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ULTRABORNOLOGICAL BOCHNER INTEGRABLE FUNCTION SPACES

J.C. FERRANDO

If (Ω, Σ, μ) is a finite measure space and X is a normed space such that X^* has the Radon-Nikodym property with respect to μ, we show first that each space $L_p(\mu, x)$, $1 < p < \infty$, is ultrabornological whenever μ is atomless. When μ is arbitrary, we prove later on that the space $L_p(\mu, X)$ is ultrabornological if X^* has the Radon-Nikodym property with respect to μ and X is itself an ultrabornological space.

In what follows, (Ω, Σ, μ) will be a finite measure space and X a normed space. If $1 \leqslant p < \infty$, $L_p(\mu, X)$ stands for the space of all (equivalence classes of) X-valued Bochner integrable functions f on Ω with $\int_\Omega \|f\|^p \, d\mu < \infty$, provided with the norm

$$\|f\| = \left(\int_\Omega \|f(\omega)\|^p \, d\mu(\omega) \right)^{1/p}.$$

In this paper a Hausdorff locally convex spaces is said to be ultrabornological if it can be represented as the locally convex hull of all its Banach subspaces with a basis.

If the measure μ is atomless and the Banach space X^* has the Radon-Nikodym property with respect to μ, we are going to show first that $L_p(\mu, X)$ is an ultrabornological space for each $1 < p < \infty$, regardless of whether or not X is ultrabornological. This must not be a surprising fact if we take into account the two following results concerning finite atomless measure spaces which has been proved recently.

(A): (Drewnowski, Florencio and Paúl, [5], Theorem 3). Let (Ω, Σ, μ) be a finite measure space and X be a normed space. If the measure μ is atomless, then $L_p(\mu, X)$ with $1 \leqslant p < \infty$ is barrelled.

(B): (Díaz, Florencio and Paúl [2, Main Theorem]). Let (Ω, Σ, μ) be a finite measure space and X be a normed space. If the measúre μ is atomless, then every weak bounded subset of the dual of $L_\infty(\mu, X)$ is bounded in norm; in other words $L_\infty(\mu, X)$ is barrelled.

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Supposing that μ is an arbitrary finite measure on Ω and X is an ultrabornological space such that X^* has the Radon-Nikodym property with respect to μ, we shall prove later on that the dense subspace $S_p(\mu, X)$ of $L_p(\mu, X)$, with $1 < p < \infty$, of all countably valued functions, is ultrabornological. In the particular case when μ is purely atomic, this implies that each normed space $\ell_p\{X\}$, $1 < p < \infty$, of absolutely ℓ_p-summable sequences [**6**, p.139] with the usual norm, is ultrabornological. Since each finite measure space decomposes into a purely atomic part and an atomless part, the previous results imply that if μ is an arbitrary finite measure on Ω, X is ultrabornological and X^* has the Radon-Nikokym property with respect to μ, then the space $L_p(\mu, X)$, $1 < p < \infty$, is ultrabornological.

THEOREM 1. *Let $1 < p < \infty$. If the measure μ is atomless and X^* has the Radon-Nikodym property with respect to μ, then $L_p(\mu, X)$ is ultrabornological.*

PROOF: Suppose there exists some p, $1 < p < \infty$ such that $L_p(\mu, X)$ is not an ultrabornological space. Then, there is an absolutely convex set V in $L_p(\mu, X)$ that meets each Banach subspace F with a basis of $L_p(\mu, X)$ in a neighbourhood of the origin in F, but V is not itself a neighbourhood of the origin in $L_p(\mu, X)$. We are going to begin the proof building inductively a sequence $\{(\Omega_n, \Sigma_n, \mu_n), \quad n = 1, 2, \ldots\}$ of atomless finite measure spaces satisfying for each $n \in \mathbb{N}$ the following properties

(i) $\Omega_n \in \Sigma_{n-1}$, $\Omega_{n+1} \subseteq \Omega_n$, $\mu(\Omega_n) = \mu(\Omega_{n-1})/2$

(ii) $\Sigma_n = \Sigma_{n-1|\Omega_n}$, $\mu_n = \mu_{n-1|\Sigma_n}$

(iii) V does not absorb the unit sphere of $L_p(\mu, X)$.

The atomless character of μ assures the existence of an $A \in \Sigma$ such that $\mu(A) = \mu(\Omega)/2$. By restricting Σ and μ to A and its complement $\Omega \setminus A$ we obtain the atomless measure spaces $(A, \mathcal{A}, \lambda)$ and $(\Omega \setminus A, \mathcal{A}', \lambda')$ so that $L_p(\mu, X)$ is the topological direct sum of $L_p(\lambda, X)$ and $L_p(\lambda', X)$. If W and W' denote the unit spheres of $L_p(\lambda, X)$ and $L_p(\lambda', X)$, respectively, then it is clear that V either does not absorb W or it does not absorb W'. Thus, setting $(\Omega_1, \Sigma_1, \mu_1)$ to be either $(A, \mathcal{A}, \lambda)$, or $(\Omega \setminus A, \mathcal{A}', \lambda')$, depending on whether V does not absorb W or does not absort W', we have accomplished the first step of our induction process.

Now assuming $(\Omega_i, \Sigma_i, \mu_i)$, $1 \leqslant i \leqslant n$, are already defined, we proceed to build $(\Omega_{n+1}, \Sigma_{n+1}, \mu_{n+1})$ with the former requirements.

The atomless character of μ_n yields a certain $A_n \in \Sigma_n$ such that $\mu_n(A_n) = \mu_n(\Omega_n)/2$. As before, $L_p(\mu_n, X)$ can be descomposed as the topological direct sum of $L_p(\lambda_n, X)$ and $L_p(\lambda'_n, X)$, where $(A_n, \mathcal{A}_n, \lambda_n)$ and $(\Omega_n \setminus A_n, \mathcal{A}'_n, \lambda'_n)$ are the respective restrictions of Σ_n and μ_n to A_n and $\Omega_n \setminus A_n$. Since V does not absorb the unit sphere V_n of $L_p(\mu_n, X)$, it cannot absorb both W_n and W'_n (the unit spheres of $L_p(\lambda_n, X)$ and $L_p(\lambda'_n, X)$, respectively). Hence, we choose again $(\Omega_{n+1}, \Sigma_{n+1}, \mu_{n+1})$ to be either $(A_n, \mathcal{A}_n, \lambda_n)$ or $(\Omega_n \setminus A_n, \mathcal{A}'_n, \lambda'_n)$, according to whether V does not absorb W_n or does not absorb W'_n. This ends our induction process.

For each $n \in \mathbb{N}$ we select an element $h_n \in V_n$ such that $h_n \notin nV$, and then we form the sequence (h_n) contained in the unit sphere of $L_p(\mu, X)$, with the support of each h_n contained in Ω_n.

Define $E := \bigcap_{k=1}^{\infty} \Omega_k$ and note that $\mu(E) = \lim_{n\to\infty} \mu(\Omega_n) = 0$.

Since X^* has the Radon-Nikodym property with respect to μ, then $L_q(\mu, X^*)$ with $1/p + 1/q = 1$ is the topological dual of $L_p(\mu, X)$, (see [**3**, p.98]) and we are going to see next that $\langle f, h_n\rangle \to 0$ for each $f \in L_q(\mu, X^*)$.

In fact, if $p > 1$ and $f \in L_q(\mu, X^*)$, then

$$\begin{aligned}|\langle f, h_n\rangle|^q &= \left|\int_\Omega \langle f(\omega), h_n(\omega)\rangle d\mu(\omega)\right|^q = \left|\int_{\Omega_n} \langle f(\omega), h_n(\omega)\rangle d\mu(\omega)\right|^q \\ &= |\langle fe(\Omega_n), h_n\rangle|^q \leqslant \|fe(\Omega_n)\|^q = \int_{\Omega_n} \|f(\omega)\|^q \, d\mu(\omega).\end{aligned}$$

But, as $\omega \to \|f(\omega)\|^q \in L_1(\mu)$ and $\mu(\Omega_n) \to 0$, it follows that

$$\lim_{n\to\infty} \int_{\Omega_n} \|f(\omega)\|^q \, d\mu(\omega) = 0$$

and, therefore, $\langle f, h_n\rangle \to 0$.

This shows that (h_n) is a weakly null normalised sequence in $L_p(\mu, X)$. Consequently, the Bessaga-Pelczynski selection principle, [**3**], guarantees the existence of a subsequence (h_{n_i}) of (h_n) which is a basic sequence in $L_p\left(\mu, \widehat{X}\right)$.

Define finally $g_j(\omega) = h_j(\omega)$ if $\omega \notin E$ and $g_j(\omega) = 0$ otherwise. So g_j is equivalent to h_j for each $j \in \mathbb{N}$ and on the one hand we have

$$g_n \notin nV$$

for each $n \in \mathbb{N}$ while, on the other hand, (g_{n_i}) is a basic sequence in $L_p(\mu, X)$. Now, since $\operatorname{supp} g_{n_i} \subseteq \Omega_{n_i} \setminus E$, $\Omega_{n_{i+1}} \subseteq \Omega_{n_i}$ for each $i \in \mathbb{N}$ and $\cap\{\Omega_{n_i} \setminus E, i \in \mathbb{N}\} = \emptyset$, it is easy to notice that every element g of the closed linear cover G of the sequence (g_{n_i}) in $L_p\left(\mu, \widehat{X}\right)$ belongs to $L_p(\mu, X)$. Indeed, $g(\omega) \in X$ for each $\omega \in \Omega$. Hence, G is a Banach subspace of $L_p(\mu, X)$.

Given that V meets G in a neighbourhood of the origin in G, this leads to the existence of some positive integer q such that

$$g_{n_q} \in n_q V.$$

This contradiction ends our proof. ∎

REMARK. If we apply the previous argument to the dense subspace $S_p(\mu, X)$ of $L_p(\mu, X)$ formed by all X-valued functions with countably many values, it also shows that this subspace is ultrabornological whenever μ is atomless and X^* has the Radon-Nikodym property with respect to μ.

In the sequel (Ω, Σ, μ) will be an arbitrary finite measure space. If M is a set of a linear space L, $\langle M\rangle$ stands for the linear hull of M in L. By $S(\mu, X)$ we shall denote the space of all X-valued μ-simple functions defined on Ω. On the other hand, given $1 < p < \infty$ and $E \in \Sigma$, the spaces $S(\mu\,|_E, X)$ and $S_p(\mu\,|_E, X)$ will be denoted simply by $S(E, X)$ and $S_p(E, X)$, respectively.

Given $f \in S_p(\mu, X)$, suppose that g is some appropriate canonical representing of f having only countably many values. If $x \in \Im g$ then by [1, p.167] we have that $g^{-1}(x) \in \Sigma$. Hence, $\{g^{-1}(x),\, x \in \Im g\}$ is a countable partition of Ω formed by pairwise non-empty elements of Σ such that f is essentially constant in each of its members.

LEMMA 1. *Let $\{A_n, n \in \mathbb{N}\}$ be a pairwise disjoint sequence of non-empty elements of Σ. Assume that X^* has the Radon-Nikodym property with respect to μ. If V is an absolutely convex set in $S_p(\mu, X)$ which meets each Banach subspace F with a basis in a neighbourhood of the origin in F, then there exists an $m \in \mathbb{N}$ such that V absorbs the closed unit ball of $S_p(\cup\{A_n, n > m\}, X)$.*

PROOF: If the property is not true, V, does not absorb the closed unit ball of $S_p(\cup\{A_n, n > m\}, X)$ for each $m \in \mathbb{N}$. Hence, for each $m \in \mathbb{N}$ there is some $f_m \in S_p(\cup\{A_n, n > m\}, X)$ such that $\|f_n\| = 1$ and

$$f_m \notin mV.$$

If we set $\Omega_m := \cup\{A_n, n > m\}$, then (f_m) is a normalised sequence in $S_p(\mu, X)$ such that $\operatorname{supp} f_m \subseteq \Omega_m$ for each $m \in \mathbb{N}$. Since X^* has the Radon-Nikodym property with respect to μ and $\mu(\Omega_m) \to 0$, then we may proceed as in Theorem 1 in order to show that $\langle h, f_n\rangle \to 0$ for every $h \in L_q(\mu, X^*)$ with $1/p + 1/q = 1$. Therefore, (f_n) is a weakly null normailised sequence in $L_p(\mu, X)$ and the Bessaga-Pelczynski selection principle establishes the existence of a subsequence $\left(f_{n_i}\right)$ of (f_n) which is a basic sequence in $L_p\left(\mu, \widehat{X}\right)$. Now, as $\Omega_{n+1} \subseteq \Omega_n$ for each $n \in \mathbb{N}$ and $\cap\{\Omega_n, n \in \mathbb{N}\} = \emptyset$, it is easy to notice that every element f of the closed linear span F of the sequence $\left(f_{n_i}\right)$ in $L_p\left(\mu, \widehat{X}\right)$ is X-valued and has countably many values. Consequently, F is a Banach subspace of $S_p(\mu, X)$.

Given that V meets F in a neighbourhood of the origin in F, the Baire category theorem leads to the existence of some positive integer q such that

$$f_{n_q} \in n_q V.$$

This contradiction ends our proof. ∎

Lemma 2. *Let V be an absolutely convex set in $S_c(\mu, X)$ meeting each Banach subspace F with a basis of $S_p(\mu, X)$ in a neighbourhood of the origin in F. If X is ultrabornological and X^* has the Radon-Nikodym property with respect to μ, then V absorb the closed unit ball of $S(\mu, X)$.*

Proof: Suppose the property is not true. Since V does not absorb the unit sphere of $S(\mu, X)$, there is some $f_1 \in S(\mu, X)$ with $\|f_1\| = 1$ such that

$$f_1 \notin 2V.$$

Let $\{Q_{11}, Q_{12}, \dots, Q_{1k(1)}\}$ be a partition of Ω by non-empty sets of Σ such that f_1 takes a different constant value μ-almost everywhere in each Q_{11} with $1 \leqslant i \leqslant k(1)$. Now, as $S(\mu, X)$ is the topological direct sum of the subspaces $S(Q_{11}, X)$, $1 \leqslant i \leqslant k(1)$, there is some $m(1) \in \{1, 2, \dots, k(1)\}$ such that V does not absorb the unit sphere of $S(Q_{1m(1)}, X)$. Thus, there is some $f_2 \in S(Q_{1m(1)}, X)$ with $\|f_2\| = 1$ such that

$$f_2 \notin 4V.$$

Again, there exists a finite partition $\{Q_{21}, Q_{22}, \dots, Q_{2k(2)}\}$ of $Q_{1m(1)}$ by non-empty sets of Σ such that f_2 is constant μ-almost everywhere in each set Q_{2i}, $1 \leqslant i \leqslant k(2)$ and takes a different value. As $S(Q_{1m(1)}, X)$ is the topological direct sum of the subspaces $S(Q_{2i}, X)$, $1 \leqslant i \leqslant k(2)$ and V does not absorb the unit sphere of $S(Q_{1m(1)}, X)$, there is some $m(2) \in \{1, 2, \dots, k(2)\}$ such that V does not absorb the unit sphere of $S(Q_{2m(2)}, X)$. Hence, there is some $f_3 \in S(Q_{2m(2)}, X)$ with $\|f_3\| = 1$ such that

$$f_3 \notin 6T.$$

Proceeding by recurrence we obtain a sequence (f_n) of μ-simple functions and a sequence (Ω_n), with $\Omega_n = Q_{n,m(n)}$ for each $n \in \mathbb{N}$, of sets in Σ, verifying that, for each $n \in \mathbb{N}$,

(i) $\|f_n\| = 1$
(ii) $\operatorname{supp} f_{n+1} \subseteq \Omega_n$
(iii) f_n is essentially constant in Ω_n
(iv) $\Omega_{n+1} \subseteq \Omega_n$
(v) $f_n \notin 2nV$

Let $P := \bigcap_{i=1}^{\infty} \Omega_i$. Two cases are in order depending on whether $\mu(P)$ is or is not different from zero.

If $\mu(P) \neq 0$ and for each $n \in \mathbb{N}$ x_n denotes the constant value of f_n μ-almost everywhere in Ω_n, we define $h_j(\omega) = f_j(\omega)$ if $\omega \notin P$ and $h_j(\omega) = x_j$ if $\omega \in P$ for each $j \in \mathbb{N}$. Then we write

$$h'_j := h_j - x_j e(P)$$

for all $j \in \mathbb{N}$.

Since $x \to e(P)x$ is an isometry from X into $S_p(\mu, X)$, the ultrabornology of X guarantees that the set $V \cap e(P)X$ is a neighbourhood of the origin in $e(P)X$, which leads to the existence of some $r \in \mathbb{N}$ such that $x_i e(P) \in rV$ for each $i \in \mathbb{N}$. Hence,

$$x_n e(P) \in nV$$

for each $n \geqslant r$, consequently,

$$h'_j \notin jV$$

for each $j \geqslant r$. Finally, we define $g_j = h'_j / \|h'_j\|$ for each $j \in \mathbb{N}$. Since $\|h'_j\| \leqslant 1$ for each $j \in \mathbb{N}$, it follows that

$$g_j \notin jV$$

for each $j \geqslant r$. Clearly (g_n) is a normalised sequence with $\operatorname{supp} g_n \subseteq \Omega_n \setminus P$ for each $n \in \mathbb{N}$ and $\cap\{\Omega_n \setminus P, n \in \mathbb{N}\} = \emptyset$.

If $\mu(P) = 0$, for each $j \in \mathbb{N}$, we take $g_j(\omega) = f_j(\omega)$ if $\omega \notin P$ and $g_j(\omega) = 0$ otherwise, for each $j \in \mathbb{N}$. Then g_j is equivalent to f_j for each $j \in \mathbb{N}$ and consequently,

$$g_j \notin jV$$

for each $j \in \mathbb{N}$. As $\|g_j\| = \|f_j\| = 1$ for each j, (g_n) is normalised. On the other hand, we have again that $\operatorname{supp} g_j \subseteq \Omega_j \setminus P$ for each $j \in \mathbb{N}$.

Since $\mu(\operatorname{supp} g_n) \to 0$ in both cases, we proceed as in the previous results to show that (g_n) is weakly null in $L_p(\mu, X)$ and consequently that it contains a basic sequence (g_{n_i}) in $L_p\left(\mu, \widehat{X}\right)$. As $\operatorname{supp} g_n \subseteq \Omega_n \setminus P$ for each $n \in \mathbb{N}$ and $\cap\{\Omega_n \setminus P, n \in \mathbb{N}\} = \emptyset$, it is easy to show that the closed linear span $[g_{n_i}]$ in $L_p\left(\mu, \widehat{X}\right)$ is contained in $S_p(\mu, X)$. This yields a contradiction. ∎

Theorem 2. *Let $1 < p < \infty$ and suppose that X is ultrabornological. If X^* has the Radon-Nikodym property with respect to μ, then $S_p(\mu, X)$ is ultrabornological.*

Proof: Assume that X is ultrabornological but $S_p(\mu, X)$ is not. Then there is some $f_1 \in S_p(\mu, X)$ with $\|f_1\| = 1$ such that

$$f_1 \notin 2V.$$

As we noticed before, there is a partition $\{Q_{1i}, i \in \mathbb{N}\}$ of Ω by non-empty sets of Σ such that f_1 is essentially constant in each set Q_{1i}. By Lemma 1 there is an $n_1 \in \mathbb{N}$

such that V does absorb the closed unit ball of $S_p(\cup\{Q_{1n}, n > n_1\}, X)$. So V does not absorb the unit sphere of $S_p(\cup\{Q_{1n}, n \leqslant n_1\}, X)$.

Let $\Omega_1 := \cup\{Q_{1n}, n \leqslant n_1\}$. Since V does not absorb the unit sphere of $S_p(\Omega_1, X)$, there is some $f_2 \in S_p(\Omega_1, X)$ with $\|f_2\| = 1$ such that

$$f_2 \notin 3V.$$

Let $\{Q_{2i}, i \in \mathbb{N}\}$ be a partition of Ω_1 formed by non-empty sets of Σ such that f_2 is essentially constant in each Q_{2i}. Applying Lemma 1 again, there is an $n_2 \in \mathbb{N}$ such that V does absorb the closed unit ball of the subspace $S_p(\cup\{Q_{2n}, n \geqslant n_2\}, X)$. Hence, defining $\Omega_2 := \cup\{Q_{2i}, i \leqslant n_2\}$, V cannot absorb the unit sphere of $S_p(\Omega_2, X)$ and thus there is some $f_3 \in S_p(\Omega_2, X)$ with $\|f_3\| = 1$ such that

$$f_3 \notin 4V.$$

Proceeding by recurrence we obtain a sequence (f_n) of functions of $S_p(\Omega, X)$ and a sequence (Ω_n) of sets in Σ verifying for each $n \in \mathbb{N}$ the following properties

(i) $\|f_n\| = 1$
(ii) $\operatorname{supp} f_{n+1} \subseteq \Omega_n$
(iii) $e(\Omega_n) f_n \in S(\Omega_n, X)$
(iv) $\Omega_{n+1} \subseteq \Omega_n$
(v) $f_n \notin (n+1)V$.

For each $j \in \mathbb{N}$ we set

$$g_j := f_j - e(\Omega_j) f_j.$$

Clearly, $e(\Omega_j) f_j \in S(\mu, X)$ for each $j \in \mathbb{N}$ and taking into account Lemma 2, there is no loss of generality in assuming that

$$e(\Omega_j) f_j \in V$$

for each $j \in \mathbb{N}$. This implies that

$$g_j \notin jV$$

for each $j \in \mathbb{N}$.

It is clear that $\operatorname{supp} g_i \cap \operatorname{supp} g_j = \emptyset$ if $i \neq j$, and it is not difficult to see from this fact that the closed linear span $[g_j]$ in $L_p\left(\mu, \widehat{X}\right)$ of the sequence (g_j) is a copy of ℓ_p which is contained in $S_p(\mu, X)$. Now the Baire category theorem leads to the existence of some $k \in \mathbb{N}$ such that

$$g_k \in kV.$$

This contradiction ends the proof. $\square$

Corollary. *Suppose that X is ultrabornological and μ is a purely atomic, measure on Ω. Then $L_p(\mu, X)$ is ultrabornological.*

Proof: If μ is purely atomic X^* has the Radon-Nikodym property with respect to μ and $L_p(\mu, X)$ is isometric to $\ell_p\{X\}$. So, the conclusion follows from Theorem 2. ∎

Theorem 4. *Let (Ω, Σ, μ) be an arbitrary finite measure space and let X be an ultrabornological normed space. If X^* has the Radon-Nikodym property with respect to μ, then $L_p(\mu, X)$ with $1 < p < \infty$ is ultrabornological.*

Proof: Since each finite measure space containing some atom decomposes into a purely atomic part and an atomless part, this is an obvious consequence of Theorem 1 and the previous Corollary. ∎

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E.U. Informática
Depto. Matematica Aplicada
Universidad Politécnica de Valencia
Apartado 22012, E-46071 Valencia
Spain

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