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ON COVERING THE UNIT BALL IN NORMED SPACES

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By compactness, the unit ball B^n in R^n has a finite covering by translates of rB^n, for any $r>0$. The main theorem of this note shows that a weaker covering property does not hold in any infinite-dimensional normed space.

THEOREM. *Let E be an infinite-dimensional normed linear space, B the unit ball in E, and $\{r_i\}$ a sequence of nonnegative numbers such that*

(i) $r_i < 1$ *for each i,*

(ii) $\sum_{i=1}^{\infty} r_i^\alpha < \infty$ *for some $\alpha > 0$.*

Then B cannot be covered by a union of the form

$$\bigcup_{i=1}^{\infty} \{x_i+r_iB\}, \quad x_i \in E.$$

Proof. To simplify things we assume (without loss of generality) that α is a positive integer. Also, note that it suffices to prove the theorem for sequences $\{r_i\}$ such that $\sum_{i=1}^{\infty} r_i^\alpha<\infty$ and $0 \le \sup\{r_i\}<\sigma_0<1$, for some σ_0: for, assume that $\{r_i\}$ is a sequence satisfying (i) and (ii) of the theorem and that

$$B \subset \bigcup_{i=1}^{\infty} \{x_i+r_iB\}.$$

Let $\sup\{r_i\}=\sigma<1$. We obtain a new covering of B as follows: for each i such that $r_i \ge \sigma_0$, cover the ball-translate x_i+r_iB by a union of the form $\bigcup_{j=1}^{\infty}\{r_{ij}+r_ir_jB\}$. Thus, we obtain a new countable covering of B by ball-translates with radii $\{r'_k\}$, where $0 \le r'_k \le \sup\{\sigma^2, \sigma_0\}$, and (since only finitely many of the original r_i are greater than σ_0), $\sum_{i=1}^{\infty}(r'_i)^\alpha<\infty$.

We repeat this process; only a finite number of repetitions are necessary because there exists a positive integer k such that $\sigma^k<\sigma_0$. After the kth repetition we obtain a countable covering $\bigcup_{i=1}^{\infty}\{x_i^{(k)}+r_i^{(k)}B\}$ of B such that $0 \le \sup\{r_i^{(k)}\}<\sigma_0$, and $\sum_{i=1}^{\infty}(r_i^{(k)})^\alpha<\infty$.

Thus in the rest of the proof we may assume that $\{r_i\}$ satisfies condition (i): $0 \le \sup\{r_i\}<\frac{1}{4}$, and condition (ii) of the theorem; and that α is a positive integer.

We shall need the following well-known lemma [1, p. 59]:

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LEMMA. *Let E be a normed linear space, and $F\subset E$ a closed proper subspace. For each $\beta<1$, there exists $x\in E$ of norm 1 such that $\|x-F\|>\beta$.*

Using the lemma we construct a sequence $\{y_i\}$ of points in B such that

$$\|y_i\| = 1 \quad \text{and} \quad \|y_k-\text{span}\{y_1,\ldots,y_{k-1}\}\| > \tfrac{3}{4}.$$

If the unit ball is covered by a union of the form $\bigcup_{j=1}^{\infty}\{x_j+r_jB\}$, then no two of the y_i lie in the same ball-translate x_j+r_jB. Because $\sum r_i^\alpha<\infty$, we may find an integer M so large that if $J=\inf\{j \mid x_j+r_jB$ does not contain any of the y_i, $1\le i\le M\}$, then

$$\sum_{j\ge J} r_j^\alpha < (\tfrac{1}{4})^\alpha.$$

Let F_α be an α-dimensional subspace of E. Let $\phi: R^\alpha\to F_\alpha$ be a vector-space isomorphism. The set $\phi^{-1}(B)$ is closed, convex, and balanced in R^α, and has finite α-dimensional measure v. If $r\ge 0$, the set $\phi^{-1}(rB)$ has measure $r^\alpha v$. Thus we conclude the following: if $\bigcup_{n=1}^{\infty}\{z_n+r_nB\}\supseteq B\cap F_\alpha$, where $z_n\in F_\alpha$, then $\sum_{n=1}^{\infty} r_n^\alpha\ge 1$. A small generalization (left to the reader) of this implication is the following:

REMARK. If $\bigcup_{n=1}^{\infty}\{z_n+r_nB\}\supset rB\cap F_\alpha$, where $z_n\in E$, then $\sum_{n=1}^{\infty} r_n^\alpha\ge r^\alpha$. Now consider the set $B_0=\frac{3}{4}y_{n+1}+\frac{1}{16}(B\cap F_\alpha)$: by the hypotheses concerning M and the sequence $\{y_i\}$, we know that $B_0\subset B$, and $B_0\cap(\bigcup_{j=1}^{J-1}\{x_j+r_jB\})=\phi$. Thus $B_0\subset\bigcup_{j\ge J}\{x_j+r_jB\}$, which from the remark, is impossible, since $\sum_{j\ge J} r_j^\alpha<(\frac{1}{4})^\alpha$. Hence the theorem is proved.

COROLLARY 1. *Let E be an infinite-dimensional normed linear space, B the unit ball, and $0\le r<1$. Then no finite union of translates of rB will cover B.*

COROLLARY 2. *Let $f: R^1\to E$ be a Lipschitz-continuous map, where E is an infinite-dimensional normed space. Then $E-f(R^1)$ is dense in E.*

Proof. It suffices to show that for arbitrary $\delta>0$, δB is not contained in $f(R^1)$. Let $L>0$ be the Lipschitz constant for f. Divide R^1 into a countable union of subintervals $\bigcup_{n=1}^{\infty} I_n$, where

$$0 \le l(I_n) < \inf\left\{\frac{\delta}{(n+1)L}, \delta L\right\}.$$

For each n, choose some $\xi_n\in I_n$. Then

$$f(R^1) \subset \bigcup_{n=1}^{\infty}\left(f(\xi_n)+\frac{\delta}{n+1}B\right).$$

From the main theorem, with $\alpha=2$, we conclude that δB is not contained in $f(R^1)$.

Q.E.D.

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REMARK. This last corollary can, of course, be generalized to maps $f: R^n \to E$ such that

$$\|f(x)-f(y)\| \leq L\|x-y\|^{\beta},$$

where $L \geq 0$, $\beta > 0$.

REFERENCE

1. L. V. Kantorovich and G. P. Akilov, *Functional analysis in normed spaces*, Macmillan, New York, 1964.

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