Dynamics of Populations of Planetary Systems
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Symmetric periodic orbits in proto-stellar systems

Vasile Mioc[1], **Mira-Cristiana Anisiu**[2] **and Magda Stavinschi**[1]

[1]Astronomical Institute of the Romanian Academy, Str. Cuţitul de Argint 5, RO-040557 Bucharest, RO
email: vmioc@aira.astro.ro, magda@aira.astro.ro

[2]T. Popoviciu Institute of Numerical Analysis of the Romanian Academy, P.O. Box 68, RO-400110 Cluj-Napoca, RO
email: mira@math.ubbcluj.ro

Abstract. We approach the dynamics in proto-stellar systems via the two-body problem associated to an anisotropic Schwarzschild-type potential. On the basis of the natural symmetries of the characteristic vector field, and using variational methods (particularly the classical lower-semicontinuity method), we prove the existence of infinitely many families of symmetric periodic orbits.
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1. Introduction

Dynamics in proto-stellar systems is a topic that can be tackled via anisotropic two-body problems (Saslaw 1978). Actually, astronomy provides a much larger class of problems tractable by this model (e.g., Mioc *et al.* 2003).

The physical framework of our model is a central (proto-)star in fast rotation, hence oblate, but still axisymmetric, surrounded by a nonuniformly dense accretion disk. We are interested in the dynamics of a body that moves in the plane of the disk, without gravitationally interacting with the disk particles, but being influenced by the nonuniform (due to the presence of the disk) radiation of the central star that falls on the body. The oblateness of the star, created by the fast rotation, leads to the existence of a Schwarzschild-type potential. The influence of the disk makes the Newtonian-type term of the potential to be anisotropic.

Our previous attmepts (Mioc *et al.* 2003) to prove the existence or nonexistence of periodic orbits within such a Schwarzschild-type problem were unsuccessful. Here we solve this problem, even if within a little bit restricted framework. To this end, we use the natural symmetries of the problem, and a variational principle: the extrema of the action integral are genuine periodic solutions.

In this paper we present results without proofs. A complete presentation will be published elsewhere.

2. Basic equations and properties

The problem is described by a two-degrees-of-freedom system of ODE with the Hamiltonian $H(\mathbf{q},\mathbf{p}) = |\mathbf{p}|^2/2 - W(\mathbf{q})$, in which the potential $W : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ is (cf.

Mioc *et al.* 2003):

$$W\left(q_1,q_2\right)=\left(\mu q_1^2+q_2^2\right)^{-1/2}+b\left(q_1^2+q_2^2\right)^{-3/2}, \tag{2.1}$$

where $\mu>0$ and $b>0$ are parameters. The presence of the disk and the oblateness of the star lead to this expression of the potential inside the disk. The corresponding Lagrangian, $L\left(\mathbf{q},\mathbf{p}\right)=\left|\mathbf{p}\right|^2/2+W\left(\mathbf{q}\right)$, is always positive.

The equations of motion define the two-body problem dynamics in an anisotropic plane, in which, for $\mu>1$, the attraction is the strongest in the q_1-direction and the weakest in the q_2 -direction; for $\mu<1$ the situation is inverse. We consider, without loss of generality, that $\mu>1$.

The model admits the integral of energy, but not the angular momentum integral (because of the anisotropy).

An important property of the potential W is that it generates a *strong force* according to Gordon's (1975) definition (using an alternative definition, one immediately proves that $W\left(\mathbf{q}\right)\geqslant b/\left|\mathbf{q}\right|^2$ for $0<\left|\mathbf{q}\right|<1$). This makes the variational methods easier to apply.

Another important property is that the vector field that characterizes the problem benefits of eight natural symmetries $S_i=S_i\left(q_1,q_2,p_1,p_2,t\right),\ i=\overline{0,7}$, as follows:

$$\begin{aligned}
&S_0=\left(q_1,q_2,p_1,p_2,t\right)=I\ \text{(identity)},\ S_1=\left(q_1,q_2,-p_1,-p_2,-t\right),\\
&S_2=\left(q_1,-q_2,-p_1,p_2,-t\right),\ S_3=\left(-q_1,q_2,p_1,-p_2,-t\right),\\
&S_4=\left(q_1,-q_2,p_1,-p_2,t\right),\ S_5=\left(-q_1,q_2,-p_1,p_2,t\right),\\
&S_6=\left(-q_1,-q_2,-p_1,-p_2,t\right),\ S_7=\left(-q_1,-q_2,p_1,p_2,-t\right)
\end{aligned} \tag{2.2}$$

It is easy to translate these symmetries in physical terms. For instance, S_1 implies that, for every solution, there is another solution with the same coordinates and with inverse velocities, all in reversed time, and so forth.

3. Main steps

The anisotropy of the potential is a strongly destabilizing factor for the motion we study. Within a slightly more general model (Mioc et al. 2003), we found collision/ejection-type or escape/capture-type orbits, but did not succeed in proving the existence or nonexistence of periodic orbits. Here, taking into account those previous results, we use some symmetries and topological constraints in connection with a variational principle to get periodic orbits as extrema of the action. To this end, we resort to results concerning periodic solutions of fixed period for symmetric, singular, Lagrangian systems (Ambrosetti and Coti Zelati 1993), as it is the case for the system associated to (2.1).

We first choose a value $T>0$, and dwell upon the space of T-periodic C^∞ cycles $f:[0,T]\to\mathbb{R}^2$. Let L^2 be the space of square integrable functions, and let H^1 be the Sobolev space of all absolutely continuous T-periodic functions with L^2 derivatives defined almost everywhere.

The potential (2.1) is singular at $(0,0)$ (collision). Let $\Lambda=\{f\in H^1\mid f(t)\neq(0,0),\ \forall t\in[0,T]\}$ be the open subset of noncollisional cycles in H^1. We define the *winding number* $w(f)$, which shows how many times the continuous cycle f winds around the origin. It easily follows that $\Lambda=\cup_{k\in\mathbb{Z}}\Lambda_k$, with $\Lambda_k=\{f\in\Lambda\mid w(f)=k\}$. In other words, the set of noncollisional cycles is partitioned by the winding number.

Coming back to the natural symmetries of the system, denote the subsets of H^1 formed by S_i-symmetric cycles by Σ_i.

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As a first step, we proved that Σ_i, $i \in \{1,2,3,7\}$, are Sobolev spaces, which are the natural framework for finding periodic solutions by variational methods. Moreover, we proved that $H^1 = \Sigma_2 \oplus \Sigma_3 = \Sigma_1 \oplus \Sigma_7$ (orthogonal decomposition), hence such a couple is sufficient to cover the whole loop space H^1.

In the second step, in order to avoid cycles that pass through origin (collision) or have zero winding number (escape or quasiperiodic orbits), we examined all subsets Σ_i. Only Σ_2 and Σ_3 fulfil these requirements. With the above results, Σ_2 and Σ_3 can provide periodic solutions for the whole H^1.

We define now the action integral (whose extremal values will provide periodic orbits) $A_T : \Lambda \to \mathbb{R}$ between the instants 0 and T, along a cycle f whose Euclidean coordinate representation is $\mathbf{q} = (q_1, q_2)$ as

$$A_T(f) = \int_0^T L(\mathbf{q}(t), \mathbf{p}(t))\, dt, \tag{3.1}$$

with the positive Lagrangian specified in Section 2. To obtain periodic solutions, we are forced to minimize A_T on subsets Λ_k of Λ, chosen via symmetries). After selecting a suitable subset, we use the *lower-semicontinuity method* (e.g., Struwe 1996) to get a minimizer in that subset, which we prove to be an extremal value of A_T..

A specification from the astronomer's standpoint is necessary here: why the minimization of the action leads to periodic orbits? Recall that the Lagrangian of our problem is the sum of two positive terms: the kinetic energy K and the force function W (the negative of the potential energy). Also note that K and W are not independent each other, being related by the energy integral. Since $K, W > 0$, any minimization of their sum involves the minimization of both K and W; both push the trajectory away from the field-generating centre. But the limit imposed by the fixed energy-level and by the fixed value of T stops the orbit expansion to a finite value, which can lead to a periodic orbit.

A new step concerns the connection between solutions of our problem and the extremals (critical points) of A_T. We proved that, if a cycle f is a critical point of A_T on Λ, then f is a classical periodic solution of the problem.

Next we showed that the elements for which the action is bounded are bounded away from zero. This prevents critical points from being collisional solutions. Another property of the action, the *coercivity*, avoids critical points at infinity. The fact that our procedure also avoids quasiperiodic solutions makes our results lead to genuine periodic solutions.

4. Main results

Provided all previous results (Σ_i, $i \in \{1,2,3,7\}$, are Sobolev spaces; $H^1 = \Sigma_2 \oplus \Sigma_3 = \Sigma_1 \oplus \Sigma_7$; only the cycles belonging to Σ_2 and Σ_3 are noncollisional, nonescape, and do not represent quasiperiodic orbits; Σ_2 and Σ_3 can provide periodic solutions for the whole H^1; the minimizer in a Λ_k subset is an extremal of the action; a cycle that is aa critical point of A_T is a classical periodic solution of the problem), we proved the following:

THEOREM 1. *For any $T > 0$ and any $k \in \mathbb{Z}\backslash\{0\}$, there exists at least one S_i-symmetric ($i = 2, 3$) periodic orbit with period T and winding number k.*

THEOREM 2. *By the existence theorems of a minimal period τ of an autonomous system, and of a period $\tau/2$ (e.g., Amann 1990), there exist infinitely many families of distinct T-periodic orbits.*

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Remark 3. Consider the present model to be a perturbation of the isotropic case (Stoica and Mioc 1997) via the parameter $\mu > 1$. This anisotropy, no matter how large its size, deforms the S_i-symmetric ($i = 2, 3$) periodic orbits of the isotropic problem, but does not destroy them. This makes the symmetries S_i ($i = 2, 3$) constitute an indicator of the robustness of the system to perturbations.

5. Conclusions

From the astronomical point of view, a central proto-star in fast rotation (hence oblate), surrounded by a nonhomogeneous accretion disk, and a body moving under this combined influence constitute a good concrete situation to be studied via a two-body problem associated to the anisotropic potential (2.1).

Our results prove that initially far orbits can collide with the central star or escape from the system (cf. Mioc *et al.* 2003), but genuine periodic orbits exist, too.

Even if outside our framework, problems as dynamics and structure of planetary rings, or satellite dynamics under the influence of the re-emitted solar/stellar radiation, can be tackled via the same mathematical tools.

The results we presented add new features as regards structure of proto-stellar systems, but can also serve to the understanding of the dynamics in some concrete situations in the Solar system.

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