65. NON-LINEAR DENSITY WAVES IN PRESSURELESS DISKS

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Abstract. A method is described for the construction of a density wave of finite amplitude and in the form of a tightly wound spiral in a pressureless, self-gravitating disk of infinitesimal thickness in an external gravitational field. Waves of one kind are found for systems in which the law of rotation departs only slightly from that of a solid body. Waves of a second kind are found in systems possessing appreciable differential rotation; however, such waves can occur only if the self-gravitation of the disk is small compared with the external gravitation. The latter waves are to be identified with the waves described by linear theories.

1. Introduction

This is a report on our first results in a non-linear theory of density waves. We have considered the hydrodynamics of a pressureless, self-gravitating disk of infinitesimal thickness in a given external gravitational field which is time-independent and axi-symmetric; and we have constructed spiral waves of finite amplitude in that disk.

The system envisaged is a model of a galaxy in which the disk represents the sub-systems of low-velocity stars and interstellar gas which can participate appreciably in a density wave whereas the external field is attributed to those subsystems of high-velocity stars which cannot participate appreciably in the wave. In *linear* theories of density waves a basis for this model is provided by the manner in which the peculiar motions of the stars (or the pressure of the gas) tend to inhibit the participation of a subsystem in a density wave (see, e.g., Lin *et al.*, 1969).

2. Construction of Density Waves of Finite Amplitude

We consider a wave which has a stationary spiral pattern in a uniformly rotating frame of reference. In that frame the disk is in a steady state. Its structure and the pattern of flow are governed by the hydrodynamical equations of continuity and motion and Poisson's equation. Solutions of the linearized form of this problem have been described by Fujimoto (1968).

We assume that the pattern is tightly wound. Our solutions of the basic equations are asymptotic solutions of the form of series in powers of a parameter λ ($\ll 1$) whose smallness characterizes the tightness of the winding. In order to explain certain aspects of our results, we must describe our solution of Poisson's equation in some detail. This analysis is a generalization of the asymptotic solution of Poisson's equation given by Lin and Shu (1964) in their linear theory of density waves.

In the absence of the density wave, let the surface density and gravitational potential of the disk be $\sigma^{(\infty)}(\varpi)$ and $\mathfrak{B}^{(\infty)}(\varpi, z)$, respectively, where the system of cylindrical polar coordinates (ϖ, θ, z) is defined in the frame of reference rotating uniformly with the angular velocity Ω of the pattern, the z-axis is the axis of rotation, and the plane

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$z=0$ contains the disk. To construct the wave, we write the potential of the perturbed disk in the form

$$\mathfrak{B}(\varpi, \theta, z) = \mathrm{Re}\left[\mathfrak{B}^{(\infty)}(P, z)\right], \tag{1}$$

where

$$P = \varpi + \lambda F(\varpi, z, \chi), \tag{2}$$

$$\chi = m\theta + \lambda^{-1} u(\varpi, z), \tag{3}$$

and m is an integer equal to the number of arms in the pattern. The functions F and u are complex in general. The condition that $\mathfrak{B}$ is a single-valued function of position implies that F is periodic in the real part of χ with period 2π.

To make the geometry of the disk that of a tightly wound spiral, we require F and u to be slowly varying functions of ϖ and z, and we require u to be real in the plane $z=0$. In that plane, the curves $\chi=$constant are the spirals. The spiral component of the field diminishes rapidly with distance from the disk. Accordingly, the function F is required to vanish at infinity.

Outside the plane $z=0$, the potential is governed by Laplace's equation. We have obtained a formal solution by letting

$$u(\varpi, z) = u(\varpi \pm iz), \quad \text{an arbitrary function}, \tag{4}$$

and writing $F(\varpi, z, \chi)$ as a series

$$F = F^{(0)} + \lambda F^{(1)} + \cdots \tag{5}$$

in powers of λ. The requirement that the potential must satisfy Laplace's equation in each order in λ separately leads to a hierachy of equations governing $F^{(0)}$, $F^{(1)}$, etc. The solution of the first member of the hierarchy is

$$F^{(0)}(\varpi, z, \chi) = \left[\varpi^{1/2}\mathfrak{B}_{\varpi}^{(\infty)}(\varpi, z)\right]^{-1} G(\varpi \pm iz, \chi), \tag{6}$$

where $G(\varpi \pm iz, \chi)$ is an arbitrary function, and the subscript denotes differentiation with respect to ϖ. On each side of the plane $z=0$, the ambiguity of sign in Equations (4) and (6) can be removed with the aid of the condition that F vanishes at infinity.

The density distribution which gives rise to this potential is given by

$$\begin{aligned}\sigma(\varpi, \theta, z) &= \frac{1}{2\pi G} \lim_{z\to 0+} \mathfrak{B}_z(\varpi, \theta, z) \\ &= \sigma^{(\infty)}(\varpi) \mp \frac{1}{2\pi G} \lim_{z\to 0+} \left[\mathfrak{B}_{\varpi}^{(\infty)}(\varpi, z)\, \mathrm{Im}(u_{\varpi} F_{\chi}^{(0)})\right] \\ &\quad + O(\lambda),\end{aligned} \tag{7}$$

where subscripts again denote differentiation. Equation (7) expresses the density as a superposition of the unperturbed density and the density of the wave.

In the lowest order the structure of the wave is characterized by the 'amplitude' $F^{(0)}(\varpi, z, \chi)$ and a wave number

$$k(\varpi) = \lambda^{-1} u_{\varpi}(\varpi). \tag{8}$$

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The indeterminacy of these functions, arising from the arbitrariness of the functions $u(\varpi \pm iz)$ and $G(\varpi \pm iz, \chi)$, is reduced in the solution of the hydrodynamical equations. We encounter an integrability condition in the form of an equation involving Ω, $k(\varpi)$, and $F^{(0)}(\varpi, z, \chi)$. This integrability condition determines the amplitude of the wave once the angular velocity of the pattern and the dependence of the wave number on ϖ have been chosen. These choices remain arbitrary.

We have been able to construct solutions of only two kinds along these lines.

3. Density Waves of the First Kind

A necessary condition for the construction of a solution of the first kind is that the pattern of the wave must rotate more slowly than the interior of the disk. The wave extends from the origin to the radial distance at which the pattern and the disk are in corotation; and the amplitude of the wave vanishes at both the origin and the co-rotation point. In practice, we have been able to construct such a wave only over a central region of the disk in which the angular velocity of the unperturbed rotation departs only slightly (of the order of 10%) from a constant. Thus, a wave of the first kind can extend over a large region of the disk only if the law of rotation in that region departs but slightly from that of a solid body. These waves do not possess inner Lindblad resonances.

4. Density Waves of the Second Kind

Solutions of a second kind have been found for waves in disks in which differential rotation is appreciable. A necessary condition for the construction of these solutions is that

$$\mathfrak{V}^{(\infty)}(\varpi, z) = O\left[\lambda \mathfrak{V}^{(B)}(\varpi, z)\right], \tag{9}$$

where $\mathfrak{V}^{(B)}(\varpi, z)$ is the external potential. When we interpret the external field in terms of subsystems of high-velocity stars which cannot participate in a density wave, this conditions implies that a density wave of finite amplitude can occur in a galaxy in differential rotation only if the subsystems of gas and low-velocity stars which can support the wave make only a small contribution, of order λ, to the total mass of the galaxy. It further implies that the smaller the fraction of the total mass of a galaxy in subsystems which can support a density wave, the more tightly wound will be the spiral pattern of the wave. Waves of the second kind are to be identified with the density waves described by linear theories; when we consider a wave of infinitesimal amplitude and linearize our integrability condition in that amplitude, we recover the dispersion relation for Ω and $k(\varpi)$ which was given by Lin and Shu (1964) in the earliest version of their theory.

An example of a density wave of the second kind is illustrated in Figure 1. Here the external field is that of a disk of surface density

$$\sigma^{(B)}(\varpi) = \frac{\mathfrak{M}a}{2\pi}\left(a^2 + \varpi^2\right)^{-3/2}; \tag{10}$$

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and the disk which supports the wave has an unperturbed surface density

$$\sigma^{(\infty)}(\varpi) = \frac{3\lambda\mathfrak{M}}{2\pi R^2}\left(1 - \frac{\varpi^2}{R^2}\right)^{1/2}, \quad \varpi \leqslant R,$$
$$\sigma^{(\infty)}(\varpi) = 0, \quad \varpi > R, \tag{11}$$

where $\mathfrak{M}$, a, and R are constants. In this example, $\lambda = 0.15$ and $R = 3a$. We have constructed a wave of angular velocity Ω given by

$$\Omega^2 = 0.08944\, G\mathfrak{M}/a^3; \tag{12}$$

the corotation point is then $\varpi = 2a$. Outside an annulus in which

$$\nu^2 \equiv m^2(\Omega - \Omega_c)^2/\kappa^2 < \nu_0^2 = 0.49456, \tag{13}$$

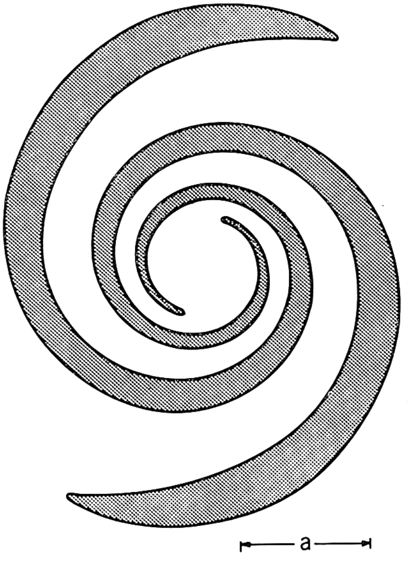

Fig. 1. Pattern of the density wave in the example described in the text.

where Ω_c is the angular velocity of the unperturbed disk and κ is the epicyclic frequency, we let $k(\varpi)$ be given by the dispersion relation for linear waves. In accordance with the remark at the end of the preceding paragraph, the consequence of this choice is that the amplitude of the wave vanishes. Within the annulus, we permit the wave to have a finite amplitude by choosing $k(\varpi)$ in the manner

$$|k| = \frac{\kappa^2}{2\pi G\sigma^{(\infty)}}(1 - \nu^2)\left[1 - \alpha\nu^2(\nu_0^2 - \nu^2)\right], \tag{14}$$

with $\alpha = 0.5$. (With $\alpha = 0$, Equation (14) is the dispersion relation for linear waves.) Figure 1 shows the spiral pattern of the wave; the boundary of the shaded region is a contour of constant density (equal to the central density) in the disk which supports

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the wave. The amplitude of the wave is typically of the order of 30–40% of the unperturbed density. This wave does not possess an inner Lindblad resonance, and the outer Lindblad resonance lies outside the annulus to which the wave has been confined.

5. Concluding Remarks

A number of problems remains to be investigated in this work. We conclude this report by commenting on two of them.

(1) The role of resonances in non-linear waves is not clear. Within the framework of the present analysis, we can avoid the effects of resonances by constructing the wave in such a way that it is confined, as in the preceding example, to an annulus which does not contain resonance points. In doing so, however, we are probably exploiting an artificial property of the asymptotic approximation. For this reason, we intend to investigate the role of resonances.

(2) Leading and trailing spiral patterns are allowed without distinction in the present theory. It seems, therefore, that to account for a preference for trailing patterns, we would have to distinguish leading and trailing patterns in terms of the circumstances of their origin or in terms of their stability.

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Discussion

R. Graham: D. J. Carson has carried out an analysis of static non-linear axisymmetric waves in a differentially rotating thin stellar sheet. A local approximation is used; it is only valid for moderately large amplitude waves. He finds that these static waves have a greater velocity dispersion than static (i.e. zero frequency) linear waves of the same wavelength, which suggests that non-linear waves are less stable than linear ones. He has also carried out a similar analysis for a differentially rotating incompressible fluid sheet, and finds that static non-linear waves exist for sheets thicker than those of the corresponding linear waves, which produces a similar suggestion.

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