EXTENSIONS OF VANDERMONDE TYPE CONVOLUTIONS WITH SEVERAL SUMMATIONS AND THEIR APPLICATIONS - II

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1. Summary. In earlier papers [3] and [4] some Vandermonde type convolution identities with several summations were established which were the extensions of Gould's work that involved one summation. Furthermore, these identities were utilized to obtain some inverse series relations for functions of several variables. As a continuation, this paper deals with some further generalizations of Gould's work in [1] and [2].

2. Introduction. In [1], Gould has developed some inverse series relations for the generalized Humbert polynomial $P_n(m, x, y, p, C)$ defined by

$$(1) \qquad (C - mxt + yt^m)^p = \sum_{n=0}^{\infty} t^n P_n(m, x, y, p, C) ,$$

where $m \geq 1$ is an integer and the other parameters are unrestricted. Let us consider the polynomial of degrees $n_1, n_2, \ldots, n_k$ in k variables $x_1, x_2, \ldots, x_k$ as the coefficient of $\prod_{i=1}^{k} t_i^{n_i}$ in the expansion of

$$(2) \qquad H_k = (C - m \sum_{i=1}^{k} x_i t_i + \sum_{i=1}^{k} y_i t_i^m)^p .$$

If we denote the polynomial by

$$P \begin{pmatrix} x_1, \ldots, x_k; m \\ y_1, \ldots, y_k; C \\ n_1, \ldots, n_k; p \end{pmatrix} \qquad \text{or briefly} \qquad P(n_1, \ldots, n_k),$$

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then

(3) $$H_k = \sum_{n_k=0}^{\infty} \cdots \sum_{n_1=0}^{\infty} P(n_1, \ldots, n_k) \prod_{i=1}^{k} t_i^{n_i},$$

where $m \geq 1$ is an integer and the other parameters are unrestricted. Observe that (3) is a natural generalization of (1).

In Section 3, some properties of $P(n_1, \ldots, n_k)$ and the inverse series relation for such polynomials are derived along the lines of [1].

Section 4 of the paper deals with the generalized notion of quasi-orthogonality in [2] and [6], which is defined as follows. Consider two multi-dimensional number sets $\{A(n_1, \ldots, n_k; j_1, \ldots, j_k)\}$ and $\{B(n_1, \ldots, n_k; j_1, \ldots, j_k)\}$ briefly denoted by $\{A(n_i, j_i; k)\}$ and $\{B(n_i, j_i; k)\}$ respectively, which arise in the expansion of ϕ- and ψ-polynomials given below;

$$\phi(x_1, \ldots, x_k; n_1, \ldots, n_k) = \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} A(n_i, j_i; k) \prod_{i=1}^{k} x_i^{j_i},$$

and

$$\psi(x_1, \ldots, x_k; n_1, \ldots, n_k) = \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} B(n_i, j_i; k) \prod_{i=1}^{k} x_i^{j_i}.$$

Set $A(n_i, j_i; k) = 0 = B(n_i, j_i; k)$, if either $n_i < 0$ or $j_i < 0$, or $n_i < j_i$, $i = 1, 2, \ldots, k$. We say that the number sets $\{A(n_i, j_i; k)\}$ and $\{B(n_i, j_i; k)\}$ are quasi-orthogonal when

(4) $$\sum_{j_k=m_k}^{n_k} \cdots \sum_{j_1=m_1}^{n_1} A(n_i, j_i; k)\, B(j_i, m_i; k) = \prod_{i=1}^{k} \delta_{m_i}^{n_i},$$

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$$(5) \qquad \sum_{j_k=m_k}^{n_k} \cdots \sum_{j_1=m_1}^{n_1} B(n_i, j_i; k)\; A(j_i, m_i; k) = \prod_{i=1}^{k} \delta_{m_i}^{n_i} ,$$

where

$$\delta_{m_i}^{n_i} = \begin{cases} 1 & \text{if} \quad m_i = n_i \\ 0 & \text{if} \quad m_i \neq n_i \end{cases}$$

are satisfied for all m_i and n_i.

We denote the multinomial coefficient

$$\frac{x(x-1)\ \ldots\ (x - \sum_{i=1}^{k} j_i + 1)}{\prod_{i=1}^{k} j_i\,!}$$

for any x and non-negative integral values of the j_i by $(x; j_1, \ldots, j_k)$ with the convention that $(x; 0, \ldots, 0) = 1$.

For notational simplicity $\sum_1^k$ and $\prod_1^k$ will be replaced by Σ and Π, respectively, throughout this paper.

3. Generalized Humbert polynomials and the inverse series relation. Expanding (2) in powers of $t_1, \ldots, t_k$ by Maclaurin's theorem for a function of several variables and comparing with (3), one finds the expression for $P(n_1, \ldots, n_k)$ as

$$(6) \quad P(n_1, \ldots, n_k) = \sum_{j_k=0}^{\left[\frac{n_k}{m}\right]} \cdots \sum_{j_1=0}^{\left[\frac{n_1}{m}\right]} (p; j_1, \ldots, j_k)(p - \Sigma j_i; n_1 - mj_1, \ldots, n_k - mj_k)$$

$$\times\ C^{p - \Sigma(n_i - (m-1)j_i)}\ \Pi\{ (-mx_i)^{n_i - mj_i} \cdot y_i^{j_i}\} ,$$

where $[x]$ is the greatest integer less than or equal to x. As a special

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case, when $n_i < m$, $i = 1, 2, \ldots, k$, $P(n_1, \ldots, n_k) = (p; n_1, \ldots, n_k) \times$
$C^{p - \Sigma n_i} \Pi(-m x_i)^{n_i}$.

Next, we derive some recurrence relations for $P(n_1, \ldots, n_k)$. Differentiation of (2) with respect to t_i yields

$$(7) \qquad t_i(C - m\Sigma x_i t_i + \Sigma y_i t_i^m) \frac{\partial H_k}{\partial t_i} = -mp(x_i t_i - y_i t_i^m) H_k ,$$

for $i = 1, 2, \ldots, k$, which, due to (3), implies the recurrence relation

$$(8) \qquad Cn_i P(n_1, \ldots, n_k) + mx_i(p+1) P(n_1, \ldots, n_i - 1, \ldots, n_k)$$

$$- \; m y_i(p+1) P(n_1, \ldots, n_i - m, \ldots, n_k)$$

$$- \; n_i \sum_{j=1}^{k} \{mx_j P(n_1, \ldots, n_j - 1, \ldots, n_k) - y_j P(n_1, \ldots, n_j - m, \ldots, n_k)\}$$

$$= \; 0, \text{ for } 1 \le m \le n_i, \; i = 1, \ldots, k .$$

Furthermore, since

$$t_i \frac{\partial H_k}{\partial t_i} = (x_i - y_i t_i^{m-1}) \frac{\partial H_k}{\partial x_i} ,$$

we have

$$(9) \qquad n_i P(n_1, \ldots, n_k) = x_i \frac{\partial P(n_1, \ldots, n_k)}{\partial x_i} - y_i \frac{\partial P(n_1, \ldots, n_i - m + 1, \ldots, n_k)}{\partial x_i}$$

for $i = 1, \ldots, k$.

Other recurrence relations similar to (2.6), (2.7) and (2.8) in [1], can be derived, but the form is too cumbersome to be of any interest. By substituting $m = 2$, $p = -1$, $C = 1$ and $x_i = \frac{1}{2}$, $y_i = -1$ for all i in (8), the following recurrence relation is obtained:

(10) $$P(n_1, \ldots, n_k) = \sum_{j=1}^{k} \{P(n_1, \ldots, n_j - 1, \ldots, n_k) + P(n_1, \ldots, n_j - 2, \ldots, n_k)\}.$$

Thus, the well known recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

for Fibonacci numbers, can be viewed as a special case of (10).

The generalized expressions corresponding to (3.4) and (3.5) in [1] are

(11) $$\frac{\partial^s P'(n_1, \ldots, n_i + s, \ldots, n_k)}{\partial x_i^s}$$

$$= \left(\frac{m}{q}\right)^s \Pi(1 + jq) \Sigma^* \{P'(j_{1,1}, \ldots, j_{k,1}) P'(j_{1,2}, \ldots, j_{k,2}) \ldots P'(j_{1,sq+1}, \ldots, j_{k,sq+1})\}$$

valid for $q = 1, 2, \ldots,$ where $P'(n_1, \ldots, n_k)$ is $P(n_1, \ldots, n_k)$ with $p = -1/q$ and Σ^* denotes the summation over all j such that

$$j_{i,1} + j_{i,2} + \ldots + j_{i,sq+1} = n_i; \; i = 1, \ldots, k$$

and

(12) $$\frac{\partial^s}{\partial x_i^s} \left\{ P \begin{pmatrix} x_1, \ldots, x_k; & m \\ y_1, \ldots, y_k; & C \\ n_1, \ldots n_k; & p \end{pmatrix} \right\} = (-m)^s \binom{p}{s} s! \, P \begin{pmatrix} x_1, \ldots, x_k; & m \\ y_1, \ldots, y_k; & C \\ n_1, \ldots, n_k; & p-s \end{pmatrix}.$$

Next, we prove the main inversion formula which is the generalization of (6.3) and (6.4) in [1].

THEOREM 1.

(13) $$F(n_1, \ldots, n_k) = \sum_{j_k=0}^{\left[\frac{n_k}{m}\right]} \ldots \sum_{j_1=0}^{\left[\frac{n_1}{m}\right]} (p - \Sigma(n_i - mj_i); \; j_1, \ldots, j_k)$$
$$\times \; f(n_1 - mj_1, \ldots, n_k - mj_k)$$

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if and only if

$$(14)\quad f(n_1,\ldots,n_k) = \sum_{j_k=0}^{\left[\frac{n_k}{m}\right]} \cdots \sum_{j_1=0}^{\left[\frac{n_1}{m}\right]} (-1)^{\Sigma j_i} \frac{p+\Sigma(mj_i-n_i)}{p-\Sigma(n_i-j_i)} (p-\Sigma(n_i-j_i); j_1,\ldots,j_k)$$

$$\times \quad F(n_1-mj_1,\ldots,n_k-mj_k)\ ,$$

where m, p are any numbers with $m > 0$ and the n_i are non-negative integers.

As usual, the proof depends on the orthogonality relations

$$(15)\quad \sum_{r_k=0}^{j_k} \cdots \sum_{r_1=0}^{j_1} (-1)^{\Sigma r_i} \frac{p+\Sigma(mr_i-n_i)}{p-\Sigma(n_i-r_i)} (p-\Sigma(n_i-r_i); r_1,\ldots,r_k)$$

$$\times \quad (p-\Sigma(n_i-mj_i); j_1-r_1,\ldots,j_k-r_k) = \delta^0_{j_i}\ ,$$

and

$$(16)\quad \sum_{j_k=0}^{r_k} \cdots \sum_{j_1=0}^{r_1} (-1)^{\Sigma j_i} (p-\Sigma(n_i-mj_i); j_1,\ldots,j_k)$$

$$\times \quad \frac{p+\Sigma(mr_i-n_i)}{p+\Sigma\{r_i-n_i+(m-1)j_i\}} (p+\Sigma\{r_i-n_i+(m-1)j_i\}; r_1-j_1,\ldots r_k-j_k)$$

$$= \quad \delta^0_{r_i}\ .$$

The proofs of (15) and (16), which are based on arguments analogous to that in [1], make use of convolution identities in [4] and the obvious property of multinomial coefficients

$$(17)\quad (-1)^{\Sigma r_i} (-x; r_1,\ldots,r_k) = (x+\Sigma r_i-1; r_1,\ldots,r_k)\ .$$

Rewriting (6) in the form

$$(18)\quad \Pi\left(\frac{C}{y_i}\right)^{n_i/m} P(n_1,\ldots,n_k) = \sum_{j_k=0}^{[n_k/m]} \cdots \sum_{j_1=0}^{[n_1/m]} (p-\Sigma(n_i-mj_i); j_1,\ldots,j_k)$$

$$\times \quad (p; n_1-mj_1,\ldots,n_k-mj_k)\; C^{p-(1-\frac{1}{m})\Sigma(n_i-mj_i)}\; \Pi\{y_i^{-\frac{1}{m}(n_i-mj_i)}$$

$$\times \quad (-mx_i)^{n_i-mj_i}\},$$

and applying Theorem 1, leads to the inversion

$$(19)\quad (p; n_1,\ldots,n_k)\,\Pi(-mx_i)^{n_i} = \sum_{r_k=0}^{[n_k/m]} \cdots \sum_{r_1=0}^{[n_1/m]} (-1)^{\Sigma r_i} (p-\Sigma(n_i-r_i); r_1,\ldots,r_k)$$

$$\times \quad \frac{p+\Sigma(mr_i-n_i)}{p-\Sigma(n_i-r_i)}\,\Pi y_i^{r_i}\; C^{-p+\Sigma(n_i-r_i)}\quad P(n_1-mr_1,\ldots,n_k-mr_k)\;.$$

Finally, we state the generalization of the last theorem of [1] as follows.

Let

$$g(x_1,\ldots,x_k) = \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} a(j_1,\ldots,j_k)\,\Pi x_i^{j_i},$$

be any arbitrary polynomial in $x_1,\ldots,x_k$ of degrees $n_1,\ldots,n_k$, respectively. Then $g(x_1,\ldots,x_k)$ can be expressed in terms of the polynomials $P(n_1,\ldots,n_k)$ by the formula

$$(20)\qquad g(x_1,\ldots,x_k) = \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} V(j_1,\ldots,j_k)\,P(j_1,\ldots,j_k),$$

where

$$V(j_1,\ldots,j_k) = \sum_{r_k=0}^{\left[\frac{n_k-j_k}{m}\right]} \cdots \sum_{r_1=0}^{\left[\frac{n_1-j_1}{m}\right]} (-1)^{\Sigma(r_i+j_i+mr_i)} \frac{(p-\Sigma(j_i+mr_i-r_i);\, r_1,\ldots,r_k)}{(p;\, j_1+mr_1,\ldots,j_k+mr_k)}$$

$$\times \frac{p-\Sigma j_i}{p-\Sigma(j_i+mr_i-r_i)} \Pi y_i^{r_i}\, C^{\Sigma(j_i+mr_i-r_i)-p} \frac{a(j_1+mr_1,\ldots,j_k+mr_k)}{m^{\Sigma(j_i+mr_i)}} .$$

4. Generalized quasi-orthogonality. In this section we extend the results in [2] and [7] on quasi-orthogonal sets of numbers.

Consider two sets of quasi-orthogonal numbers $\{A(n_i, j_i; k)\}$ and $\{B(n_i, j_i; k)\}$, in the sense that both (4) and (5) are satisfied. Then it is easy to check that

$$\psi(x_1,\ldots,x_k;\, n_1,\ldots,n_k) = \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} A(n_i, j_i; k)\, \Pi\, x_i^{j_i} \tag{21}$$

if and only if

$$\Pi\, x_i^{n_i} = \sum_{r_k=0}^{n_k} \cdots \sum_{r_1=0}^{n_1} B(n_i, r_i; k)\, \psi\,(x_1,\ldots,x_k;\, r_1,\ldots,r_k) . \tag{22}$$

Next, we construct new sets of quasi-orthogonal numbers from known sets of such numbers.

THEOREM 2. *Let $\{A(n_i, j_i; k)\}$ and $\{B(n_i, j_i; k)\}$ be two sets of quasi-orthogonal numbers. Furthermore, suppose that $Q(j_1,\ldots,j_k)$, $j_i = 0, 1, \ldots,$ for all i be a k-dimensional sequence of numbers (real or complex) such that neither Q, nor $Q^{-1} = 1/Q$ is zero for any relevant value of the index. Then the set of numbers defined by*

$$\begin{cases} A^*(n_i, j_i; k) = A(n_i, j_i; k)\; Q(j_1,\ldots,j_k) ; \\ B^*(n_i, j_i; k) = B(n_i, j_i; k)\; Q^{-1}(n_1,\ldots,n_k) ; \end{cases} \tag{23}$$

forms a quasi-orthogonal set of numbers.

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Proof. The proof is trivial since

$$\sum_{j_k=m_k}^{n_k} \cdots \sum_{j_1=m_1}^{n_1} A^*(n_i, j_i; k)\, B^*(j_i, m_i; k) = \sum_{j_k=m_k}^{n_k} \cdots \sum_{j_1=m_1}^{n_1} A(n_i, j_i; k)\, B(j_i, m_i; k),$$

and

$$\sum_{j_k=m_k}^{n_k} \cdots \sum_{j_1=m_1}^{n_1} B^*(n_i, j_i; k)\, A^*(j_i, m_i; k)$$

$$= Q^{-1}(n_1, \ldots, n_k)\, Q(m_1, \ldots, m_k) \sum_{j_k=m_k}^{n_k} \cdots \sum_{j_1=m_1}^{n_1} B(n_i, j_i; k)\, A(j_i, m_i; k).$$

We remark that the above theorem unifies the generalizations of several results (viz. Theorems 1, 2, and 3) in [2].

A special case of Theorem 2(a) in [4], when

(i) $a_i = m_i$ is an integer and $b_i = 0$ for all i

(ii) $F(m_1, \ldots, m_k) = (-1)^{\Sigma m_i} g(m_1, \ldots, m_k)$

(iii) $f(m_1, \ldots, m_k) = \Pi\, x_i^{m_i}$

is the inversion relation

$$(24) \quad g(m_1, \ldots, m_k) = \sum_{j_k=0}^{m_k} \cdots \sum_{j_1=0}^{m_1} (-1)^{\Sigma j_i} (\Sigma m_i; m_1 - j_1, \ldots, m_k - j_k)\, \Pi\, x_i^{j_i}$$

if and only if

$$(25) \quad \Pi\, x_i^{m_i} = \sum_{j_k=0}^{m_k} \cdots \sum_{j_1=0}^{m_1} (-1)^{\Sigma j_i} (\Sigma m_i; m_1 - j_1, \ldots, m_k - j_k)\, g(j_1, \ldots, j_k).$$

Comparing (24) and (25) with (21) and (22) respectively, we observe that the two sets given by

$$(26)\qquad \begin{cases} A(n_i, j_i; k) = (-1)^{\Sigma j_i} (\Sigma n_i; n_1 - j_1, \ldots, n_k - j_k); \\ B(n_i, j_i; k) = (-1)^{\Sigma j_i} (\Sigma n_i; n_1 - j_1, \ldots, n_k - j_k); \end{cases}$$

are quasi-orthogonal. An application of Theorem 2 shows that

$$(27)\qquad \begin{cases} A(n_i, j_i; k) = (-1)^{\Sigma j_i} (\Sigma n_i; n_1 - j_1, \ldots, n_k - j_k)\,(\Sigma j_i; j_1, \ldots, j_k); \\ B(n_i, j_i; k) = (-1)^{\Sigma j_i} (\Sigma n_i; n_1 - j_1, \ldots, n_k - j_k)/(\Sigma n_i; n_1, \ldots, n_k); \end{cases}$$

are quasi-orthogonal, which are essentially the sets as suggested by Tauber in [6].

Again, set $C = 1$ and $y_i = 1$, for all i in (18) and (19) and after some elementary simplifications we obtain

$$(28)\qquad P(n_1, \ldots, n_k) = \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} \left(p - \Sigma j_i; \frac{n_1 - j_1}{m}, \ldots, \frac{n_k - j_k}{m}\right)$$

$$\times\ (p; j_1, \ldots, j_k)\, \Pi \left\{(-mx_i)^{j_i}\, g_m(n_i - j_i)\right\}$$

if and only if

$$(29)\qquad \Pi\,(-mx_i)^{n_i} = \sum_{j_k=0}^{n_k} \cdots \sum_{j_1=0}^{n_1} (-1)^{\Sigma\left(\frac{n_i - j_i}{m}\right)} \frac{p - \Sigma j_i}{p - \Sigma n_i + \Sigma\left(\frac{n_i - j_i}{m}\right)}$$

$$\times\ \frac{\left(p - \Sigma n_i + \Sigma\left(\frac{n_i - j_i}{m}\right); \frac{n_1 - j_1}{m}, \ldots, \frac{n_k - j_k}{m}\right)}{(p; n_1, \ldots, n_k)}\, P(j_1, \ldots, j_k)\, \Pi\, g_m(n_i - j_i),$$

where

$$g_k(r) = \begin{cases} 1 & \text{if } k \mid r, \\ 0 & \text{if } k \nmid r. \end{cases}$$

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This leads to the quasi-orthogonal sets of numbers

$$(30)\quad \begin{cases} A(n_i, j_i; k) = \left(p - \Sigma j_i;\ \frac{n_1-j_1}{m}, \ldots, \frac{n_k-j_k}{m}\right)(p; j_1, \ldots, j_k)\,\Pi\, g_m(n_i-j_i); \\ B(n_i, j_i; k) = (-1)^{\Sigma\left(\frac{n_i-j_i}{m}\right)} \cdot \dfrac{p - \Sigma j_i}{p - \Sigma n_i + \Sigma\left(\frac{n_i-j_i}{m}\right)}\ \Pi\, g_m(n_i - j_i) \\ \qquad \times \dfrac{\left(p - \Sigma n_i + \Sigma\left(\frac{n_i-j_i}{m}\right);\ \frac{n_1-j_1}{m}, \ldots, \frac{n_k-j_k}{m}\right)}{(p;\ n_1, \ldots, n_k)} \quad ; \end{cases}$$

and by the use of Theorem 2 with $Q(j_1, \ldots, j_k) = 1/(p; j_1, \ldots, j_k)$, one finds the following quasi-orthogonal sets:

$$(31)\quad \begin{cases} A(n_i, j_i; k) = \left(p - \Sigma j_i;\ \frac{n_1-j_1}{m}, \ldots, \frac{n_k-j_k}{m}\right)\Pi\ g_m(n_i-j_i)\ ; \\ B(n_i, j_i; k) = (-1)^{\Sigma\left(\frac{n_i-j_i}{m}\right)} \dfrac{p - \Sigma j_i}{p - \Sigma n_i + \Sigma\left(\frac{n_i-j_i}{m}\right)} \\ \qquad \times \left(p - \Sigma n_i + \Sigma\left(\frac{n_i-j_i}{m}\right);\ \frac{n_1-j_1}{m}, \ldots, \frac{n_k-j_k}{m}\right)\Pi\ g_m(n_i-j_i)\ . \end{cases}$$

Lastly, we state that for any integer $m \geq 1$

$$(32)\quad \begin{cases} A(n_i, j_i; k) = (-1)^{\Sigma\left(\frac{n_i-j_i}{m}\right)} \cdot \dfrac{X(n_1, \ldots, n_k)}{\Pi\left(\frac{n_i-j_i}{m}\right)!}\ \Pi\ g_m(n_i-j_i)\ ; \\ B(n_i, j_i; k) = \dfrac{1}{X(j_1, \ldots, j_k)\ \Pi\left(\frac{n_i-j_i}{m}\right)!}\ \Pi\ g_m(n_i-j_i)\ ; \end{cases}$$

are sets of quasi-orthogonal numbers for any sequence of numbers $X(n_1, \ldots, n_k)$. This result corresponds to Theorem 2 in [2].

Other special cases are obvious and therefore omitted. Concluding the discussion, we remark that it might be of interest to see the corresponding generalizations of Riordan's results in [5] which are comparable to the results of this paper.

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