamond p$$

$$\mathbf{4} : \Diamond\Diamond p \to \Diamond p \qquad\qquad\qquad \mathbf{MP} : \frac{\varphi \quad \varphi \to \psi}{\psi}$$

$$\mathbf{N} : \neg\Diamond\bot \qquad\qquad\qquad\qquad\quad \mathbf{Mon} : \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}.$$

The logic $\mathsf{S5}^\lambda$ is then obtained by adding the axiom

$$\mathbf{5} : \Diamond p \to \Box\Diamond p.$$

Given a logic Λ over $\mathcal{L}_\Diamond^\lambda$, we obtain a new logic $\Lambda\mathsf{U}$ by adding the S5 axioms and rules for $\forall$ (or rather for $\exists := \neg\forall\neg$) and the connecting axiom $\Diamond\varphi \to \exists\varphi$. We will specifically be interested in S5U.

The language $\mathcal{L}_\Diamond^\lambda$ has familiar Kripke semantics based on frames $\mathfrak{F} = (W, R)$, where $W \neq \varnothing$ and $R \subseteq W \times W$ (see, e.g., [7, 6]). We will not review this semantics in detail, and instead regard it as a special case of topological or, more generally, algebraic semantics.

Closure algebras provide the algebraic semantics for S4 and its *normal* extensions, meaning those logics containing the axioms of S4 and closed under the rules of S4.

DEFINITION 2.5. An *algebraic model* of S4U is a structure $\mathfrak{M} = (B, [\![\cdot]\!])$, where B is a closure algebra and $[\![\cdot]\!] : \mathcal{L}_{\Diamond\forall}^\lambda \to B$ is a *valuation for* $\mathcal{L}_{\Diamond\forall}^\lambda$; that is, a function such that $[\![p]\!] \in B$ for each $p \in \mathbb{P}^\lambda$ and

$$\begin{aligned}
[\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \wedge [\![\psi]\!] & [\![\neg\varphi]\!] &= -[\![\varphi]\!] \\
[\![\Diamond\varphi]\!] &= \mathbf{c}[\![\varphi]\!] & [\![\forall\varphi]\!] &= \begin{cases} 1 & \text{if } [\![\varphi]\!] = 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

We may also say that $\mathfrak{M}$ is an *algebraic model of* S4, and that the restriction of $[\![\cdot]\!]$ to $\mathcal{L}_\Diamond^\lambda$ is a *valuation for* $\mathcal{L}_\Diamond^\lambda$.

Validity, soundness, and completeness are then defined in the usual way:

DEFINITION 2.6. Let $\mathfrak{M} = (B, [\![\cdot]\!])$ be an algebraic model and $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Diamond\forall}^\lambda$.

1. We say that φ is *valid* in $\mathfrak{M}$, written $\mathfrak{M} \models \varphi$, if $[\![\varphi]\!] = 1$. If $\mathfrak{M} \models \gamma$ for each $\gamma \in \Gamma$, then we say that Γ is *valid* in $\mathfrak{M}$ and write $\mathfrak{M} \models \Gamma$.
2. We write $B \models \varphi$ if $(B, [\![\cdot]\!]) \models \varphi$ for every valuation $[\![\cdot]\!]$ on B, and define $B \models \Gamma$ similarly.
3. If Ω is a class of closure algebras, we say that φ is *valid in* Ω provided $B \models \varphi$ for each $B \in \Omega$. That Γ is *valid in* Ω is defined similarly.
4. For $S \subseteq \{\Diamond, \forall\}$, we denote the set of valid $\mathcal{L}_S^\lambda$-formulas in Ω by $\mathsf{Log}_S(\Omega)$. If $\Omega = \{B\}$, we simply write $\mathsf{Log}_S(B)$.
5. A logic Λ is *sound* for Ω if $\Lambda \subseteq \mathsf{Log}_S(\Omega)$, and *complete* if $\Lambda \supseteq \mathsf{Log}_S(\Omega)$.

We will also be interested in *strong* completeness of logics.

DEFINITION 2.7. Let φ be a formula, Γ a set of formulas, and Λ a logic in $\mathcal{L}_{\Diamond\forall}^\lambda$.

1. We write $\Gamma \vdash_\Lambda \varphi$ if there is a finite $\Delta \subseteq \Gamma$ such that $(\forall \bigwedge \Delta) \to \varphi \in \Lambda$. We omit Γ when $\Gamma = \varnothing$. If Λ is a logic in $\mathcal{L}_\Diamond^\lambda$, $\forall \bigwedge \Delta$ should be replaced by $\Box \bigwedge \Delta$.

2. For a closure algebra B, we write $\Gamma \models_B \varphi$ if $B \models \Gamma$ implies $B \models \varphi$.[2]
3. We say that Λ is *strongly complete* for a closure algebra B if $\Gamma \models_B \varphi$ implies $\Gamma \vdash_\Lambda \varphi$, and that Λ is *strongly complete* for a class of structures Ω if $\Gamma \models_\Omega \varphi$ implies $\Gamma \vdash_\Lambda \varphi$.

As we had mentioned previously, Kripke frames can be seen as a special case of Kuratowski algebras. If W is a set and $R \subseteq W \times W$, then R induces an operator on $\wp(W)$ given by

$$A \mapsto R^{-1}(A) := \{w \in W \mid wRa \text{ for some } a \in A\}.$$

If W is nonempty and R reflexive and transitive, we say that (W, R) is an S4-*frame*. It is well known and not hard to check that in this case $(\wp(W), R^{-1})$ is a Kuratowski algebra, and that it is the Kuratowski algebra of the topology on W whose open sets are those $U \subseteq W$ that satisfy $R(U) \subseteq U$, recalling that

$$R(U) = \{w \in W \mid uRw \text{ for some } u \in U\}.$$

If in addition R is symmetric, then we say that (W, R) is an S5-*frame*. We will tacitly identify an S4-frame with the associated topological space and even with the associated Kuratowski algebra.

It is well known that the topologies arising from S4-frames are *Alexandroff spaces*; that is, topologies where arbitrary intersections of open sets are open. It is a consequence of the results of McKinsey and Tarski [24] and Kripke [21] that S4 is sound for the class of closure algebras, and complete for the class of S4-frames. It is well known that S4 and S5 are strongly complete, with respect to both global and local consequence relations; both consequence relations are discussed in [6]; on the other hand, [25] only focuses on the global consequence relation (using different terminology), which is the route we take here. The literature typically considers countable languages, but the adaptation of these results to uncountable languages is straightforward; we provide a sketch below.

THEOREM 2.8. *For an infinite cardinal λ, we have*:

(1) $\mathsf{S4}^\lambda$ *and* $\mathsf{S4U}^\lambda$ *are strongly complete for the class of all* S4-*frames of cardinality at most λ.*
(2) $\mathsf{S5}^\lambda$ *and* $\mathsf{S5U}^\lambda$ *are strongly complete for the class of all* S5-*frames of cardinality at most λ.*

PROOF SKETCH. We outline how a standard proof, as given, e.g., in [6, p. 203], can be adapted to obtain the results as stated. Let Λ be one of the logics mentioned above, and suppose that $\Gamma \nvdash_\Lambda \varphi$. The completeness proof by the relativized canonical model $\mathfrak{M}_c^\Gamma = (W_c^\Gamma, R_c^\Gamma, \llbracket \cdot \rrbracket_c^\Gamma)$, where W_c^Γ is the set of maximal consistent sets containing Γ, carries over mostly verbatim for uncountable languages. The relativized canonical model has the property that $\mathfrak{M}_c^\Gamma \models \Gamma$ and $\mathfrak{M}_c^\Gamma \nmodels \varphi$. The only caveat is that the proof of the Lindenbaum lemma, that any consistent set of formulas Δ can be extended to a maximal consistent set $\Delta' \supseteq \Delta$, requires the use of Zorn's lemma.

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[2]This is the 'global consequence relation', which should not be confused with the 'local consequence relation', defined pointwise on Kripke frames. The former is more natural in algebraic semantics.

The set of worlds W_c^Γ may have cardinality greater than λ. However, viewing $\mathcal{L}_{\Diamond\forall}^\lambda$ as a fragment of the first-order logic via the standard translation [6, p. 84], we may apply the downward Löwenheim–Skolem theorem to obtain a model $\mathfrak{M}$ of size at most λ such that $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \not\models \varphi$, as required. $\dashv$

For a topological space X, it is well known (and not hard to see) that the following are equivalent:

1. $X \models \mathsf{S5}$;
2. every open set of X is closed;
3. X has a basis which is also a partition of X;
4. X is an S5-frame.

Thus, unlike the case of S4, no generality is achieved by passing to topological semantics of S5. As we will see, the story is quite different for algebraic semantics.

We conclude this preliminary section by recalling the extensions of S5. For each natural number $n > 0$, let

$$\mathrm{Alt}_n := \bigwedge_{i=1}^{n+1} \Diamond p_i \to \bigvee_{1 \leq i < j \leq n+1} \Diamond(p_i \wedge p_j).$$

The logic $\mathsf{S5}_n$ is the extension of S5 by the above axiom. We note that an S5-frame validates Alt_n iff each point has at most n distinct successors, or 'alternatives'.

The next result is well known, and was originally proved by Scroggs [26]. We remark that in this article we use $\subset$ to denote strict inclusion.

THEOREM 2.9 (Scroggs' Theorem). *The consistent extensions of* S5 *form the following* $(\omega + 1)$-*chain* (*with respect to* $\supset$) :

$$\mathsf{S5}_1 \supset \cdots \supset \mathsf{S5}_n \supset \cdots \supset \mathsf{S5}.$$

Note that Scroggs' Theorem applies to $\mathcal{L}_\Diamond^\lambda$ for any infinite λ since any logic Λ over $\mathcal{L}_\Diamond^\lambda$ is uniquely determined by $\Lambda \cap \mathcal{L}_\Diamond^\omega$ (as logics are closed under substitution). However, it is possible to exhibit infinite sets of formulas that may only be satisfied on uncountable models. For example, to satisfy

$$\{\Diamond p_\alpha \mid \alpha < \lambda\} \cup \{\Box\neg(p_\alpha \wedge p_\beta) \mid \alpha < \beta < \lambda\}$$

requires one point for each p_α.

DEFINITION 2.10. Let (W, R) be an S5-frame. Then it is a disjoint union of equivalence classes, or *clusters*, $\bigcup_{\iota < \kappa} C_\iota$, where each C_ι is of the form $R(\{w\})$ (which we henceforth write as $R(w)$) for some $w \in W$. We define:

- the *number of clusters* to be κ;
- the *lower cluster size* to be $\min_{\iota < \kappa} |C_\iota|$, and
- the *upper cluster size* to be $\sup_{\iota < \kappa} |C_\iota|$.

The unique (up to isomorphism) frame with one cluster of size λ is the λ-*cluster,* and we denote it by $\mathfrak{C}_\lambda$.

Theorem 2.8 can be sharpened using the structures $\mathfrak{C}_\lambda$.

THEOREM 2.11. *For each finite $n > 0$, $\mathsf{S5}_n$ is sound and strongly complete for the n-cluster, while for infinite λ, $\mathsf{S5}^\lambda$ is sound and strongly complete for any κ-cluster with $\kappa \geq \lambda$.*

PROOF SKETCH. It follows from Scroggs' Theorem that $\mathsf{S5}_n$ is sound and complete for the n-cluster. Strong completeness follows from compactness of the first-order logic (applied to the standard translations of the theorems of $\mathsf{S5}_n$) and using the formula stating that there are at most n elements. For infinite λ, we have that $\mathsf{S5}^\lambda$ is sound and strongly complete for the class of $\mathsf{S5}$-frames. So, if $\Gamma \nvdash \varphi$, there is a model $\mathfrak{M}$ and a world w of $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma$ but $w \notin [\![\varphi]\!]$. The submodel of $\mathfrak{M}$ generated by w is the cluster $R(w)$ and satisfies the same formulas. We may assume that $R(w)$ is infinite by adding duplicates of a point if needed, which does not affect modal formulas. Then using the upwards or downwards Löwenheim–Skolem theorem, we can assume that this cluster is isomorphic to $\mathfrak{C}_\kappa$. ⊣

§3. Baire algebras as a new semantics for $\mathsf{S5}$**.** In this section we introduce the main concept of the paper, that of Baire algebras, which provide a new semantics for $\mathsf{S5}$. Baire algebras are obtained from topological spaces X by modding out $\wp(X)$ by the σ-ideal $\mathbf{M}$ of meager subsets of X. We show that the closure operator $\mathbf{c}$ on X gives rise to a closure operator on $\wp(X)/\mathbf{M}$, so that $(\wp(X)/\mathbf{M}, \mathbf{c})$ is a monadic algebra.

We start by the following well-known definition.

DEFINITION 3.1. Let X be a topological space and $A \subseteq X$.

1. A is *nowhere dense* if $\mathbf{ic}A = \varnothing$.
2. A is *meager* if it is a σ-union of nowhere dense sets.
3. $\mathbf{M}$ denotes the set of meager subsets of X.
4. X is a *Baire space* if for each nonempty open subset U of X, we have that $U \notin \mathbf{M}$.

It is easy to see that $\mathbf{M}$ is a σ-ideal of $\wp(X)$. Therefore, the quotient $\wp(X)/\mathbf{M}$ is a σ-algebra (see, e.g., [17, Section 13]). However, $\wp(X)/\mathbf{M}$ is not always a complete Boolean algebra (see, e.g., [17, Section 25]).

We recall that elements of $\wp(X)/\mathbf{M}$ are equivalence classes of the equivalence relation given by

$$A \approx B \text{ iff } A \setminus B, B \setminus A \in \mathbf{M}.$$

We write $[A]$ for the equivalence class of A under $\approx$. Then $[A] \sqsubseteq [B]$ iff $A \setminus B \in \mathbf{M}$, and the operations on $\wp(X)/\mathbf{M}$ are given by

$$\prod_{n<\omega} [A_n] = \left[\bigcap_{n<\omega} A_n \right] \text{ and } -[A] = [X \setminus A]. \tag{1}$$

On the other hand, as we already pointed out above, if $S \subseteq \wp(X)/\mathbf{M}$ is uncountable, then $\prod S$ may not always exist.

EXAMPLE 3.2.

1. Let (W, R) be an $\mathsf{S5}$-frame, and identify it with the topological space whose opens are unions of equivalence classes of the equivalence relation R. For each $w \in W$, since $R(w)$ is the least open containing w, which is also closed, $\varnothing$ is

the only nowhere dense subset of W. Therefore, $\mathbf{M} = \{\varnothing\}$, and so $\wp(W)/\mathbf{M}$ is isomorphic to $\wp(W)$.

2. More generally, if (W, R) is an S4-frame, a point $w \in W$ is *quasi-maximal* if wRv implies vRw. Let $\mathsf{qmax}W$ be the set of quasi-maximal points. Then $w \in \mathsf{qmax}W$ iff $R(w) \subseteq R^{-1}(w)$. Therefore, $w \in \mathsf{qmax}W$ iff $\{w\}$ is not nowhere dense. Thus, if W is countable, $\mathbf{M} = \{A \subseteq W \mid A \subseteq W \setminus \mathsf{qmax}W\}$, and hence $\wp(W)/\mathbf{M}$ is isomorphic to $\wp(\mathsf{qmax}W)$ (via $[A] \mapsto A \cap \mathsf{qmax}W$).

DEFINITION 3.3. Let X be a topological space.

1. We call $a \in \wp(X)/\mathbf{M}$ *open* if $a = [U]$ for some open $U \in \wp(X)$, and *closed* if $a = [F]$ for some closed $F \in \wp(X)$. An element that is both open and closed is *clopen*.
2. We denote the set of closed elements of $\wp(X)/\mathbf{M}$ by Γ.

It is easy to see that Γ is a bounded sublattice of $\wp(X)/\mathbf{M}$. We next show that Γ is in fact a Boolean subalgebra of $\wp(X)/\mathbf{M}$. For this we point out that if U is open, then $\mathbf{c}U \setminus U$ is nowhere dense (since it is closed and does not have any nonempty open subsets), so $[\mathbf{c}U] = [U]$. Similarly, if F is closed, then $[F] = [\mathbf{i}F]$.

THEOREM 3.4. *For every topological space X, we have that Γ is a Boolean subalgebra of $\wp(X)/\mathbf{M}$.*

PROOF. Since Γ is a bounded sublattice of $\wp(X)/\mathbf{M}$, it is sufficient to show that $a \in \Gamma$ implies $-a \in \Gamma$. From $a \in \Gamma$ it follows that $a = [F]$ for some F closed in X. Then $-a = [X \setminus F]$. Since $X \setminus F$ is open, we have that $X \setminus F \approx \mathbf{c}(X \setminus F)$. Thus, $-a = [X \setminus F] = [\mathbf{c}(X \setminus F)]$, and hence $-a \in \Gamma$. ⊣

In fact, Γ is a σ-subalgebra of $\wp(X)/\mathbf{M}$, which follows from Equation (1) and the fact that intersection of closed sets is closed. But we can say more. For this we recall that a subset of X is a *Borel set* if it belongs to the σ-algebra generated by the open sets. We write $\mathbf{Borel}(X)$ for the σ-algebra of Borel sets and $\mathbf{Borel}(X)/\mathbf{M}$ for the σ-subalgebra of $\wp(X)/\mathbf{M}$ isomorphic to $\mathbf{Borel}(X)/(\mathbf{M} \cap \mathbf{Borel}(X))$.

THEOREM 3.5. *For every topological space X, we have that $\Gamma = \mathbf{Borel}(X)/\mathbf{M}$.*

PROOF. Since each closed set is Borel, we have $\Gamma \subseteq \mathbf{Borel}(X)/\mathbf{M}$. For the reverse inclusion, every open set U is equivalent to its own closure, so $[U] \in \Gamma$. But Γ is a σ-algebra, so we conclude that $\mathbf{Borel}(X)/\mathbf{M} \subseteq \Gamma$, hence the equality. ⊣

Since $\mathbf{Borel}(X)/\mathbf{M}$ is always a complete Boolean algebra (see [28, p. 75]), we obtain:

COROLLARY 3.6. *For every topological space X, we have that Γ is a complete Boolean algebra.*

In addition, we show that Γ is a relatively complete subalgebra of $\wp(X)/\mathbf{M}$:

THEOREM 3.7. *For an arbitrary topological space X, we have that Γ is a relatively complete subalgebra of $\wp(X)/\mathbf{M}$.*

PROOF. Let $a = [A] \in \wp(X)/\mathbf{M}$ and let $\mathcal{C}_a$ be the collection of all closed sets C such that $[A] \sqsubseteq [C]$. By [28, p. 75], $\bigsqcap\{[C] \mid C \in \mathcal{C}_a\} = [\bigcap \mathcal{C}_a]$. Thus, $\mathcal{C}_a$ has a least element, and hence Γ is a relatively complete subalgebra of $\wp(X)/\mathbf{M}$. ⊣

Theorems 3.4 and 3.7 yield:

THEOREM 3.8. *If X is a topological space, then $\mathbf{Baire}(X) := (\wp(X)/\mathbf{M}, \mathbf{c})$ is a monadic algebra, where for $a \in \wp(X)/\mathbf{M}$, $\mathbf{c}a$ is defined as the least closed element above a.*

REMARK 3.9. In general, for a closure algebra $(B, \mathbf{c})$, an element $a \in B$ is *closed* if $a = \mathbf{c}a$, *open* if $a = \mathbf{i}a$, and *clopen* if it is both closed and open. Since $\mathbf{Baire}(X)$ is a closure algebra, it is not hard to check that these notions coincide with the ones given in Definition 3.3.

§4. Baire resolvability. To obtain completeness results with respect to this new semantics we require to introduce the concept of Baire resolvability, which refines the concept of resolvability introduced by Hewitt [18]. A space is *resolvable* if it can be partitioned into two disjoint dense sets. This notion readily extends to the algebraic setting:

DEFINITION 4.1. Let B be a Boolean algebra. We call $a, b \in B$ *orthogonal* if $a \wedge b = 0$, and a family $\{a_i \mid i \in I\}$ *pairwise orthogonal* if $a_i \wedge a_j = 0$ for $i \neq j$.

DEFINITION 4.2. Let $(B, \mathbf{c})$ be a closure algebra and κ a nonzero cardinal.

1. We call $a \in B$ *dense* if $\mathbf{c}a = 1$.
2. We call $(B, \mathbf{c})$ κ-*resolvable* if there is a pairwise orthogonal family of dense elements whose cardinality is κ.

DEFINITION 4.3. Let $(B, \mathbf{c})$ be a closure algebra, $a \in B$, and κ a nonzero cardinal. We call a κ-*resolvable* if there is a pairwise orthogonal family $\mathcal{R} = \{b_\imath \mid \imath < \kappa\}$ such that $b_\imath \leq a$ and $a \leq \mathbf{c}b_\imath$ for each $\imath < \kappa$. The family $\mathcal{R}$ is a κ-*resolution of* a.

If $\{[A_\imath] \mid \imath < \kappa\}$ is a κ-resolution of $\mathbf{Baire}(X)$, we remark that the collection of representatives $A_\imath$ need not come from a partition of X; for example, two of them may have nonempty (but meager) intersection. However, sometimes it will be useful to have 'nice' representatives, as made precise in the following definition.

DEFINITION 4.4. Let X be a Baire space and let $Y \subseteq X$. If $A \subseteq Y$ is such that $[Y] \sqsubseteq \mathbf{c}[A]$, we say that A is *Baire-dense in* Y. For a nonzero cardinal κ, we call $Y \subseteq X$ *Baire κ-resolvable* if there is a partition $\{A_\imath \mid \imath < \kappa\}$ of Y such that, for each $\imath$, $A_\imath$ is Baire-dense in Y. We call $\{A_\imath \mid \imath < \kappa\}$ a *Baire κ-resolution of* Y.

If $\{A_\imath \mid \imath < \kappa\}$ is a Baire κ-resolution of Y, then $\{[A_\imath] \mid \imath < \kappa\}$ is a κ-resolution of $[Y]$. Thus, Baire κ-resolvability of Y implies κ-resolvability of $[Y]$. As we have mentioned, the converse is not true a priori since the $A_\imath$ do not have to form a partition, but it is not hard to check that the two notions are equivalent if κ is countable: just define recursively $A'_0 = A_0$ and $A'_{i+1} = A_{i+1} \setminus \bigcup_{j<i} A'_j$. Note that $a_\imath = [A_\imath]$ is dense in $\mathbf{Baire}(X)$ provided for each nonempty open $U \in \wp(X)$ we have $A_\imath \cap U \notin \mathbf{M}$. We call such sets 'nowhere meager'. More generally, we adopt the following conventions.

DEFINITION 4.5. Let X be a topological space and κ a cardinal. We say that:

- $A \subseteq X$ is *nowhere meager* if for each nonempty open $U \subseteq X$ we have that $A \cap U \notin \mathbf{M}$.
- $A \subseteq X$ is *somewhere meager* if it is not nowhere meager.

- X is *everywhere Baire κ-resolvable* if each open $U \subseteq X$ is Baire κ-resolvable (seen as a subspace of X).
- X is *somewhere Baire κ-resolvable* if there is some nonempty open $U \subseteq X$ that is Baire κ-resolvable.
- X is *nowhere Baire κ-resolvable* if it is not somewhere Baire κ-resolvable.

So, $A \subseteq X$ is somewhere meager provided that there is a nonempty open $U \subseteq X$ with $A \cap U \in \mathbf{M}$. We have the following characterization of Baire resolvability:

LEMMA 4.6. *Let X be a Baire space and κ a nonzero cardinal. Then X is Baire κ-resolvable iff it is everywhere Baire κ-resolvable.*

PROOF. Clearly if X is everywhere Baire κ-resolvable, then X is Baire κ-resolvable. Conversely, suppose that X is Baire κ-resolvable and $\{A_\iota \mid \iota < \kappa\}$ is a Baire κ-resolution of X. Let $U \subseteq X$ be nonempty open. Since A_ι is nowhere meager, $A_\iota \cap U$ is nowhere meager in U. Thus, $\{A_\iota \cap U \mid \iota < \kappa\}$ is a Baire κ-resolution of U. ⊣

Let X be a Baire space and $\kappa \leq \omega$. If X is Baire κ-resolvable, then X is κ-resolvable, given that Baire-dense sets must be dense. The next example shows that the converse is not true in general.

EXAMPLE 4.7. Let $X = \omega + 1$ with the topology whose nonempty open sets are of the form $[n, \omega]$ where $n < \omega$. Then $\{\omega\}$ is dense in X, and being a singleton, it cannot be written as a countable union of nowhere dense sets, so $\{\omega\} \notin \mathbf{M}$. Therefore, every nonempty open set in X is not meager, and hence X is a Baire space.

Each infinite subset of $[0, \omega)$ is dense. Therefore, if we write $[0, \omega)$ as a countable union of infinite disjoint subsets of $[0, \omega)$, we obtain that X is ω-resolvable. On the other hand, $[0, \omega) = \bigcup_{n < \omega} [0, n]$ and each $[0, n]$ is nowhere dense, so $[0, \omega) \in \mathbf{M}$. Thus, no subset of $[0, \omega)$ is nowhere meager, yielding that X is not Baire 2-resolvable.

As we pointed out in the Introduction, our main Baire resolvability results follow from the Disjoint Refinement Lemma (see [10, Lemma 7.5] and [10, Notes for Section 7] for the history of this result). This lemma has been used in proofs of resolvability in other contexts (see, e.g., [9]); ours follows a similar pattern.

LEMMA 4.8 (Disjoint Refinement Lemma). *Let κ be an infinite cardinal and $\{A_\iota \mid \iota < \kappa\}$ a family of sets such that $|A_\iota| = \kappa$ for each $\iota < \kappa$. Then there is a family $\{B_\iota \mid \iota < \kappa\}$ of pairwise disjoint sets such that $B_\iota \subseteq A_\iota$ and $|B_\iota| = \kappa$ for each $\iota < \kappa$.*

We will use the following version of the Disjoint Refinement Lemma:

LEMMA 4.9. *Let κ be an infinite cardinal, X a set, and $\mathcal{K}$ a family of subsets of X such that $|\mathcal{K}| \leq \kappa$ and $|K| \geq \kappa$ for each $K \in \mathcal{K}$. Then there is a partition $\{A_\iota \mid \iota < \kappa\}$ of X such that $A_\iota \cap K \neq \varnothing$ for each $K \in \mathcal{K}$ and $\iota < \kappa$.*

PROOF. Enumerate $\mathcal{K}$ as $\{K_\iota \mid \iota < \kappa\}$, and for each $\iota < \kappa$ let K_ι' be a subset of K_ι with $|K_\iota'| = \kappa$. By the Disjoint Refinement Lemma, there is a family $\{B_\iota \mid \iota < \kappa\}$ of pairwise disjoint sets such that $B_\iota \subseteq K_\iota'$ and $|B_\iota| = \kappa$ for each $\iota < \kappa$. Enumerating each B_ι as $\{b_{\iota v} \mid v < \kappa\}$, for $v < \kappa$, let $A_v' = \{b_{\iota v} \mid \iota < \kappa\}$. To ensure that we obtain a partition, let $A_0 = A_0' \cup \left(X \setminus \bigcup_{v < \kappa} A_v'\right)$ and for $v > 0$, let $A_v = A_v'$. Then $\{A_v \mid v < \kappa\}$ is the desired partition, since for each $\iota < \kappa$ we have that $b_{\iota v} \in A_v \cap K_\iota' \subseteq A_v \cap K_\iota$. ⊣

To apply Lemma 4.9 to Baire resolvability, we use the notion of a κ-*witnessing family.*

DEFINITION 4.10. Let X be a Baire space and κ a cardinal. We call $\mathcal{K} \subseteq \wp(X)$ a κ-*witnessing family* for X provided $|\mathcal{K}| \leq \kappa$, $|K| \geq \kappa$ for each $K \in \mathcal{K}$, and if $A \subseteq X$ is somewhere meager, then there is $K \in \mathcal{K}$ with $K \cap A = \varnothing$.

THEOREM 4.11. *Let κ be an infinite cardinal. Each Baire space that has a κ-witnessing family is Baire κ-resolvable.*

PROOF. Let X be a Baire space with a κ-witnessing family $\mathcal{K}$. By Lemma 4.9, there is a family $\{A_\iota \mid \iota < \kappa\}$ of pairwise disjoint subsets of X such that $A_\iota \cap K \neq \varnothing$ for each $K \in \mathcal{K}$ and $\iota < \kappa$. We claim that each A_ι is nowhere meager. Otherwise, since $\mathcal{K}$ is a κ-witnessing family, there is $K \in \mathcal{K}$ such that $K \cap A_\iota = \varnothing$, a contradiction. Thus, $\{A_\iota \mid \iota < \kappa\}$ gives the desired Baire κ-resolution of X. ⊣

In the next two lemmas we show that the uncountable compact sets provide a witnessing family in many Baire spaces. These lemmas are folklore, but we state them in the precise form we need and briefly recall their proofs.

LEMMA 4.12. *Let X be a crowded topological space which is either locally compact Hausdorff or completely metrizable. If $A \subseteq X$ is somewhere meager, then there is a compact $K \subseteq X$ such that $|K| \geq \mathfrak{c}$ and $A \cap K = \varnothing$.*

PROOF. The proof follows the standard tree construction method (see, e.g., [11, Exercise 3.12.11]). We provide a sketch in the case when X is a crowded locally compact Hausdorff space. Suppose that $A \subseteq X$ is somewhere meager, so there is a nonempty open $U \subseteq X$ such that $A \cap U \in \mathbf{M}$. Then $A \cap U = \bigcup_{n<\omega} B_n$, with each B_n nowhere dense. Using the assumption that X is crowded locally compact Hausdorff, to each finite sequence $\mathbf{b} = (b_0, \dots, b_n) \in \{0,1\}^{<\omega}$, we can assign a compact set $K_\mathbf{b} \subseteq U$ so that:

- $K_\mathbf{b}$ has nonempty interior;
- $B_i \cap K_\mathbf{b} = \varnothing$ for each $i \leq n$;
- if $\mathbf{b}$ is an initial segment of $\mathbf{b}'$, then $K_{\mathbf{b}'} \subseteq K_\mathbf{b}$;
- if $\mathbf{b}, \mathbf{b}'$ are incomparable (i.e., disagree on some coordinate), then $K_\mathbf{b} \cap K_{\mathbf{b}'} = \varnothing$.

Let $K = \bigcap_{n<\omega} \bigcup_{|\mathbf{b}|=n} K_\mathbf{b}$. Being an intersection of compact sets, this is a compact set. It follows from the construction that each $K_\mathbf{b}$ is contained in U and that $A \cap K = \varnothing$. The intersections along infinite sequences $\mathbf{b} \in \{0,1\}^\omega$ are nonempty and disjoint, witnessing that $|K| \geq \mathfrak{c}$.

The proof for a crowded completely metrizable X is essentially the same, but we let each $K_\mathbf{b}$ be a ball of radius at most $2^{-|\mathbf{b}|}$ so that we can use that X is complete. ⊣

LEMMA 4.13. *Let X be either Hausdorff and second-countable or metrizable and of cardinality continuum. Then X contains at most $\mathfrak{c}$ compact sets.*

PROOF. First suppose that X is Hausdorff and second-countable. The latter implies that X contains at most $\mathfrak{c}$ open sets, and so at most $\mathfrak{c}$ closed sets. Since X is Hausdorff, compact sets are closed, and hence there can be at most $\mathfrak{c}$ compact sets.

Next suppose that X is metrizable and of cardinality continuum. Then every compact set $K \subseteq X$ is the closure of some countable subset of X (see, e.g., [11,

Theorem 4.1.18]). Since X has $\mathfrak{c}$-many countable subsets, there can be at most $\mathfrak{c}$ values for K.   ⊣

Putting together these results, we arrive at the following:

THEOREM 4.14.   *Suppose that X is a crowded space which is either locally compact, Hausdorff, and second-countable or completely metrizable and continuum sized. Then X is Baire $\mathfrak{c}$-resolvable.*

PROOF.   Note that X is a Baire space. By Lemmas 4.12 and 4.13, the uncountable compact subsets of X form a $\mathfrak{c}$-witnessing family for X. Therefore, by Theorem 4.11, X is Baire $\mathfrak{c}$-resolvable.   ⊣

REMARK 4.15.   Lemma 4.12 can be generalized to the setting of Čech-complete spaces, under the assumption that no point has a least neighborhood (equivalently, that every point has infinite character: see, e.g., [11, Exercise 3.12.11(b)]). This allows us to extend Theorem 4.14 to include any second-countable Čech-complete space with the latter property.

§5. New completeness results for S5.   In this section we derive new completeness results for S5 and its extensions using our new semantics of Baire algebras. We start by recalling that a map $h\colon A \to B$ between two closure algebras $(A, \mathbf{c}_A)$ and $(B, \mathbf{c}_B)$ is a *homomorphism of closure algebras* if h is a homomorphism of Boolean algebras and $h(\mathbf{c}_A a) = \mathbf{c}_B h(a)$ for each $a \in A$. We say that h is an *embedding* if it is injective, and that h is an *isomorphism* if it is bijective.

Recall that if X and Y are topological spaces and $f\colon X \to Y$, then f is *continuous* if $f^{-1}(B)$ is open whenever $B \subseteq Y$ is open, *open* if $f(A)$ is open whenever $A \subseteq X$ is open, and an *interior map* if it is both continuous and open. It is well known (see, e.g., [25, p. 99]) that if $f\colon X \to Y$ is an interior map, then $f^{-1}\colon \mathbf{Kur}(Y) \to \mathbf{Kur}(X)$ is a homomorphism of closure algebras. Moreover, if f is surjective, then f^{-1} is an embedding. To obtain analogous results for Baire algebras, it is sufficient to work with partial maps. We remind the reader that $[A]$ denotes the equivalence class of A modulo the meager sets.

DEFINITION 5.1.   Let X, Y be arbitrary topological spaces and $f\colon X \to Y$ a partial map.

1. We say that f is defined *almost everywhere* if the complement of the domain of f is meager.
2. We call f *non-degenerate* if $B \subseteq Y$ meager implies that $f^{-1}(B)$ is meager.
3. We define $h\colon \mathbf{Baire}(Y) \to \mathbf{Baire}(X)$ by

$$h[A] = [f^{-1}(A)] \text{ for each } A \subseteq Y.$$

LEMMA 5.2.   *Let X, Y be arbitrary topological spaces and $f\colon X \to Y$ a partial map. If f is defined almost everywhere and is non-degenerate, then $h\colon \mathbf{Baire}(Y) \to \mathbf{Baire}(X)$ is a well-defined homomorphism of Boolean algebras.*

PROOF.   Let $A, B \subseteq Y$ with $[A] = [B]$. Then $A \setminus B$ is meager, so $f^{-1}(A) \setminus f^{-1}(B) = f^{-1}(A \setminus B)$ is meager since f is non-degenerate. By a symmetric argument,

$f^{-1}(B) \setminus f^{-1}(A)$ is also meager. Thus, $[f^{-1}(A)] = [f^{-1}(B)]$, and hence h is well defined.

In addition,

$$h([A] \sqcap [B]) = [f^{-1}(A \cap B)] = [f^{-1}(A)] \sqcap [f^{-1}(B)] = h(A) \sqcap h(B).$$

Therefore, h commutes with binary meets. To see that it also commutes with complements, let M be the complement of the domain of f. Then $X \setminus f^{-1}(A) = f^{-1}(Y \setminus A) \cup M$. Since f is defined almost everywhere, M is meager, so $[X \setminus f^{-1}(A)] = [f^{-1}(Y \setminus A)]$. Thus,

$$- h[A] = [X \setminus f^{-1}(A)] = [f^{-1}(Y \setminus A)] = h(- [A]).$$

Consequently, h is a well-defined homomorphism of Boolean algebras.   ⊣

Note, however, that h may fail to be injective. To obtain an embedding of Boolean algebras, we need an additional condition on f.

DEFINITION 5.3. We call a partial map $f : X \to Y$ *exact* if $f^{-1}(A)$ meager implies that A is meager for each $A \subseteq Y$.

LEMMA 5.4. *Let X, Y be Baire spaces and let $f : X \to Y$ be non-degenerate, exact, and defined almost everywhere. Then $h \colon$ **Baire**$(Y) \to$ **Baire**(X) is an embedding of Boolean algebras.*

PROOF. By Lemma 5.2, h is a homomorphism of Boolean algebras. To see that it is an embedding, let $h[A] = h[B]$. Then $f^{-1}(A) \setminus f^{-1}(B) = f^{-1}(A \setminus B)$ is meager. Since f is exact, $A \setminus B$ is meager. A symmetric argument yields that $B \setminus A$ is meager. Thus, $[A] = [B]$, and hence h is an embedding.   ⊣

Next we detail some conditions that will ensure that h is not only a Boolean homomorphism, but indeed a homomorphism of closure algebras.

DEFINITION 5.5. Let $f : X \to Y$ be a partial map. We say that:

1. f is *Baire-continuous* if V an open subset of Y implies that $[f^{-1}(V)]$ is an open element of **Baire**(X);
2. f is *Baire-open* if U a nonempty open subset and M a meager subset of X imply that $[f(U \setminus M)]$ is a nonzero open element of **Baire**(Y);
3. f is a *Baire map* if it is defined almost everywhere and is non-degenerate, Baire-continuous, and Baire-open.

REMARK 5.6. If a Baire map $f : X \to Y$ exists, then X must be a Baire space. This is because if $U \subseteq X$ were a nonempty open meager set, then $[f(U \setminus U)] = 0$ in **Baire**(Y), so f could not be Baire-open.

LEMMA 5.7. *Let X, Y be topological spaces. If $f : X \to Y$ is a Baire map, then $h \colon$ **Baire**$(Y) \to$ **Baire**(X) is a homomorphism of closure algebras. If in addition f is exact, then h is an embedding of closure algebras.*

PROOF. By Lemma 5.2, h is a homomorphism of Boolean algebras. Let $A \subseteq Y$. We first show that $\mathbf{c}h[A] \sqsubseteq h\mathbf{c}[A]$. Since $\mathbf{c}[A]$ is closed, there is a closed set $D \subseteq Y$ such that $[D] = \mathbf{c}[A]$. Because h is a homomorphism of Boolean algebras, it is order preserving. Therefore, $[A] \sqsubseteq \mathbf{c}[A]$ implies $h[A] \sqsubseteq h\mathbf{c}[A] = [f^{-1}(D)]$. Let $U = Y \setminus D$

and recall that M is the complement of the domain of f. Since U is open in Y and M is meager in X, we have

$$[f^{-1}(D)] = [f^{-1}(Y \setminus U)] = [f^{-1}(Y \setminus U) \cup M] = [X \setminus f^{-1}(U)] = -[f^{-1}(U)].$$

Because f is Baire-continuous and U is open, $[f^{-1}(U)]$ is open, hence $[f^{-1}(D)]$ is closed in $\mathbf{Baire}(X)$. Thus, $\mathbf{c}h[A] \sqsubseteq [f^{-1}(D)] = h\mathbf{c}[A]$ by the definition of closure.

We next show that $h\mathbf{c}[A] \sqsubseteq \mathbf{c}h[A]$. There is a closed set $C \subseteq X$ such that $\mathbf{c}h[A] = [C]$. Let D be as above and, using the fact that f is Baire-continuous, write $[f^{-1}(D)] = [F]$, where $F \subseteq X$ is closed. Let $V = \mathbf{i}F \setminus C$. Since F is closed in X, we have $[\mathbf{i}F] = [F] = [f^{-1}(D)]$. Therefore,

$$[V] = [f^{-1}(D)] - [C] = h\mathbf{c}[A] - \mathbf{c}h[A],$$

so it suffices to show that $V = \varnothing$. Suppose otherwise. Since $[f^{-1}(A)] \sqsubseteq [C]$, we have that $f^{-1}(A) \setminus C$ is meager, so $L := f^{-1}(A) \cap V$ is meager. Because f is Baire-open, L is meager, and V is nonempty open, $[f(V) \setminus A] = [f(V \setminus f^{-1}(A))] = [f(V \setminus L)]$ is nonzero and open in $\mathbf{Baire}(Y)$. Thus, $\mathbf{c}[A] - [f(V) \setminus A] \sqsubset \mathbf{c}[A]$. Recall that we chose D so that $\mathbf{c}[A] = [D]$. In particular, $[A] \sqsubseteq [D]$. Since $A \setminus (D \setminus (f(V) \setminus A)) = A \setminus D$ and the latter is meager, we see that

$$[A] \sqsubseteq [D \setminus (f(V) \setminus A)] = \mathbf{c}[A] - [f(V) \setminus A] \sqsubset \mathbf{c}[A].$$

As $\mathbf{c}[A] - [f(V) \setminus A]$ is closed in $\mathbf{Baire}(Y)$, this contradicts the definition of $\mathbf{c}[A]$. Consequently, $V = \varnothing$, as required.

Finally, if f is exact, then it follows from Lemma 5.4 that h is an embedding. $\dashv$

If Y is an S5-frame, it follows from Example 3.2(1) that $\mathbf{Kur}(Y)$ is isomorphic to $\mathbf{Baire}(Y)$, so we may regard h as a map from $\mathbf{Kur}(Y)$ to $\mathbf{Baire}(X)$. Since the only meager subset of an S5-frame is $\varnothing$, the next lemma is immediate.

LEMMA 5.8. *Let X be a Baire space, Y an S5-frame, and $f : X \to Y$ a Baire map. Then f is exact iff $f^{-1}(y)$ is non-meager for each $y \in Y$.*

Recall that for a nonzero cardinal κ, $\mathfrak{C}_\kappa$ denotes the κ-cluster. By [2, Lemma 5.9], a space X is κ-resolvable iff there is an interior map from X onto $\mathfrak{C}_\kappa$. We have the following analogue of this result for Baire algebras.

LEMMA 5.9. *Let X be a Baire space and κ a nonzero cardinal. Then X is Baire κ-resolvable iff there is an exact Baire map $f : X \to \mathfrak{C}_\kappa$.*

PROOF. Let $(w_\iota)_{\iota < \kappa}$ be an enumeration of $\mathfrak{C}_\kappa$. First suppose that X is Baire κ-resolvable, and let $(A_\iota)_{\iota < \kappa}$ be a Baire κ-resolution of X. Define $f : X \to \mathfrak{C}_\kappa$ by setting $f(x) = w_\iota$ provided $x \in A_\iota$. Then f is a total onto map. Since f is total, it is defined almost everywhere, and f is non-degenerate because $\varnothing$ is the only meager subset of $\mathfrak{C}_\kappa$. Since $\varnothing, \mathfrak{C}_\kappa$ are the only opens of $\mathfrak{C}_\kappa$, it is clear that f is continuous, hence Baire-continuous. To see that f is Baire-open, let $U \subseteq X$ be nonempty open and $M \subseteq X$ meager. Since each A_ι is Baire-dense, $A_\iota \cap (U \setminus M)$ is non-meager, hence nonempty, so $w_\iota \in f(U \setminus M)$, and hence $f(U \setminus M) = \mathfrak{C}_\kappa$. Therefore, f is a Baire map. Finally, $f^{-1}(w_\iota) = A_\iota$, which is non-meager. Thus, f is exact by Lemma 5.8.

Next suppose that $f : X \to \mathfrak{C}_\kappa$ is an exact Baire map. By Lemma 5.8, each $f^{-1}(w_\iota)$ is non-meager, and we can enlarge $f^{-1}(w_0)$ to obtain a partition of X if needed.

By identifying $\mathbf{Baire}(\mathfrak{C}_\kappa)$ with $\mathbf{Kur}(\mathfrak{C}_\kappa)$ and applying Lemma 5.7, $h : \mathbf{Kur}(\mathfrak{C}_\kappa) \to \mathbf{Baire}(X)$ is a homomorphism of closure algebras. Therefore, for each $\iota < \kappa$,

$$\mathbf{c}[f^{-1}(w_\iota)] = \mathbf{c}h(w_\iota) = h\mathbf{c}(w_\iota) = h(\mathfrak{C}_\kappa) = [f^{-1}(\mathfrak{C}_\kappa)] = [X].$$

Thus, $(f^{-1}(w_\iota))_{\iota<\kappa}$ witnesses that X is Baire κ-resolvable. ⊣

We are ready to prove our first completeness result. For this we recall that $\mathsf{S5}^\lambda$ denotes the variant of S5 with λ-many variables, and if $\lambda < \omega$, then $\mathsf{S5}_\lambda$ is the extension of S5 axiomatized by Alt_λ.

THEOREM 5.10. *Let X be a Baire space, λ a nonzero cardinal, and suppose that X is Baire λ-resolvable.*
(1) *If $\lambda \geq \omega$, then $\mathsf{S5}^\lambda$ is sound and strongly complete for $\mathbf{Baire}(X)$.*
(2) *If $\lambda < \omega$ and X is nowhere Baire $(\lambda + 1)$-resolvable, then $\mathsf{S5}_\lambda$ is sound and strongly complete for $\mathbf{Baire}(X)$.*

PROOF. (1) Soundness follows from the fact that $\mathbf{Baire}(X)$ is a monadic algebra (see Theorem 3.8). For completeness, let $\Gamma \models_{\mathbf{Baire}(X)} \varphi$. By Lemma 5.9, there is an exact Baire map $f : X \to \mathfrak{C}_\lambda$. Since $\mathbf{Kur}(\mathfrak{C}_\lambda)$ is isomorphic to $\mathbf{Baire}(\mathfrak{C}_\lambda)$, by Lemma 5.7 $\mathbf{Kur}(\mathfrak{C}_\lambda)$ embeds into $\mathbf{Baire}(X)$. Therefore, $\Gamma \models_{\mathfrak{C}_\lambda} \varphi$. Thus, $\Gamma \vdash_{\mathsf{S5}^\lambda} \varphi$ by Theorem 2.11.

(2) We first prove completeness. Suppose $\Gamma \models_{\mathbf{Baire}(X)} \varphi$. By Lemma 5.9, there is an exact Baire map $f : X \to \mathfrak{C}_\lambda$. By Lemma 5.7, $\mathbf{Kur}(\mathfrak{C}_\lambda)$ embeds into $\mathbf{Baire}(X)$. Therefore, $\Gamma \models_{\mathfrak{C}_\lambda} \varphi$. Thus, $\Gamma \vdash_{\mathsf{S5}_\lambda} \varphi$ by Theorem 2.11.

To prove soundness, suppose that $\mathsf{S5}_\lambda$ is not sound for X, and fix a valuation $\llbracket \cdot \rrbracket$ falsifying the $\mathsf{S5}_\lambda$ axiom Alt_λ. Since this axiom is a Boolean combination of formulas $\Diamond\psi$, we have that $\llbracket \mathrm{Alt}_\lambda \rrbracket$ is clopen, so we can choose nonempty open U with $[U] = \llbracket \neg\mathrm{Alt}_\lambda \rrbracket$. Then

$$[U] \sqsubseteq \left\llbracket \bigwedge_{i=1}^{\lambda+1} \Diamond p_i \right\rrbracket \tag{$*$}$$

and

$$[U] \sqcap \left\llbracket \bigvee_{1 \leq i < j \leq \lambda+1} \Diamond(p_i \wedge p_j) \right\rrbracket = 0. \tag{$\dagger$}$$

For $i \leq \lambda + 1$, let P_i' be such that $\llbracket p_i \rrbracket = [P_i']$. From $(*)$ we obtain for each $i \leq \lambda + 1$ that $[U] \sqsubseteq \mathbf{c}[P_i']$, which means that P_i' is nowhere meager in U, i.e., if $U' \subseteq U$ is nonempty open then $P_i' \cap U'$ is non-meager. From $(\dagger)$ it follows that $[U] \sqcap \mathbf{c}\llbracket p_i \wedge p_j \rrbracket = 0$ whenever $i < j \leq \lambda + 1$, so $U \cap P_i' \cap P_j'$ is meager. Since λ is finite, we can then replace each P_i' by some P_i in such a way that $[P_i] = [P_i']$ and $\{P_i \mid i \leq \lambda + 1\}$ forms a partition of U, witnessing that U is Baire $(\lambda + 1)$-resolvable, hence X is somewhere Baire $(\lambda + 1)$-resolvable, contrary to our assumption. ⊣

As a consequence of Theorems 5.10(1) and 4.14, we arrive at the following analogue of the McKinsey–Tarski theorem:

COROLLARY 5.11. *Let X be a crowded space which is either completely metrizable and of cardinality continuum, or second-countable locally compact Hausdorff. Then $\mathsf{S5}^\mathfrak{c}$ is sound and strongly complete for $\mathbf{Baire}(X)$.*

In particular, we obtain:

COROLLARY 5.12. S5 *is the logic of* **Baire**(X) *whenever X is*:

(1) $\mathbb{R}^n$ *for any* $n \geq 1$.
(2) *Any perfect closed subset of* $\mathbb{R}^n$, *including the Cantor space and the interval* $[0, 1]$.
(3) *The Banach space* ℓ^p *for* $p \in [1, \infty]$.

Recall that ℓ^p is the normed vector space whose elements are functions $f : \mathbb{N} \to \mathbb{R}$ whose p-norm converges (see [8] for details). Note that we may allow $p = \infty$ since $|\ell^\infty| = \mathfrak{c}$, despite not being second-countable.

REMARK 5.13. On the other hand, since $\mathbb{Q}$ is meager, **Baire**$(\mathbb{Q})$ is the trivial algebra, and hence its logic is the contradictory logic, i.e., the logic axiomatized by $\perp$. This is in contrast with the McKinsey–Tarski theorem that the logic of **Kur**$(\mathbb{Q})$ is S4.

By Theorem 5.10(2), if X is a Baire space that is Baire n-resolvable and nowhere Baire $(n + 1)$-resolvable, then the logic of **Baire**(X) is S5$_n$. As we next show, we can avoid verifying whether X is nowhere Baire $(n + 1)$-resolvable by constructing a subalgebra of **Baire**(X) whose logic is S5$_n$. For this we require the following decomposition lemma, which follows from similar decompositions in the theory of ℓ-groups; see, e.g., [5] and the references therein. To keep the paper self-contained, we give a proof.

LEMMA 5.14. *Let A be a monadic algebra, A_0 its subalgebra of all clopen elements, B a Boolean subalgebra of A, and C the Boolean subalgebra of A generated by $A_0 \cup B$.*

(1) *C is a monadic subalgebra of A.*
(2) *Each $c \in C$ can be written as $c = \bigvee_{i=1}^n (a_i \wedge b_i)$, where $a_1, \dots, a_n \in A_0$ are pairwise orthogonal and $b_1, \dots, b_n \in B$.*
(3) *Each $c^1, \dots, c^n \in C$ can be written in the compatible form $c^j = \bigvee_{i=1}^n (a_i \wedge b_i^j)$ where $a_1, \dots, a_n \in A_0$ are pairwise orthogonal and each $b_i^j \in B$.*

PROOF. (1) For $a \in C$ we have $\mathbf{c}a \in A_0 \subseteq C$ (since closed elements are clopen by Theorem 3.4). Thus, C is a monadic subalgebra of A.

(2) Since C is a Boolean subalgebra of A, it is well known (see, e.g., [25, p. 74]) that each $c \in C$ can be written as $c = \bigvee_{i=1}^n \bigwedge_{j=1}^{m_j} d_{ij}$ where either $d_{ij} \in A_0 \cup B$ or $-d_{ij} \in A_0 \cup B$. Since both A_0 and B are Boolean subalgebras, we may assume that $d_{ij} \in A_0 \cup B$; and gathering together the elements in A_0 and B, we may write $\bigwedge_{j=1}^{m_j} d_{ij} = a_i \wedge b_i$ where $a_i \in A_0$ and $b_i \in B$. Therefore, each $c \in C$ can be written as $c = \bigvee_{i=1}^n (a_i \wedge b_i)$ where $a_i \in A_0$ and $b_i \in B$. It is left to prove that the a_i can be chosen pairwise orthogonal. But this is a standard argument. Indeed, if $c = (a_1 \wedge b_1) \vee (a_2 \wedge b_2)$, then we may write

$$c = (a_1 \wedge b_1) \vee (a_2 \wedge b_2)$$
$$= \left[\left((a_1 - a_2) \wedge b_1\right) \vee \left((a_1 \wedge a_2) \wedge b_1\right)\right] \vee \left[\left((a_2 \wedge a_1) \wedge b_2\right) \vee \left((a_2 - a_1) \wedge b_2\right)\right]$$
$$= \left((a_1 - a_2) \wedge b_1\right) \vee \left((a_1 \wedge a_2) \wedge (b_1 \vee b_2)\right) \vee \left((a_2 - a_1) \wedge b_2\right),$$

where $a_1 - a_2, a_1 \wedge a_2, a_2 - a_1 \in A_0$ are pairwise orthogonal and $b_1, b_1 \vee b_2, b_2 \in B$. Now a simple inductive argument finishes the proof.

(3) We only consider the case of two elements c, d as the general case follows by simple induction. By (2), we may write $c = \bigvee_{i=1}^{n}(a_i \wedge b_i)$ where $a_1, \dots, a_n \in A_0$ are pairwise orthogonal and $b_1, \dots, b_n \in B$. Similarly $d = \bigvee_{j=1}^{m}(e_j \wedge f_j)$ where $e_1, \dots, e_m \in A_0$ are pairwise orthogonal and $f_1, \dots, f_m \in B$. By letting $a_{n+1} = \neg \bigvee_{i=1}^{n} a_i$ and $b_{n+1} = 0$ we may assume that $\bigvee_{i=1}^{n} a_i = 1$, and similarly $\bigvee_{j=1}^{m} e_j = 1$. But then

$$c = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m}((a_i \wedge e_j) \wedge b_i) \text{ and } d = \bigvee_{j=1}^{m} \bigvee_{i=1}^{n}((a_i \wedge e_j) \wedge f_j).$$

Clearly

$$a_1 \wedge e_1, \dots, a_1 \wedge e_m, \dots, a_n \wedge e_1, \dots, a_n \wedge e_m$$

are pairwise orthogonal and are in A_0; also each of b_i, f_j is in B (and may appear multiple times in the decomposition). ⊣

THEOREM 5.15. *Let X be a Baire space and $n < \omega$. If X is Baire n-resolvable, then* **Baire**(X) *has a subalgebra A containing all clopens of* **Baire**(X) *such that* $\mathsf{S5}_n$ *is sound and strongly complete for A.*

PROOF. Let $\mathfrak{C}_n = \{w_1, \dots, w_n\}$ be the n-cluster. By Lemmas 5.7 and 5.9, there is an embedding $h : \mathbf{Kur}(\mathfrak{C}_n) \to \mathbf{Baire}(X)$. Let $A = h[\mathbf{Kur}(\mathfrak{C}_n)]$ and $H = \{h(w) \mid w \in \mathfrak{C}_n\}$. Then A is a monadic subalgebra of **Baire**(X) generated by H, and each $h(w)$ is an atom of A. Let B be the Boolean subalgebra of **Baire**(X) generated by A and the clopen elements of **Baire**(X). By Lemma 5.14(1), B is a monadic subalgebra of **Baire**(X). We claim that the logic of B is $\mathsf{S5}_n$.

Since $\mathsf{S5}_n$ is complete for $\mathfrak{C}_n$, it follows that $\mathsf{S5}_n$ is complete for B. It remains to check that $\mathsf{S5}_n$ is also sound for B. For this it is sufficient to show that Alt_n is valid in B. Let $\llbracket \cdot \rrbracket$ be a valuation on B. By Lemma 5.14(3), we may write each $\llbracket p_i \rrbracket$ in the form $\bigsqcup_{\ell=1}^{m}(u_\ell \sqcap a_\ell^i)$, where the u_ℓ are orthogonal clopens and the a_ℓ^i are elements of A. Therefore, since each u_ℓ is clopen, $\mathbf{c}(u_\ell \sqcap a_\ell^i) = u_\ell \sqcap \mathbf{c}a_\ell^i$, so we have

$$\left\llbracket \bigwedge_{i=1}^{n+1} \Diamond p_i \right\rrbracket = \prod_{i=1}^{n+1} \bigsqcup_{\ell=1}^{m} \mathbf{c}(u_\ell \sqcap a_\ell^i) = \prod_{i=1}^{n+1} \bigsqcup_{\ell=1}^{m}(u_\ell \sqcap \mathbf{c}a_\ell^i) = \bigsqcup_{\ell=1}^{m}\left(u_\ell \sqcap \prod_{i=1}^{n+1} \mathbf{c}a_\ell^i\right),$$

where the last equality uses distributivity plus the orthogonality of the u_ℓ, so that all cross-terms cancel. Similarly,

$$\left\llbracket \bigvee_{i \neq j} \Diamond(p_i \wedge p_j) \right\rrbracket = \bigsqcup_{i \neq j} \bigsqcup_{\ell=1}^{m}(u_\ell \sqcap \mathbf{c}(a_\ell^i \sqcap a_\ell^j)) = \bigsqcup_{\ell=1}^{m}\left(u_\ell \sqcap \bigsqcup_{i \neq j} \mathbf{c}(a_\ell^i \sqcap a_\ell^j)\right).$$

So to show that $\llbracket \varphi \rrbracket = 1$, it suffices to show that

$$\bigsqcup_{\ell=1}^{m}\left(u_\ell \sqcap \prod_{i=1}^{n+1} \mathbf{c}a_\ell^i\right) \sqsubseteq \bigsqcup_{\ell=1}^{m}\left(u_\ell \sqcap \bigsqcup_{i \neq j} \mathbf{c}(a_\ell^i \sqcap a_\ell^j)\right). \tag{3}$$

Given that the u_ℓ are orthogonal, (3) holds iff, for each ℓ,

$$u_\ell \sqcap \prod_{i=1}^{n+1} \mathbf{c}a_\ell^i \sqsubseteq u_\ell \sqcap \bigsqcup_{i \neq j} \mathbf{c}(a_\ell^i \sqcap a_\ell^j),$$

and for this it suffices to show that

$$\prod_{i=1}^{n+1} \mathbf{c}a_\ell^i \sqsubseteq \bigsqcup_{i \neq j} \mathbf{c}(a_\ell^i \sqcap a_\ell^j). \tag{4}$$

So fix ℓ. Note that (4) holds trivially if $a_\ell^i = 0$ for some i (as the left-hand side is then zero), so we assume otherwise. Since A is generated by H, for each i there is $w \in \mathfrak{C}_n$ such that $h(w) \sqsubseteq a_\ell^i$. Because $|\mathfrak{C}_n| = n$, the pigeonhole principle yields that there are $i \neq j$ and w so that $h(w) \sqsubseteq a_\ell^i \sqcap a_\ell^j$. But $\mathbf{c}h(w) = 1$, so $\mathbf{c}(a_\ell^i \sqcap a_\ell^j) = 1$. Therefore,

$$\prod_{i=1}^{n+1} \mathbf{c}a_\ell^i \sqsubseteq 1 = \mathbf{c}(a_\ell^i \sqcap a_\ell^j) \sqsubseteq \bigsqcup_{i \neq j} \mathbf{c}(a_\ell^i \sqcap a_\ell^j).$$

Thus, (4) holds and, since ℓ was arbitrary, we conclude that (3) holds, as needed.   ⊣

As a consequence of Theorems 5.15 and 4.14 we obtain:

COROLLARY 5.16. *Let X be a crowded space which is either a complete metric space of cardinality $\mathfrak{c}$ or second-countable locally compact Hausdorff. Then for each nonzero $n < \omega$ there is a subalgebra of $\mathbf{Baire}(X)$ containing all clopen elements such that $\mathsf{S5}_n$ is sound and strongly complete for $\mathbf{Baire}(X)$.*

§6. New completeness results for $\mathsf{S5U}$. In this final section, we extend our completeness results to the language with the universal modality (recall that the latter can be interpreted in arbitrary algebraic models as per Definition 2.5). As in the previous section, these results utilize standard Kripke completeness results. In particular, the following completeness result is well known (see, e.g., [15]), and can be lifted to strong completeness via standard methods.

THEOREM 6.1. $\mathsf{S5U}$ *is sound and strongly complete for the class of countable $\mathsf{S5}$-frames.*

In the topological setting, the language $\mathcal{L}_{\Diamond\forall}$ is able to express connectedness by Shehtman's axiom

$$\forall(\Box p \vee \Box \neg p) \to (\forall p \vee \forall \neg p)$$

(see [27]). Therefore, $\mathsf{S4U}$ is not complete for any connected space. The situation is quite different in the setting of Baire algebras, and in order to see this, we will need to discuss disconnectedness in this context. This notion has already been considered in the context of closure algebras by McKinsey and Tarski [24].

DEFINITION 6.2. A closure algebra $(B, \mathbf{c})$ is *disconnected* if there are open $a, b \in B$ such that $a, b \neq 0$, $a \wedge b = 0$, and $a \vee b = 1$.

LEMMA 6.3. *If X is a Hausdorff Baire space with at least two points, then* $\mathbf{Baire}(X)$ *is disconnected.*

PROOF. Let $x \neq y$ in X, and let U, V be the disjoint open neighborhoods of x, y. Set $a = [U]$. Since U is open, a is open, hence clopen in $\mathbf{Baire}(X)$. Because X is a Baire space, U is non-meager, so $a \neq 0$. Also, $X \setminus U$ is non-meager because $V \subseteq X \setminus U$. Thus, $a \neq 1$, and hence $\mathbf{Baire}(X)$ is disconnected. ⊣

As was the case with resolvability, disconnectedness readily extends to κ-disconnectedness for any cardinal κ.

DEFINITION 6.4. A closure algebra $(B, \mathbf{c})$ is κ-*disconnected* if there is a sequence $(a_\iota)_{\iota < \kappa} \subseteq B$ of nonzero pairwise orthogonal clopen elements such that $\bigvee_{\iota < \kappa} a_\iota = 1$.

LEMMA 6.5. *If X is an infinite Hausdorff Baire space, then* $\mathbf{Baire}(X)$ *is* ω-*disconnected.*

PROOF. First suppose that X has no limit points. Then every point is isolated, so each singleton $\{x_i\}$ is open, hence clopen and non-meager. Since X is infinite, we can choose a sequence of distinct points $(x_i)_{i<\omega}$. Let $a_0 = [X \setminus \{x_i\}_{1 \leq i < \omega}]$ and for $i > 0$ let $a_i = [\{x_i\}]$. Then $(a_i)_{i<\omega}$ is a sequence of nonzero pairwise orthogonal clopens of $\mathbf{Baire}(X)$, witnessing that X is ω-disconnected.

Next suppose that X has a limit point, say x_*. We define a sequence of points $(x_i)_{i<\omega}$ and two sequences of open sets $(U_i)_{i\leq n}$ and $(V_i)_{i\leq n}$ as follows.

To begin, choose $x_0 \neq x_*$, and let U_0, V_0 be disjoint open neighborhoods of x_0, x_*, respectively. For the inductive step, suppose that $(x_i)_{i\leq n}$, $(U_i)_{i\leq n}$, $(V_i)_{i\leq n}$ are already chosen and satisfy the following conditions:

1. for all $i \leq n$, we have $x_i \in U_i$, $x_* \in V_i$, and U_i, V_i are open;
2. if $i < j \leq n$, then $U_i \cap U_j = \varnothing$, and
3. if $i \leq j \leq n$, then $U_i \cap V_j = \varnothing$.

Since x_* is a limit point, choose $x_{n+1} \in V_n \setminus \{x_*\}$. Let U, V be disjoint neighborhoods of x_{n+1}, x_*, respectively, and define $U_{n+1} = U \cap V_n$, $V_{n+1} = V \cap V_n$. It is not hard to check that the sequences $(x_i)_{i\leq n+1}$, $(U_i)_{i\leq n+1}$, $(V_i)_{i\leq n+1}$ satisfy all the desired properties.

Once we have constructed $(U_i)_{i<\omega}$, we let $a_0 = [X \setminus \bigcup_{1 \leq n < \omega} U_n]$ and for $i > 0$ we let $a_i = [U_i]$. Then $(a_i)_{i<\omega}$ is the desired sequence of nonzero pairwise orthogonal clopens of $\mathbf{Baire}(X)$. ⊣

REMARK 6.6. By Lemma 6.5, the Baire algebra of the Cantor space is ω-disconnected. In contrast, the Cantor space, while disconnected, is not ω-disconnected because, by compactness, any partition into disjoint open sets must be finite. More generally, no compact space can be ω-disconnected.

REMARK 6.7. Let X be an infinite separable Hausdorff Baire space. Then $\mathbf{Baire}(X)$ is ω-disconnected by Lemma 6.5. On the other hand, $\mathbf{Baire}(X)$ is not κ-disconnected for any cardinal $\kappa > \omega$. For, suppose that $(a_i)_{i<\kappa} \subseteq \mathbf{Baire}(X)$ are pairwise orthogonal and clopen. Then for each $i < \kappa$, there is an open set $A_i \subseteq X$ with $a_i = [A_i]$. We must have that $A_i \cap A_j = \varnothing$ if $i \neq j$, for otherwise their intersection, being open, would be non-meager. It follows that $\kappa \leq \omega$ as no separable space admits an uncountable collection of disjoint open subsets.

THEOREM 6.8. *Let X be a Baire space and W an* S5-*frame with the upper cluster size λ and the number of clusters κ. If* $\mathbf{Baire}(X)$ *is κ-disconnected and λ-resolvable, then there is an embedding $h\colon \mathbf{Kur}(W) \to \mathbf{Baire}(X)$.*

PROOF. Let $(C^\iota)_{\iota<\kappa}$ enumerate the clusters of W and let $\lambda_\iota = |C^\iota|$. Enumerate each C^ι by $(w^\iota_\nu)_{\nu<\lambda_\iota}$. Since $\mathbf{Baire}(X)$ is κ-disconnected, there are pairwise orthogonal elements $([U^\iota])_{\iota<\kappa}$ such that each U^ι is open and $\bigsqcup_{\iota<\kappa}[U^\iota] = [X]$. Because X is a Baire space, the U^ι must be pairwise disjoint as their intersection is open and meager. By Lemma 4.6, for each $\iota < \kappa$ and $\nu < \lambda_\iota$, there are $A^\iota_\nu \subseteq U^\iota$ that Baire λ_ι-resolve U^ι.

Define a partial function $f\colon X \to W$ by $f(x) = w$ if there exist $\iota < \kappa$, $\nu < \lambda_\iota$ such that $x \in A^\iota_\nu$ and $w = w^\iota_\nu$. We show that f is a Baire map. It is Baire-continuous since if $V \subseteq W$ is open, then V is a union of clusters $\bigcup_{\iota\in I} C^\iota$, hence $f^{-1}(V) = \bigcup_{\iota\in I} f^{-1}(C^\iota) = \bigcup_{\iota\in I} U^\iota$. It is Baire-open since if $V \subseteq X$ is nonempty open, $M \subseteq X$ meager, and $w \in f(V \setminus M)$, then $w = w^\iota_\nu$ for some ι, ν. Since $V \cap U^\iota \neq \varnothing$ and the sets $A^\iota_{\nu'}$ yield a Baire λ_ι-resolution of U^ι, we have that $A^\iota_{\nu'} \cap V$ is non-meager for any $\nu' < \lambda_\iota$. Therefore, $(A^\iota_{\nu'} \cap V) \setminus M$ is nonempty, and hence $w^\iota_{\nu'} \in f(V \setminus M)$. Since ν' was arbitrary, $C^\iota \subseteq f(V \setminus M)$, and since w was arbitrary, $f(V \setminus M)$ is open, hence f is Baire-open. The map f is defined almost everywhere since $[X] = \bigsqcup_{\iota<\kappa}[U^\iota]$, so $X \setminus \bigcup_{\iota<\kappa} U^\iota$ is meager. It is trivially non-degenerate since the only meager subset of W is the empty set. Finally, it is exact by Lemma 5.8 since $f^{-1}(w^\iota_\nu) = A^\iota_\nu$, which is non-meager. Thus, $h\colon \mathbf{Kur}(W) \to \mathbf{Baire}(X)$ given by $h(B) = [f^{-1}(B)]$ is an embedding by Lemma 5.7.   ⊣

THEOREM 6.9. *If X is ω-resolvable, then* S5U *is sound and strongly complete for* $\mathbf{Baire}(X)$.

PROOF. Soundness follows from $\mathbf{Baire}(X)$ being a monadic algebra. For completeness, if $\Gamma \nvdash_{\mathsf{S5U}} \varphi$, then by Theorem 6.1, there is a countable S5U-frame W and a valuation $\llbracket\cdot\rrbracket_W$ such that $\Gamma \not\models_W \varphi$. By Theorem 6.8, there is an embedding $f\colon \mathbf{Kur}(W) \to \mathbf{Baire}(X)$. Thus, $\Gamma \not\models_{\mathbf{Baire}(X)} \varphi$.   ⊣

COROLLARY 6.10. *Let X be a crowded space which is either completely metrizable and of cardinality continuum or locally compact Hausdorff. Then* S5U *is sound and strongly complete for* $\mathbf{Baire}(X)$. *In particular,* S5U *is sound and strongly complete for* $\mathbf{Baire}(\mathbb{R})$ *and* $\mathbf{Baire}(\mathbf{C})$, *where* $\mathbf{C}$ *is the Cantor space.*

Corollary 6.10 can be viewed as an analogue of the McKinsey–Tarski theorem for S5U. As mentioned in Remark 5.13, a similar result does not hold for $\mathbb{Q}$, even though S4U is complete for $\mathbb{Q}$ but not $\mathbb{R}$ (see, e.g., [27]), so the picture in the Baire setting is a bit different from the original topological semantics.

REMARK 6.11. Corollary 6.10 does not extend to uncountable languages in view of Remark 6.7 as the collection

$$\{\exists\Box p_\iota\}_{\iota<\kappa} \cup \{\neg\exists(p_\iota \wedge p_\varsigma)\}_{\iota<\varsigma<\kappa}$$

cannot be satisfied on, e.g., the real line if $\kappa > \omega$ since this would require $\llbracket p_\iota \rrbracket = [U_\iota]$ with disjoint open sets U_ι, which is impossible on $\mathbb{R}$.

§7. **Concluding remarks.** We have shown that the Baire algebra of any topological space is a closure algebra, and that the logic of these algebras is S5, providing variants of the celebrated McKinsey–Tarski theorem for the logic S5. We have identified Baire resolvability as a sufficient condition for strong completeness. For such spaces, we have also constructed subalgebras of the Baire algebra whose logic is $S5_n$ (for any finite n). Given a cardinal λ, strong completeness also holds for variants of S5 with λ-many variables provided that the space is Baire λ-resolvable. In addition, we have shown that if the Baire algebra is ω-disconnected, then strong completeness extends to S5U. Finally, we have shown that crowded spaces which are either complete continuum-sized metric or locally compact Hausdorff enjoy the above properties, leading to a large class of concrete examples of spaces for which S5 and S5U are strongly complete.

This work follows [12, 22] in studying point-free semantics for modal logic based on quotients of the powerset algebras of topological spaces. There are many other σ-ideals that may be used to define similar quotients, e.g., the σ-ideal of countable subsets. A general and systematic study of such quotients and their associated modal logics remains an interesting line of research.

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