BULL. AUSTRAL. MATH. SOC. 43A10
VOL. 33 (1986), 89-102. 42C05

HYPERGROUP STRUCTURES ON THE SET OF NATURAL NUMBERS

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Every hermitian hypergroup structure on the set of nonnegative integers can be generated by a family of real-valued continuous functions defined on a compact interval. We characterise such structures in terms of properties of the generating functions.

1. Introduction

Let X be a locally compact Hausdorff space. The notation below will be used throughout the paper.

$M(X)$	Space of bounded Radon measures on X.
$M^+(X), M^1(X)$	Subset of $M(X)$ consisting of those measures that are nonnegative, and those that are nonnegative with total variation one, respectively.
supp μ	Support of $\mu \in M(X)$.
ε_x	Point measure at $x \in X$.
N, N'	Space of nonnegative integers, and positive integers, respectively.

A nonvoid locally compact Hausdorff space K will be called a hypergroup if the following conditions are satisfied:

Received 2 May 1985

(HG1) $M(K)$ admits a binary operation $*$ under which it is a complex algebra.

(HG2) The bilinear mapping $* : M(K) \times M(K) \to M(K)$ given by $(\mu,\nu) \to \mu * \nu$ is nonnegative ($\mu * \nu \geq 0$ whenever $\mu,\nu \geq 0$) and its restriction to $M^+(K) \times M^+(K)$ is continuous when $M^+(K)$ is given the weak topology.

(HG3) Given $x,y \in K$, $\varepsilon_x * \varepsilon_y \in M(K)$ and $\operatorname{supp}(\varepsilon_x * \varepsilon_y)$ is compact.

(HG4) The mapping $(x,y) \to \operatorname{supp}(\varepsilon_x * \varepsilon_y)$ of $K \times K$ into the space of nonvoid compact subsets of K is continuous, the latter space with the topology as given in Section 2.5, Jewett [6].

(HG5) There exists a (necessarily unique) element e of K such that $\varepsilon_x * \varepsilon_e = \varepsilon_e * \varepsilon_x = \varepsilon_x$ for all $x \in K$.

(HG6) There exists a unique involution (a homeomorphism $x \to \bar{x}$ of K onto itself with the property $\bar{\bar{x}} = x$ for all $x \in K$) such that for $x,y \in K$, $e \in \operatorname{supp}(\varepsilon_x * \varepsilon_y)$ if and only if $x = \bar{y}$, and $(\mu * \nu)^- = \bar{\nu} * \bar{\mu}$ for all $\mu,\nu \in M(K)$, where $\bar{\mu} \in M(K)$ is defined by $\bar{\mu}(A) = \mu(\bar{A})$ for Borel subsets A of K and $\bar{A} = \{\bar{x} : x \in A\}$.

The study of hypergroups in harmonic analysis was put on a firm footing with the papers of Dunkl [3], Jewett [6] and Spector [11]. Since then work on hypergroups, together with the development of probability theory on these spaces, has progressed considerably. For an 'early' survey of results in this area, see the article by Ross [8], and for a more recent overview, Heyer [5].

The idea of generalising convolution had in fact been investigated by many authors. Schwartz [9] gave an axiomatic structure that leads to a generalised convolution on $M(\mathbb{N})$. With a refinement of these axioms (see below) one obtains various hypergroup structures on $\mathbb{N}$, a topic that has been treated in some detail by Lasser [7].

Throughout this paper we shall be dealing with hypergroup structures on $\mathbb{N}$, when the latter is given the discrete topology, and where the involution in (HG6) is just given by the identity mapping (such a

https://doi.org/10.1017/S0004972700002914 Published online by Cambridge University Press

hypergroup is termed hermitian). We first show how the convolution structure of Schwartz fits within this framework and then, in Section 2, we prove that for such a hypergroup having as its dual a compact subinterval of the real line, a necessary and sufficient condition for it to be generated by polynomials (see the definition following (P1)-(P6) below) is that it satisfy the assumptions of Lasser [7].

For a general reference to hypergroups see Jewett [6] and Bloom and Heyer [1]. Numerous examples are to be found in the paper by Jewett and Bloom and Heyer [2]. The more straightforward results in these papers will be used without explicit reference.

Suppose that $\mathbf{N}$ has the structure of a hermitian discrete hypergroup, which guarantees that the convolution is commutative. The dual $\mathbf{N}^{\wedge}$ is defined as the set of bounded real-valued sequences $\{\chi(n)\}_{n\in\mathbf{N}}$ satisfying

$$\chi(m)\ \chi(n) = \sum_{k=0}^{\infty} \chi(k)\ \varepsilon_m \bar{*}\ \varepsilon_n(k) \tag{1.1}$$

for all $m,n \in \mathbf{N}$. With the topology of uniform convergence on compact sets (which, in this case, is just pointwise convergence), $\mathbf{N}^{\wedge}$ is a compact space,

For $\mu \in M(\mathbf{N})$ its Fourier transform $\hat{\mu}$ is the function on $\mathbf{N}^{\wedge}$ defined by

$$\hat{\mu}(\chi) = \sum_{k=0}^{\infty} \chi(k)\ \mu(k)\ .$$

It is clear that $\hat{\mu}$ is continuous, the mapping $\mu \to \hat{\mu}$ is linear and one-to-one, and $(\mu * \nu)^{\wedge} = \hat{\mu}\ \hat{\nu}$ for all $\mu,\nu \in M(\mathbf{N})$.

Every commutative hypergroup K admits a Haar measure λ, that is, a nonnegative measure satisfying $\varepsilon_x * \lambda = \lambda$ for all $x \in K$. Levitan's theorem (Jewett [6], Theorem 7.3I) guarantees the existence of a unique nonnegative Borel measure σ on $\mathbf{N}^{\wedge}$, called the Plancherel measure associated with λ, with the property that

$$\sum_{k=0}^{\infty} f(k)\ \overline{g(k)}\ \lambda(k) = \int_{\mathbf{N}^{\wedge}} \hat{f}(\chi)\ \overline{\hat{g}(\chi)}\ d\sigma(\chi)$$

for all $f,g \in L^2(\mathbb{N},\lambda)$. (The Fourier transform is extended to $L^2(\mathbb{N},\lambda)$ in the usual way.) For each $n \in \mathbb{N}$ write $f_n = \lambda(n)^{-1}\xi_{\{n\}}$ (where $\xi_{\{n\}}$ denotes the function taking the value 1 at n and 0 elsewhere.) Then

$$\hat{f}_n(\chi) = \chi(n)\ \lambda(n)^{-1}\ \lambda(n) = \chi(n)\ .$$

For ϕ_n defined on $\mathbb{N}^\wedge$ by $\phi_n(\chi) = \chi(n)$ we have, using Levitan's theorem,

$$\int_{\mathbb{N}^\wedge} \phi_m\phi_n d\sigma = \lambda(n)^{-1}\delta_{mn}\ ,$$

where δ_{mn} is Kronecker's symbol. Thus $\{\phi_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to the Plancherel measure. Writing the convolution on $\mathbb{N}$ as

$$\varepsilon_m * \varepsilon_n = \sum_{k=0}^{\infty} \alpha_{mn}(k)\varepsilon_k\ ,$$

a further application of Levitan's theorem gives

$$\alpha_{mn}(k) = \lambda(k)\int_{\mathbb{N}^\wedge} \phi_m\phi_n\phi_k d\sigma\ .$$

Functions ϕ_n with properties derived from the above can be used to generate hermitian hypergroup structures. Let K be a discrete space, X a compact space with nonnegative Borel measure σ, and $\{\phi_a\}_{a\in K}$ a family of continuous functions on X. Write

$$\lambda(a) = \{\int_X |\phi_a|^2\ d\sigma\}^{-1}$$

$$\alpha_{ab}(c) = \lambda(c)\int_X \phi_a\phi_b\overline{\phi_c}\ d\sigma$$

for each $a,b,c \in K$. Suppose that $\{\phi_a\}_{a\in K}$ satisfies the following properties:

(P1) $\{\phi_a\}_{a\in K}$ is orthogonal in $L^2(X,\sigma)$.

(P2) There exists $a_0 \in K$ such that $\phi_{a_0} \equiv 1$.

(P3) There exists $x_0 \in X$ such that $\phi_a(x_0) = 1$ for all $a \in K$.

(P4) $\{\phi_a\}_{a\in K}$ is uniformly bounded.

(P5) For each $a,b \in K$, the function α_{ab} on K is nonnegative and has finite support.

(P6) For every $a,b \in K$,

$$\phi_a\phi_b = \sum_{c\in K} \alpha_{ab}(c)\phi_c .$$

These properties correspond to the axioms given in Schwartz [9] for generalised convolution on $\mathbf{N}$, except that Schwartz does not require each α_{mn} to be finitely supported (which is needed if the convolution of two point measures is to have compact support). It should be noted that (P6) holds automatically if $\{\phi_a\}_{a\in K}$ is further assumed to be complete in $L^2(X,\sigma)$.

With $\{\phi_a\}_{a\in K}$ satisfying (P1)-(P6), K can be made into a hermitian hypergroup by defining the convolution as

$$\varepsilon_a * \varepsilon_b = \sum_{c\in K} \alpha_{ab}(c)\varepsilon_c , \quad a,b \in K ,$$

(compare with Dunkl [3], Examples 4.4 and 4.5) and we refer to the hypergroup structure on K as being generated by $\{\{\phi_a\}_{a\in K}, X\}$. In the case $K = \mathbf{N}$ we always take $a_0 = 0$.

Writing

$$\lambda = \sum_{c\in K} \lambda(c)\varepsilon_c ,$$

(in fact, $\lambda(a) = (\sigma(X)\varepsilon_a * \varepsilon_a(0))^{-1}$) we have, for $a,b \in K$,

$$\begin{aligned}
\varepsilon_a * \lambda(b) &= \sum_{c\in K} \varepsilon_a * \varepsilon_c(b)\lambda(c) \\
&= \sum_{c\in K} \sum_{c'\in K} \alpha_{ac}(c')\varepsilon_{c'}(b)\lambda(c) \\
&= \sum_{c\in K} \alpha_{ac}(b)\lambda(c) \\
&= \sum_{c\in K} \alpha_{ab}(c)\lambda(b) \\
&= \lambda(b)
\end{aligned}$$

(the last equality following from (P6) by evaluating at x_0) and this shows that $\varepsilon_a * \lambda = \lambda$, so that λ is a Haar measure on K .

There is a natural mapping $\eta : X \to K^\wedge$ given by $\eta(x)(a) = \phi_a(x)$, where $a \in K$, $x \in X$; we write $\eta(x) = \chi_x$.

THEOREM 1.

(a) *If the linear span of* $\{\phi_a\}_{a \in K}$ *is dense in* $C(X)$ *then* η *is one-to-one.*

(b) $\eta(X)$ *is closed in* $K^\wedge$.

(c) $\eta(\sigma)$ *is the Plancherel measure on* $K^\wedge$ *associated with* λ.

(d) *If* $K^\wedge$ *is a hypergroup (compatible with pointwise multiplication of characters) then* $\eta(X) = K^\wedge$.

Proof.

(a) This is evident.

(b) Since ϕ_a is continuous we have that $x \to \eta(x)(a)$ is continuous for each $a \in K$. Thus η is continuous, as $K^\wedge$ has the topology of pointwise convergence, and the conclusion follows from the compactness of X.

(c) First consider the functions $f_a = \lambda(a)^{-1}\xi_{\{a\}}$, where $a \in K$. We have

$$\int_{K^\wedge} \hat{f}_a \overline{\hat{f}_b}\, d\,\eta(\sigma) = \int_X \hat{f}_a(\eta(x))\, \overline{\hat{f}_b}(\eta(x))\, d\sigma(x)$$

$$= \int_X \chi_x(a)\, \chi_x(b)\, d\sigma(x)$$

$$= \int_X \phi_a(x)\, \phi_b(x)\, d\sigma(x)$$

$$= \lambda(a)^{-1}\, \delta_{ab}$$

$$= \sum_{c \in K} f_a(c)\, \overline{f_b(c)}\, \lambda(c) . \qquad (1.2)$$

https://doi.org/10.1017/S0004972700002914 Published online by Cambridge University Press

Now the linear span of $\{f_a\}_{a\in K}$ is dense in $L^2(K,\lambda)$, whence it follows that (1.2) holds for all $f \in L^2(K,\lambda)$. By uniqueness, $\eta(\sigma)$ is the Plancherel measure on $K^\wedge$ associated with λ.

(d) If $K^\wedge$ is a hypergroup then (Jewett [6], Theorem 12.4) the support of its Plancherel measure is the whole space. Thus, using (c),

$$K^\wedge = \mathrm{supp}(\eta(\sigma)) \subset \eta(X) ,$$

and this completes the proof. //

A discrete commutative hypergroup could be generated by many families $\{\{\phi_a\}_{a\in K^\wedge}, X\}$. For example, let G be a compact nonabelian group and take $K = \Sigma$, its dual object. The Clebsch-Gordan formula for characters on G is

$$\psi_U\psi_V = \sum_{k=1}^{n} \frac{m_k \dim(U_k)}{\dim(U)\dim(V)} \psi_{U_k} ,$$

where $\psi_U = \dim(U)^{-1}\,\mathrm{Tr}(U)$ for $U \in \Sigma$, $\dim(U)$ denotes the dimension of the representation U , and m_k denotes the multiplicity of U_k in the decomposition $\sum_{k=1}^{n} m_k U_k$ of $U \otimes V$ as a direct sum of its irreducible components. A convolution on K can be defined by

$$\varepsilon_U * \varepsilon_V = \sum_{k=1}^{n} \frac{m_k \dim(U_k)}{\dim(U)\dim(V)} \varepsilon_{U_k} .$$

This hypergroup structure is generated by $\{\{\psi_U\}_{U\in\Sigma}, G\}$, or alternatively via the ψ_U suitably defined on the (compact) space G^G of conjugacy classes of G. It should be noted that in the former case Dunkl's Examples 4.4 and 4.5 do not apply since the span of characters would be contained in the set of continuous central functions of G , a proper closed subspace of $C(G)$.

2. Hermitian Hypergroup Structures on N

Throughout we assume that $\mathbb{N}$ has a hermitian hypergroup structure generated by $\{\{\phi_n\}_{n\in\mathbb{N}} , [a,b]\}$, where the associated map η is a homeomorphism. All of the examples given in Lasser [7] are hypergroups of this type, where each ϕ_n is a polynomial of degree n. In this section we show that this restriction on ϕ_n is necessary and sufficient for the convolution to be given by

$$\varepsilon_m * \varepsilon_n = \sum_{k=|m-n|}^{m+n} \alpha_{mn}(k)\, \varepsilon_k \,. \tag{2.1}$$

The first thing to notice is that (2.1) is equivalent to the corresponding identity with $m = 1$. One implication is obvious. In the other direction, suppose that

$$\varepsilon_1 * \varepsilon_n = \sum_{k=|1-n|}^{1+n} \alpha_{1n}(k)\, \varepsilon_k \,, \tag{2.2}$$

that is, $\operatorname{supp}(\varepsilon_1 * \varepsilon_n) \subset [|n - 1|, n + 1]$ for all $n \in \mathbb{N}$. We proceed by induction. Suppose that $\operatorname{supp}(\varepsilon_\ell * \varepsilon_n) \subset [|n - \ell|, n + \ell]$ for all $\ell \leq m - 1$, where $m \geq 2$ is given. Then, for $n \geq 1$,

$$\varepsilon_{m-1} * (\varepsilon_1 * \varepsilon_n) = \alpha_{1n}(n-1) \sum_{k=|m-n|}^{m+n-2} \alpha_{m-1,n-1}(k)\varepsilon_k + \alpha_{1n}(n) \sum_{k=|m-n-1|}^{m+n-1} \alpha_{m-1,n}(k)\, \varepsilon_k$$

$$+ \alpha_{1n}(n + 1) \sum_{k=|m-n-2|}^{m+n} \alpha_{m-1,n+1}(k)\, \varepsilon_k \tag{2.3}$$

and

$$(\varepsilon_{m-1} * \varepsilon_1) * \varepsilon_n = \alpha_{m-1,1}(m-2) \sum_{k=|m-n-2|}^{m+n-2} \alpha_{m-2,n}(k)\varepsilon_k$$

$$+ \alpha_{m-1,1}(m-1) \sum_{k=|m-n-1|}^{m+n-1} \alpha_{m-1,n}(k)\varepsilon_k + \alpha_{m-1,1}(m) \sum_{k\in\mathbb{N}} \alpha_{mn}(k)\varepsilon_k . \tag{2.4}$$

Since these two expressions are equal by associativity of convolution, we see that $\alpha_{mn}(k) = 0$ for $k \notin [|m - n|, m + n]$. This clearly holds for $n = 0$ as well, which gives the assertion. Our main result is:

THEOREM 2. *Suppose that the hermitian hypergroup structure on* $\mathbb{N}$ *is generated by* $\{\{\phi_n\}_{n\in\mathbb{N}} : [a,b]\}$ *where the associated map* η *is a homeomorphism.*

(a) *If each* ϕ_n *is a polynomial of degree* n *then* $\mathrm{supp}(\varepsilon_m * \varepsilon_n) \subset [|m-n|, m+n]$ *for all* $m, n \in \mathbb{N}$.

(b) *Suppose* $\mathrm{supp}(\varepsilon_m * \varepsilon_n) \subset [|m-n|, m+n]$ *for each* $m, n \in \mathbb{N}$. *Then there is a closed interval* $[d, 1]$ *and a sequence* $\{\phi_n^*\}_{n\in\mathbb{N}}$ *of polynomials with* $\deg \phi_n^* = n$ *such that* $\{\{\phi_n^*\}_{n\in\mathbb{N}} : [d,1]\}$ *generates the given hypergroup structure on* $\mathbb{N}$.

Proof.

(a) Using (P6)

$$\alpha_{1n}(k) = \lambda(k) \int_a^b \phi_1 \phi_n \phi_k \, d\sigma = 0$$

whenever $k > n + 1$ or $n > k + 1$, in which case $k < n - 1$. Thus $\mathrm{supp}(\varepsilon_1 * \varepsilon_n) \subset [|n - 1|, n + 1]$, and the result follows by the discussion preceding the theorem.

(b) First note that $\phi_0 = 1$ is a consequence of the assumption on the support given in the statement of the theorem.

We begin by showing that $\alpha_{n1}(n + 1) > 0$ for all $n \in \mathbb{N}'$. Since $*$ is associative on $M(\mathbb{N})$, for $m \geq 2$ and $n \in \mathbb{N}'$ we can write

$$(\varepsilon_{m-1} * \varepsilon_1) * \varepsilon_n = \varepsilon_{m-1} * (\varepsilon_1 * \varepsilon_n).$$

Equating the coefficients of $\varepsilon_{|m-n-2|}$ of both sides (refer to (2.3) and (2.4)) we obtain

$$\alpha_{n1}(n + 1)\alpha_{m-1,n+1}(|m - n - 2|) = \alpha_{m-1,1}(m-2)\alpha_{m-2,n}(|m - n - 2|).$$

In particular if $m = 2$, then we obtain

$$\alpha_{n1}(n + 1)\alpha_{n+1,1}(n) = \alpha_{11}(0)\alpha_{n0}(n) = \varepsilon_1 * \varepsilon_1(0) > 0$$

since $0 \in \mathrm{supp}(\varepsilon_1 * \varepsilon_1)$. Thus $\alpha_{n1}(n + 1) > 0$ for all $n \in \mathbb{N}'$, and in fact this inequality holds for $n = 0$ as well.

The identity for convolution ensures that

$$\phi_n(x)\phi_1(x) = \sum_{k=n-1}^{n+1} \alpha_{n1}(k)\phi_k(x)$$

https://doi.org/10.1017/S0004972700002914 Published online by Cambridge University Press

for $n \in \mathbb{N}'$ and $x \in [a,b]$. Hence

$$\alpha_{n1}(n+1)\phi_{n+1}(x) = (\phi_1(x) - \alpha_{n1}(n))\phi_n(x) - \alpha_{n1}(n-1)\phi_{n-1}(x). \quad (2.5)$$

We can use (2.5) to show that ϕ_1 is a homeomorphism of $[a,b]$ onto $[d,1]$ for some $d \in [-1,1)$. The first thing to notice is that since $\{\phi_n\}_{n\in\mathbb{N}}$ is orthogonal in $L^2([a,b],d\sigma)$, $\phi_1 \neq \phi_0 = 1$, which ensures that $\phi_1([a,b]) \neq \{1\}$. It is also easy to see that $|\phi_1(x)| \leq 1$ for all $x \in [a,b]$. Indeed, writing $M_x = \sup\{|\chi_x(n)| : n \in \mathbb{N}\}$, we have using (1.1) that for $m,n \in \mathbb{N}$, $x \in [a,b]$,

$$|\chi_x(m)||\chi_x(n)| \leq \sum_{k=0}^{\infty} |\chi_x(k)|\varepsilon_m * \varepsilon_n(k) \leq M_x ,$$

from which it follows that $M_x \leq 1$, and hence

$$|\phi_1(x)| = |\chi_x(1)| \leq 1 ,$$

as required. Thus all that needs to be shown is that ϕ_1 is one-to-one.

Suppose that $\phi_1(x) = \phi_1(y)$ for some $x,y \in [a,b]$. We proceed by induction to show that $\phi_n(x) = \phi_n(y)$ for all $n \in \mathbb{N}$, in which case $x = y$ as η is one-to-one. Assume that $\phi_m(x) = \phi_m(y)$ for all $m \leq n$. Using (2.5), we have

$$\begin{aligned}\alpha_{n1}(n+1)\phi_{n+1}(x) &= (\phi_1(x) - \alpha_{n1}(n))\phi_n(x) - \alpha_{n1}(n-1)\phi_{n-1}(x)\\ &= (\phi_1(y) - \alpha_{n1}(n))\phi_n(y) - \alpha_{n1}(n-1)\phi_{n-1}(y)\\ &= \alpha_{n1}(n+1)\phi_{n+1}(y) .\end{aligned}$$

Since $\alpha_{n1}(n+1) > 0$, $\phi_{n+1}(x) = \phi_{n+1}(y)$. This completes the induction step as required.

Write $\phi_n^* = \phi_n \circ \phi_1^{-1}$. Then (2.5) becomes

$$\alpha_{n1}(n+1)\phi_{n+1}^*(x) = (x - \alpha_{n1}(n))\phi_n^*(x) - \alpha_{n1}(n-1)\phi_{n-1}^*(x) . \quad (2.6)$$

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As $\alpha_{n1}(n+1) > 0$ for each $n \in \mathbb{N}$, it follows from (2.6) that ϕ_n^* is a polynomial of degree n. Also, since

$$\int_d^1 \phi_m^* \phi_n^* d\phi_1(\sigma) = \int_a^b \phi_m \phi_n d\sigma = \lambda(n)^{-1} \delta_{mn} ,$$

the set $\{\phi_n^*\}_{n\in\mathbb{N}}$ is orthogonal in $L^2([d,1], d\phi_1(\sigma))$. It is easy to check that $\{\phi_n^*\}_{n \mathbb{N}}$ satisfies properties (P1)-(P6), and that $\{\{\phi_n\}_{n \mathbb{N}} : [d,1]\}$ generates the given hypergroup structure on $\mathbb{N}$. //

A related result can be found in Schwartz [10], Theorem 1, proved under somewhat different conditions. Schwartz starts with an ℓ^1-convolution algebra where the convolution is not assumed to be non-negative $(\mu * \nu \geq 0$ whenever $\mu, \nu \geq 0)$; on the other hand, he does require from the outset that $\alpha_{n1}(n+1) \neq 0$.

It should be noted from the proof of Theorem 2 that for each $n \in \mathbb{N}$

$$\phi_n^*(x) = \chi_{\phi_1^{-1}(x)}(n)$$

for all $x \in [a,b]$.

We observe that for $\mathbb{N}$ to be a hermitian hypergroup with the convolution (2.1) it is necessary that the α_{mn} $(m,n \in \mathbb{N})$ be generated as in Lasser [7]. Indeed

$$\varepsilon_m * \varepsilon_n = \sum_{k=|m-n|}^{m+n} \alpha_{mn}(k)\varepsilon_k = \sum_{k=0}^{2m} g(m,n,m+n-k)\varepsilon_{m+n-k} ,$$

where $g(m,n,k) = \alpha_{mn}(k)$ for $m,n,k \in \mathbb{N}$. Then

$$\begin{aligned} g(0,n,n) &= \alpha_{0n}(n) &= 1 , \\ a_n = g(1,n,n+1) &= \alpha_{1n}(n+1) &> 0 , \\ b_n = g(1,n,n) &= \alpha_{1n}(n) &\geq 0 , \\ c_n = g(1,n,n-1) &= \alpha_{1n}(n-1) &> 0 \end{aligned}$$

and, since $\varepsilon_1 * \varepsilon_n$ is a probability measure,

$$a_n + b_n + c_n = \alpha_{1n}(n+1) + \alpha_{1n}(n) + \alpha_{1n}(n-1) = 1.$$

Thus $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy property (P) of Lasser [7], p.191. Using the associativity of the convolution, it is straightforward but tedious to check that the function g satisfies the recursive properties given in Lasser [7], Section 2, p.188.

Finally we consider the following convolution on $\mathbf{N}$, introduced in Gilewski and Urbanik [4] :

$$\varepsilon_m * \varepsilon_n = \frac{\cosh a(m-n)}{2\cosh am \cosh an}\varepsilon_{|m-n|} + \frac{\cosh a(m+n)}{2\cosh am \cosh an}\varepsilon_{m+n},$$

where $a \geq 0$ is given. It is easily shown that $\mathbf{N}$ is a hermitian hypergroup with the above convolution and 0 as unit element, and that

$$\mathbf{N}^{\wedge} = \{\chi_x : x \in [-\cosh a, \cosh a]\},$$

where

$$\chi_x(n)\cosh an = \begin{cases} (-1)^n \cosh(n \operatorname{arcosh}(-x)) & \text{if } x \in [-\cosh a, -1), \\ \cos(n \arccos x) & \text{if } x \in [-1, 1], \\ \cosh(n \operatorname{arcosh} x) & \text{if } x \in (1, \cosh a]. \end{cases}$$

Each ϕ_n is a polynomial of degree n, agreeing (up to a constant) on $[-1,1]$ with the nth Tchebychef polynomial of the first kind. The sequence $\{\phi_n\}_{n\in\mathbf{N}}$ is orthogonal on $[-\cosh a, \cosh a]$ with respect to the measure σ supported in $[-1,1]$ and satisfying

$$d\sigma(x) = (\pi(1-x^2))^{-\frac{1}{2}}\,dx, \quad x \in [-1,1].$$

It is clear that the hypergroup structure is generated by both $\{\{\phi_n\}_{n\in\mathbf{N}}, [-\cosh a, \cosh a]\}$ (with orthogonalising measure σ) and the corresponding family restricted to $[-1,1]$; however (using the notation of Theorem 1 above) in the latter case $\eta([-1,1])$ is a proper subset of $\mathbf{N}^{\wedge}$. Note that the ϕ_n cannot be extended past the interval $[-\cosh a, \cosh a]$, as then (P4) would not hold.

References

[1] Walter R. Bloom and Herbert Heyer, "The Fourier transform for probability measureson hypergroups", *Rend. Mat.* 2 (1982), 315-334.

[2] Walter R. Bloom and Herbert Heyer, "Convergence of convolution products of probability measures on hypergroups", *Rend. Mat.* 2 (1982), 547-563.

[3] Charles F. Dunkl, "The measure algebra of a locally compact hypergroup", *Trans. Amer. Math. Soc.* 179 (1973), 331-348.

[4] J. Gilewski and K. Urbanik, "Generalized convolutions and generating functions", *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 16 (1968), 481-487.

[5] Herbert Heyer, "Probability theory on hypergroups : a survey", *Probability measures on groups VII. (Proc. Conf., Oberwolfach Math. Res. Inst., Oberwolfach* 1983.*)* 481-550. (Lecture Notes in Math., 1064 Springer Verlag, Berlin, Heidelberg, New York, 1984).

[6] Robert I. Jewett, "Spaces with an abstract convolution of measures", *Adv. in Math.* 18 (1975), 1-101.

[7] Rupert Lasser, "Orthogonal polynomials and hypergroups", *Rend. Mat.* 3 (1983), 185-209.

[8] Kenneth A. Ross, "Hypergroups and centers of measure algebras", *Symposia Mathematica* 22 (1977), 189-203.

[9] Alan Schwartz, "Generalized convolutions and positive definite functions associated with general orthogonal series", *Pacific J. Math.* 55 (1974), 565-582.

[10] Alan L. Schwartz, "ℓ^1-convolution algebras : representation and factorization", *Z. Wahrsch. Verw. Gebiete* 41 (1977), 161-176.

[11] René Spector, "Aperçu de la théorie des hypergroupes", *Analyse harmonique sur les groupes de Lie.* (Sém. Nancy-Strasbourg, 1973-1975.) 643-673. (Lecture Notes in Math Springer Verlag, Berlin, Heidelberg, New York, 1975).

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