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EXISTENCE OF S-ASYMPTOTICALLY ω-PERIODIC SOLUTIONS FOR ABSTRACT NEUTRAL EQUATIONS

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Abstract

A bounded continuous function $u : [0, \infty) \to X$ is said to be S-asymptotically ω-periodic if $\lim_{t\to\infty}[u(t + \omega) - u(t)] = 0$. This paper is devoted to study the existence and qualitative properties of S-asymptotically ω-periodic mild solutions for some classes of abstract neutral functional differential equations with infinite delay. Furthermore, applications to partial differential equations are given.

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1. Introduction

In this paper we study the existence and uniqueness of S-asymptotically ω-periodic mild solutions for two classes of initial value problems modeled by abstract neutral functional differential equations with infinite delay. Throughout this paper, X denotes a Banach space endowed with a norm $\|\cdot\|$ and A is the infinitesimal generator of a strongly continuous and uniformly exponentially stable semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X. We are concerned with the problems

$$\frac{d}{dt}\,(u(t) - f(t, u_t)) = Au(t) + g(t, u_t), \quad t \geq 0, \tag{1.1}$$

and

$$\frac{d}{dt}\,D(t, u_t) = AD(t, u_t) + g(t, u_t), \quad t \geq 0, \tag{1.2}$$

with initial condition

$$u_0 = \varphi \in \mathcal{B}. \tag{1.3}$$

In these equations, $u(t) \in X$, the history $u_t : (-\infty, 0] \to X$ defined by $u_t(\theta) = u(t + \theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, $D(t, \psi) = \psi(0) - f(t, \psi)$, and $f, g : \mathbb{R} \times \mathcal{B} \to X$ are appropriate functions.

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The literature concerning S-asymptotically ω-periodic functions is very restricted and limited essentially to the study of the existence of S-asymptotically ω-periodic solutions of ordinary differential equations described on finite-dimensional spaces. The interested reader is referred to [22, 6, 8, 30, 31]. To the best of the authors' knowledge, the existence of S-asymptotically ω-periodic mild solutions for abstract functional differential equations (in particular, abstract neutral functional differential equations with unbounded delay) is a subject that has not been treated in the literature. This fact and the interesting relationship between S-asymptotically ω-periodic and asymptotically ω-periodic functions are the main motivations of this work.

Neutral differential equations arise in many areas of applied mathematics. For this reason, this type of equation has received much attention in recent years. Partial neutral differential equations with finite delay arise, for instance, in the transmission line theory. Wu and Xia have shown in [33] that a ring array of identical resistively coupled lossless transmission lines leads to a system of neutral functional differential equations with discrete diffusive coupling, which exhibit various types of discrete waves. By taking a natural limit, they obtained from this system of neutral equations a scalar partial neutral functional differential equation with finite delay defined on the unit circle. This partial neutral functional differential equation was also investigated by Hale in [9] under the more general form

$$\frac{d}{dt}\,\mathcal{D}u_t(x) = \frac{\partial^2}{\partial x^2}\mathcal{D}u_t(x) + f(u_t)(x), \qquad t \geq 0,$$
$$u_0 = \varphi \in C([-r, 0]; C(S^1, \mathbb{R})),$$

where

$$\mathcal{D}(\psi)(s) = \psi(0)(s) - \int_{-r}^{0} [d\eta(\theta)]\psi(\theta)(s)$$

for $s \in S^1$, $\psi \in C([-r, 0], C(S^1; \mathbb{R}))$ and $\eta(\cdot)$ is a function of bounded variation and non-atomic at zero.

Partial neutral differential equations with infinite delay arise in the theory elaborated by Gurtin and Pipkin [7] and Nunziato [27] for the description of heat conduction in materials with fading memory. In the classic theory of heat conduction, it is assumed that both the internal energy and the heat flux are linearly dependent on the temperature $u(\cdot)$ and its gradient $\nabla u(\cdot)$. Under these conditions, the classic heat equation is sufficient for describing the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [7, 27], the internal energy and the heat flux are described as functionals of $u(\cdot)$ and u_x. The next system,

$$\frac{d}{dt}\left[c_0u(t,x)+\int_{-\infty}^{t}k_1(t-s)u(s,x)\,ds\right] = c_1\Delta u(t,x)+\int_{-\infty}^{t}k_2(t-s)\Delta u(s,x)\,ds, \tag{1.4}$$

$$u(t,x)=0, \quad x\in\partial\Omega, \tag{1.5}$$

(see, for instance, [4, 23, 29]), has been used frequently to describe this phenomenon. In this system, $\Omega\subseteq\mathbb{R}^n$ is open, bounded with smooth boundary; $(t,x)\in[0,\infty)\times\Omega$; $u(t,x)$ represents the temperature in the position x and at the time t; c_0, c_1 are physical constants and $k_i:\mathbb{R}\to\mathbb{R}$, $i=1,2$, are the internal energy and the heat flux relaxation, respectively. By assuming that the initial distribution of temperature $u(\cdot)$ is known on $(-\infty,0]\times\Omega$ and that $k_1=k_2$ or $k_2\equiv0$, we can transform this system into an abstract neutral functional differential equation. For more details on partial neutral functional differential equations we refer the reader to Hale [9], Wu and Xia [34, 33, 32], Adimy [1] for finite delay equations, and Hernández and Henríquez [15, 16, 12] for equations with infinite delay.

Neutral systems with unbounded delay also appear in control theory. A method to stabilize lumped control systems is to use a hereditary proportional–integral–differential (PID) feedback control [2, 21]. Consider a linear distributed hereditary system with unbounded delay in the form

$$x'(t)=Ax(t)+a(t)\lambda(x_t)+Bu(t), \tag{1.6}$$

where $x(t)\in X$ represents the state, $u(t)\in\mathbb{R}^m$ denotes the control, A is the generator of an analytic semigroup on a Banach X, $B:\mathbb{R}^m\to X$ is a bounded linear operator and $a(\cdot)$, $\lambda(\cdot)$ are appropriated functions. By using a PID-hereditary control defined by

$$u(t)=K_0x(t)+\int_{-\infty}^{t}K_2(t-s)x(s)\,ds-\frac{d}{dt}\int_{-\infty}^{t}K_1(t-s)x(s)\,ds \tag{1.7}$$

where $K_0:X\to\mathbb{R}^m$ is a bounded linear operator and $K_1,\ K_2:[0,\infty)\to\mathcal{L}(X;\mathbb{R}^m)$ are strongly continuous operator valued maps, we obtain the neutral system with unbounded delay

$$\frac{d}{dt}\left[x(t)+\int_{-\infty}^{t}BK_1(t-s)x(s)\,ds\right] = (A+BK_0)x(t)+a(t)\lambda(x_t)+\int_{-\infty}^{t}BK_2(t-s)x(s)\,ds.$$

On the other hand, owing to its intrinsic mathematical interest as well as to its relevance in numerous applications to physics, biology, control theory, and so on, the existence of solutions with some periodicity property is one of the most attractive subjects of the qualitative theory of differential equations. In particular, the existence of almost-periodic, asymptotically almost-periodic, almost-automorphic, asymptotically almost-automorphic and pseudo-almost-periodic solutions, to mention some of them, have been investigated widely; see, for instance, [3, 11, 18, 20, 26]. Recent contributions on the existence of solutions with some of the previously enumerated properties or another type of almost periodicity to neutral functional differential equations have been made in [25, 35] for the case of neutral ordinary differential equations, and in [17, 14, 5] for partial functional differential systems.

Our notation follows the usual conventions in operator theory. In particular, for Banach spaces X, Y, we denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators from X into Y and $\mathcal{L}(X) = \mathcal{L}(X, X)$.

This paper is organized as follows. In the next section, we introduce definitions and we establish some preliminary properties needed to establish our results. In Section 3, we study the existence of S-asymptotically ω-periodic mild solutions for the neutral systems (1.1)–(1.3) and (1.2)–(1.3). In the same section, we discuss some relationships between S-asymptotically ω-periodic and asymptotically ω-periodic functions. Section 4 is dedicated to exhibiting some applications.

2. Preliminaries

Throughout this paper $(X, \|\cdot\|)$ is a Banach space, ω is a positive real number, $A : D(A) \subset X \to X$ is the infinitesimal generator of a uniformly stable strongly continuous semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X, and we denote by M_0, μ positive constants such that $\| T(t) \| \leq M_0 e^{-\mu t}$ for every $t \geq 0$. For the concepts and properties of strongly continuous semigroups we refer the reader to [28].

In this work, we employ the axiomatic definition of the phase space $\mathcal{B}$ introduced in [19]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and verifying the following axioms.

(A) If $x : (-\infty, \sigma + a) \mapsto X$, $a > 0$, $\sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following hold:

(i) x_t is in $\mathcal{B}$;

(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$;

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| \mid \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$;

where $H > 0$ is a constant, $K, M : [0, \infty) \mapsto [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (**A**), the function $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathcal{B}$.

(B) The space $\mathcal{B}$ is complete.

(C2) If $(\psi^n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence of continuous functions with compact support and $\psi^n \to \psi$, $n \to \infty$, in the compact-open topology, then $\psi \in \mathcal{B}$ and $\|\psi^n - \psi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

We introduce the space $\mathcal{B}_0 = \{\psi \in \mathcal{B} \mid \psi(0) = 0\}$ and the operator $S(t) : \mathcal{B} \to \mathcal{B}$ given by

$$[S(t)\psi](\theta) = \begin{cases} T(t+\theta)\psi(0), & -t \le \theta \le 0, \\ \psi(t+\theta), & -\infty < \theta < -t. \end{cases}$$

It is well known that $(S(t))_{t\ge 0}$ is a C_0-semigroup [19].

DEFINITION 2.1. The phase space $\mathcal{B}$ is called a *fading memory space* if $\|S(t)\psi\|_{\mathcal{B}} \to 0$ as $t \to \infty$ for every $\psi \in \mathcal{B}_0$.

REMARK 2.2. Since $\mathcal{B}$ satisfies axiom (C2), the space $C_b((-\infty, 0], X)$ consisting of continuous and bounded functions $\psi : (-\infty, 0] \to X$, is continuously included in $\mathcal{B}$. Thus, there exists a constant $L \ge 0$ such that $\|\psi\|_{\mathcal{B}} \le L\|\psi\|_\infty$, for every $\psi \in C_b((-\infty, 0], X)$ [19, Proposition 7.1.1].

Moreover, if $\mathcal{B}$ is a fading memory space, then K, M are bounded functions (see [19, Proposition 7.1.5]).

EXAMPLE 2.3 (The phase space $C_r \times L^p(\rho, X)$). Let $r \ge 0$, $1 \le p < \infty$ and let $\rho : (-\infty, -r] \mapsto \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5)–(g-6) in the terminology of [19]. Briefly, this means that ρ is locally integrable and there exists a non-negative locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi+\theta) \le \gamma(\xi)\rho(\theta)$, for all $\xi \le 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set whose Lebesgue measure zero.

The space $\mathcal{B} = C_r \times L^p(\rho, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \mapsto X$ such that φ is continuous on $[-r, 0]$, Lebesgue measurable and $\rho\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\rho, X)$ is defined as follows:

$$\| \varphi \|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r \le \theta \le 0\} + \left(\int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|^p \, d\theta \right)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(\rho, X)$ satisfies axioms (A), (A-1) and (B). Moreover, when $r = 0$ and $p = 2$, it is possible to choose $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and

$$K(t) = 1 + \left(\int_{-t}^{0} \rho(\theta)\, d\theta \right)^{1/2} \quad \forall\, t \ge 0$$

(see [19, Theorem 1.3.8] for details). Note that if conditions (g-5)–(g-7) of [19] hold, then $\mathcal{B}$ is a fading memory space. (See [19, Example 7.1.8].)

To study S-asymptotically ω-periodic functions, it is convenient to introduce additional notation. In the rest of this work $C_b([0, \infty), X)$, $C_0([0, \infty), X)$ and $C_\omega([0, \infty), X)$ denote the spaces

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$$C_b([0,\infty),X)=\{x\in C([0,\infty),X):\sup_{t\geq 0}\|x(t)\|<\infty\},$$
$$C_0([0,\infty),X)=\{x\in C_b([0,\infty),X):\lim_{t\to\infty}\|x(t)\|=0\},$$
$$C_\omega([0,\infty),X)=\{x\in C_b([0,\infty),X):x\text{is }\omega\text{-periodic}\},$$

endowed with the norm of the uniform convergence.

DEFINITION 2.4. A function $u\in C_b([0,\infty),X)$ is called *S-asymptotically ω-periodic* if

$$\lim_{t\to\infty}(u(t+\omega)-u(t))=0.$$

DEFINITION 2.5. A function $u\in C(\mathbb{R},X)$ is called *almost periodic* if for every $\varepsilon>0$, there exists a relatively dense subset $\mathcal{H}(\varepsilon,u)$ of $\mathbb{R}$ such that $\|u(t+\xi)-u(t)\|<\varepsilon$ for every $t\in\mathbb{R}$ and all $\xi\in\mathcal{H}(\varepsilon,u)$.

DEFINITION 2.6. A function $u\in C_b([0,\infty),X)$ is called *asymptotically almost periodic* if there exists an almost-periodic function $v(\cdot)$ and $w\in C_0([0,\infty),X)$ such that $u=v+w$. If v is ω-periodic, $u(\cdot)$ is said asymptotically ω-periodic.

In the rest of this paper, the notation $SAP_\omega(X)$ stands for the space formed by the X-valued S-asymptotically ω-periodic functions endowed with the norm of the uniform convergence. It is clear that $SAP_\omega(X)$ is a Banach space.

In the following statements, $(W,\|\cdot\|_W)$, $(Z,\|\cdot\|_Z)$ are Banach spaces.

DEFINITION 2.7. A continuous function $F:[0,\infty)\times Z\to W$ is called *uniformly S-asymptotically ω-periodic* on bounded sets if $F(\cdot,x)$ is bounded for each $x\in Z$, and for every $\varepsilon>0$ and all bounded set $K\subseteq Z$ there exists $L_{K,\varepsilon}\geq 0$ such that $\|F(t,x)-F(t+\omega,x)\|_W\leq\varepsilon$ for every $t\geq L_{K,\varepsilon}$ and all $x\in K$.

DEFINITION 2.8. A continuous function $F:[0,\infty)\times Z\to W$ is called *asymptotically uniformly continuous on bounded sets*, if for every $\varepsilon>0$ and all bounded sets $K\subseteq Z$ there exist constants $L_{K,\varepsilon}\geq 0$ and $\delta_{K,\varepsilon}>0$ such that $\|F(t,x)-F(t,y)\|_W\leq\varepsilon$ for all $t\geq L_{K,\varepsilon}$ and every $x,y\in K$ with $\|x-y\|_Z\leq\delta_{K,\varepsilon}$.

LEMMA 2.9. *Assume that $F:[0,\infty)\times Z\to W$ is a function uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let $u\in SAP_\omega(Z)$. Then,*

$$\lim_{t\to\infty}(F(t+\omega,u(t+\omega))-F(t,u(t)))=0.$$

PROOF. Since $\mathcal{R}(u)=\{u(t)\mid t\geq 0\}$ is a bounded set, for each $\varepsilon>0$ there exist constants $\delta_{\mathcal{R}(u),\varepsilon}>0$ and $L^1_{\mathcal{R}(u),\varepsilon}>0$ such that

$$\max\{\|F(t+\omega,z)-F(t,z)\|_W,\|F(t,x)-F(t,y)\|_W\}\leq\varepsilon,$$

for every $t\geq L^1_{\mathcal{R}(u),\varepsilon}$ and $x,y,z\in\mathcal{R}(u)$ with $\|x-y\|_Z\leq\delta_{\mathcal{R}(u),\varepsilon}$. Likewise, there exists $L^2_\varepsilon>0$ such that $\|u(t+\omega)-u(t)\|_Z\leq\delta_{\mathcal{R}(u),\varepsilon}$, for every $t\geq L^2_\varepsilon$. Combining these inequalities, for $t\geq\max\{L^2_\varepsilon,L^1_{\mathcal{R}(u),\varepsilon}\}$, we obtain

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$$\begin{aligned}\|F(t+\omega, u(t+\omega)) - F(t, u(t)\|_W &\le \|F(t+\omega, u(t+w)) - F(t, u(t+w))\|_W \\ &\quad + \|F(t, u(t+\omega)) - F(t, u(t))\|_W \\ &\le 2\varepsilon,\end{aligned}$$

which proves the assertion. □

For fading memory spaces the following property holds.

LEMMA 2.10. *Assume that $\mathcal{B}$ is a fading memory space. Let $u : \mathbb{R} \to X$ be a function with $u_0 \in \mathcal{B}$ and $u_{|[0,\infty)} \in SAP_\omega(X)$. Then the function $t \to u_t$ belongs to $SAP_\omega(\mathcal{B})$.*

PROOF. Since $K(t)$ and $M(t)$ are bounded functions, from axiom (A)(iii) we obtain that the function $t \to u_t$ is bounded on $[0, \infty)$. We define the function

$$y(t) = u(t+\omega) - u(t) = u_\omega(t) - u(t) \quad \text{for } t \in \mathbb{R}.$$

Clearly $y(\cdot)$ is continuous on $[0, \infty)$ and $y_0 = u_\omega - u_0 \in \mathcal{B}$. Since $y(t) \to 0$ as $t \to \infty$, it follows from [19, Proposition 7.1.3] that $\|y_t\|_{\mathcal{B}} = \|u_{t+\omega} - u_t\|_{\mathcal{B}} \to 0$ as $t \to \infty$, which completes the proof that $t \to u_t$ is S-asymptotically ω-periodic. □

3. S-asymptotically ω-periodic mild solutions for neutral systems with unbounded delay

In this section, we discuss the existence of S-asymptotically ω-periodic mild solutions for the neutral systems (1.1)–(1.3) and (1.2)–(1.3).

Initially we study the existence of S-asymptotically ω-periodic mild solutions for the neutral system (1.1)–(1.3). For completeness we recall the concept of a mild solution introduced in [15].

DEFINITION 3.1. A function $u : \mathbb{R} \to X$ is called a *mild solution* of the system (1.1)–(1.3) if $u_0 = \varphi$, $u(\cdot)$ is continuous on $[0, \infty)$, the functions $s \to AT(t-s)f(s, u_s)$ and $s \to T(t-s)g(s, u_s)$ are integrable on $[0, t)$ for every $t \ge 0$ and

$$\begin{aligned}u(t) &= T(t)(\varphi(0) - f(0, \varphi)) + f(t, u_t) + \int_0^t AT(t-s)f(s, u_s)\, ds \\ &\quad + \int_0^t T(t-s)g(s, u_s)\, ds \quad t \ge 0.\end{aligned}$$

We next introduce the following general assumptions.

(H_1) There exists a Banach space $(Y, \|\cdot\|_Y)$ continuously included in X and a function $H \in L^1([0, \infty); \mathbb{R}^+)$ such that $\|AT(t)\|_{\mathcal{L}(Y,X)} \le H(t)$ for every $t > 0$.

(H_2) The function $f \in C([0, \infty) \times \mathcal{B}, Y)$ and there exists $L_f > 0$ such that

$$\|f(t, \psi_1) - f(t, \psi_2)\|_Y \le L_f \|\psi_1 - \psi_2\|_{\mathcal{B}}, \ (t, \psi_i) \in [0, \infty) \times \mathcal{B}.$$

(H_3) The function $g \in C([0, \infty) \times \mathcal{B}, X)$ and there exists $L_g > 0$ such that

$$\|g(t, \psi_1) - g(t, \psi_2)\| \le L_g \|\psi_1 - \psi_2\|_{\mathcal{B}}, \ (t, \psi_i) \in [0, \infty) \times \mathcal{B}.$$

REMARK 3.2. The conditions (H_1) and (H_2) are linked to the integrability of the function $s \to AT(t-s)f(s, u_s)$ over $[0, t)$. We remark that except for the trivial

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cases, the operator function $t \to AT(t)$ is not integrable over $[0, a]$. If conditions (H_1) and (H_2) are verified, then from the Bochner's criterion for integrable functions and the estimate

$$\|AT(t-s)f(s,u_s)\| \leq \|AT(t-s)\|_{\mathcal{L}(Y,X)}\|f(s,u_s)\|_Y \leq H(t-s)\|f(s,u_s)\|_Y,$$

it follows that the function $s \mapsto AT(t-s)f(s,u_s)$ is integrable over $[0, t]$ for every $t > 0$. For additional remarks related to this type of condition in the theory of partial neutral differential equations, we refer the interested reader to [1, 15, 16], and in particular to [12, 13].

REMARK 3.3. We note that the condition (H_1) is verified in several situations that arise in concrete problems. If, for instance, $(T(t))_{t\geq 0}$ is an analytic semigroup, we can assume that Y is some interpolation space between X and $[D(A)]$. For concrete examples, see Lunardi [24].

LEMMA 3.4. *Assume that condition* (H_1) *holds. Let* $u \in SAP_\omega(Y)$ *and let* $v : [0, \infty) \to X$ *be the function defined by*

$$v(t) = \int_0^t AT(t-s)u(s)\,ds. \tag{3.1}$$

Then $v \in SAP_\omega(X)$.

PROOF. The estimate

$$\|v(t)\| \leq \int_0^t \|AT(t-s)\|_{\mathcal{L}(Y,X)}\|u(s)\|_Y\,ds \leq \|u\|_{Y,\infty}\int_0^\infty H(s)\,ds,$$

shows that $v \in C_b([0,\infty), X)$. Furthermore, for $t \geq L_1 > 0$, we can write

$$\begin{aligned}\|v(t+\omega)-v(t)\| \leq{}& \int_0^\omega \|AT(t+\omega-s)\|_{\mathcal{L}(Y,X)}\|u(s)\|_Y\,ds\\ &+\int_0^{L_1}\|AT(t-s)\|_{\mathcal{L}(Y,X)}\|u(s+\omega)-u(s)\|_Y\,ds\\ &+\int_{L_1}^t\|AT(t-s)\|_{\mathcal{L}(Y,X)}\|u(s+\omega)-u(s)\|_Y\,ds.\end{aligned}$$

For $\varepsilon > 0$, we select $L_1 > 0$ such that $\|u(s+\omega)-u(s)\| \leq \varepsilon$ for all $s \geq L_1$ and $\int_{L_1}^\infty H(s)\,ds < \varepsilon$. Hence, for $t \geq 2L_1$ we obtain

$$\begin{aligned}\|v(t+\omega)-v(t)\| \leq{}& \|u\|_{Y,\infty}\int_t^{t+\omega}H(s)\,ds + 2\|u\|_{Y,\infty}\int_{L_1}^t H(s)\,ds\\ &+\varepsilon\int_0^\infty H(s)\,ds\\ \leq{}& \varepsilon\|u\|_{Y,\infty}\left(3+\int_0^\infty H(s)\,ds\right).\end{aligned}$$

This completes the proof. □

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Using the fact that $(T(t))_{t\geq 0}$ is uniformly stable, and arguing as in the proof of Lemma 3.4, we can state the following result.

LEMMA 3.5. *Let $u \in SAP_\omega(X)$ and let $v : [0, \infty) \to X$ be the function defined by*

$$v(t) = \int_0^t T(t-s)u(s)\,ds. \tag{3.2}$$

Then $v \in SAP_\omega(X)$.

We can establish the main result of this section. In this result, ι denotes the inclusion from Y into X and L is the constant introduced in Remark 2.2.

THEOREM 3.6. *Assume that $\mathcal{B}$ is a fading memory space and that conditions (H_1)–(H_3) hold. If the functions f and g are uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets, and $LL_f\|\iota\|_{\mathcal{L}(Y,X)} < 1$, then there exists a unique S-asymptotically ω-periodic mild solution of problem (1.1)–(1.3).*

PROOF. We set $SAP_{\omega,0}(X) = \{x \in SAP_\omega(X) \mid x(0) = 0\}$. It is clear that $SAP_{\omega,0}(X)$ is a closed subspace of $SAP_\omega(X)$. We next identify the elements $x \in SAP_{\omega,0}(X)$ with its extension to $\mathbb{R}$ given by $x(\theta) = 0$ for $\theta \leq 0$. Moreover, we denote by $y(\cdot)$ the function defined by $y_0 = \varphi$ and $y(t) = T(t)\varphi(0)$ for $t \geq 0$. We define the map Γ on $SAP_{\omega,0}(X)$ by

$$\Gamma x(t) = -T(t)f(0,\varphi) + f(t, x_t + y_t) + \int_0^t AT(t-s)f(s, x_s + y_s)\,ds + \int_0^t T(t-s)g(s, x_s + y_s)\,ds$$

for $t \geq 0$. It follows from our hypotheses, Lemmas 3.4 and 3.5, that Γ is a map from $SAP_{\omega,0}(X)$ into $SAP_{\omega,0}(X)$. Furthermore, the estimate

$$\begin{aligned}\|\Gamma x(t) - \Gamma z(t)\| &\leq L_f\|\iota\|_{\mathcal{L}(Y,X)}\|x_t - z_t\|_{\mathcal{B}} + L_f\int_0^t H(t-s)\|x_s - z_s\|_{\mathcal{B}}\,ds \\ &\quad + M_0L_g\int_0^t e^{-\mu(t-s)}\|x_s - z_s\|_{\mathcal{B}}\,ds \\ &\leq L\left[L_f\left(\|\iota\|_{\mathcal{L}(Y,X)} + \int_0^\infty H(s)\,ds\right) + \frac{M_0L_g}{\mu}\right] \\ &\quad \times \|x - z\|_\infty \end{aligned} \tag{3.3}$$

shows that Γ is continuous.

On the other hand, we define $B : C_b([0,\infty)) \to C_b([0,\infty))$ by

$$B\alpha(t) = LL_f\|\iota\|_{\mathcal{L}(Y,X)}\alpha(t) + LL_f\int_0^t H(t-s)\alpha(s)\,ds + M_0LL_g\int_0^t e^{-\mu(t-s)}\alpha(s)\,ds$$

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for $t \geq 0$. It is clear that B is a bounded linear operator. Let $B_0 : C_b([0, \infty)) \to C_b([0, \infty))$ given by

$$B_0\alpha(t) = LL_f \int_0^t H(t-s)\alpha(s)\, ds + M_0 LL_g \int_0^t e^{-\mu(t-s)}\alpha(s)\, ds$$

for $t \geq 0$. Then $B = LL_f \|\iota\|_{\mathcal{L}(Y,X)} I + B_0$. Using that $H(\cdot)$ is integrable on $[0, \infty)$ it follows that B_0 is completely continuous with spectrum $\sigma(B_0) = \{0\}$. Therefore, $\sigma(B) = \{LL_f \|\iota\|_{\mathcal{L}(Y,X)}\}$ and B is an operator with spectral radius $r_e(B) < 1$. In addition, we define $m : C_b([0, \infty), X) \to C_b([0, \infty))$ by

$$m(x)(t) = \sup_{0 \leq s \leq t} \|x(s)\|, \quad t \geq 0.$$

From (3.3), we obtain that

$$m(\Gamma x - \Gamma z) \leq Bm(x - z)$$

and, applying [10, Theorem 1], we conclude that Γ has a unique fixed point $x \in SAP_{\omega,0}(X)$. Defining $u(t) = y(t) + x(t)$ for $t \in \mathbb{R}$, we can confirm that $u \in SAP_\omega(X)$ is a mild solution of problem (1.1)–(1.3). □

Using similar arguments we can obtain the existence of S-asymptotically ω-periodic mild solutions for the neutral system (1.2)–(1.3). We begin by defining our concept of a mild solution.

DEFINITION 3.7. A function $u : \mathbb{R} \to X$ is called a *mild solution of the neutral system* (1.2)–(1.3) if $u_0 = \varphi$, u is continuous on $[0, \infty)$ and

$$u(t) = T(t)(\varphi(0) - f(0, \varphi)) + f(t, u_t) + \int_0^t T(t-s) g(s, u_s)\, ds, \quad t \geq 0.$$

In the following result we consider the condition (H_2) with $Y = X$.

THEOREM 3.8. *Assume that $\mathcal{B}$ is a fading memory space and that conditions (H_2)–(H_3) hold. If the functions f and g are uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets, and $LL_f < 1$, then there exists a unique S-asymptotically ω-periodic mild solution of problem (1.2)–(1.3).*

3.1. On S-asymptotically ω-periodic and asymptotically ω-periodic solutions. To finish this section we establish conditions under which an S-asymptotically ω-periodic function is asymptotically ω-periodic. Using these conditions, we discuss the existence of asymptotically ω-periodic mild solutions.

REMARK 3.9. In the rest of this section, for $\tau > 0$ and $p \in C_b([0, \infty), X)$, we denote by $\mathcal{T}_\tau p$ the left shift defined by $\mathcal{T}_\tau p(t) = p(t + \tau)$. In addition, for $t \geq 0$, we consider the decomposition $t = \xi(t) + \tau(t)\omega$ where $\xi(t) \in [0, \omega)$, $\tau(t) \in \mathbb{N} \cup \{0\}$.

We first study some qualitative properties of S-asymptotically ω-periodic functions.

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LEMMA 3.10. *Let $u \in SAP_\omega(X)$ and let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence in $[0, \infty)$ with $\tau_n \to \infty$ as $n \to \infty$. If $\mathcal{T}_{\tau_n} u \to v$ as $n \to \infty$ uniformly on compact subsets of $[0, \infty)$, then $v \in C_\omega([0, \infty), X)$.*

PROOF. It is clear that v is continuous. On the other hand, for each $t \geq 0$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \|v(s) - u(s + \tau_n)\| &\leq \varepsilon, \quad s \in [t, t + \omega], \\ \|u(\mu + \tau_n + \omega) - u(\mu + \tau_n)\| &\leq \varepsilon, \quad \mu \geq 0, \end{aligned}$$

for every $n \geq n_0$. Consequently, for $n \geq n_0$

$$\begin{aligned} \|v(t + \omega) - v(t)\| &\leq \|v(t + \omega) - u(t + \omega + \tau_n)\| + \|u(t + \omega + \tau_n) - u(t + \tau_n)\| \\ &\quad + \|u(t + \tau_n) - v(t)\| \\ &\leq 3\varepsilon, \end{aligned}$$

which implies that $v(t + \omega) = v(t)$. □

PROPOSITION 3.11. *If u is an S-asymptotically ω-periodic and asymptotically almost periodic function, then $u(\cdot)$ is asymptotically ω-periodic.*

PROOF. We can decompose $u = p + q$, where the function p is almost periodic and $q \in C_0([0, \infty), X)$. We select a sequence $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \to \infty$ and $\mathcal{T}_{\tau_n} p \to p$ as $n \to \infty$ uniformly on $[0, \infty)$. Therefore, $\mathcal{T}_{\tau_n} u = \mathcal{T}_{\tau_n} p + \mathcal{T}_{\tau_n} q \to p$ as $n \to \infty$. Applying Lemma 3.10 we infer that p is ω-periodic and u is asymptotically ω-periodic. □

COROLLARY 3.12. *Let $u \in C_b([0, \infty), X)$, and assume that there exists a non-decreasing sequence of natural numbers $(n_j)_{j\in\mathbb{N}}$ such that $n_1 = 1$, $\alpha = \sup_{j\in\mathbb{N}}(n_{j+1} - n_j) < \infty$ and*

$$\lim_{t\to\infty} (u(t + n_j\omega) - u(t)) = 0,$$

uniformly for $j \in \mathbb{N}$. Then $u(\cdot)$ is asymptotically ω-periodic.

PROOF. The hypotheses imply that $u \in SAP_\omega(X)$ is asymptotically almost periodic. The assertion follows directly from Proposition 3.11. □

REMARK 3.13. From Corollary 3.12, we note that if $u \in C_b([0, \infty), X)$ is a function such that $\lim_{t\to\infty}(u(t + n\omega) - u(t)) = 0$, uniformly for $n \in \mathbb{N}$, then $u(\cdot)$ is asymptotically ω-periodic.

It is stated in [8, Lemma 2.1] that every S-asymptotically ω-periodic scalar function is asymptotically ω-periodic. This assertion is not true, as the following example shows.

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EXAMPLE 3.14. Let $(b_n)_{n\in\mathbb{N}_0}$ be a sequence of real numbers such that $b_n \neq 0$ for every $n \in \mathbb{N}$, $b_n \to 0$ and $n \to \infty$, and the sequence $(a_n)_{n\in\mathbb{N}} = (\sum_{i=0}^{n} b_i)_{n\in\mathbb{N}}$ is bounded and non-convergent. Let $w : [0, \infty) \to \mathbb{R}$ be the function defined by $w(n) = a_n$ for $n \in \mathbb{N}_0$ and

$$w(t) = a_{n+1} + (a_{n+1} - a_n)(t - n - 1), \tag{3.4}$$

for $n \leq t \leq n + 1$. Consequently, the graph of w consists of the segments of lines with corners at the points (n, a_n). It is clear from this geometrical description that f is bounded and continuous. Furthermore, w is uniformly continuous. In fact, we set $c = \max_{n\geq 1} |a_n - a_{n-1}|$. Employing (3.4) for $s \in [n, n + 1]$ and $t \in [n, n + 2]$, we obtain that $|w(t) - w(s)| \leq c|t - s|$. On the other hand, turning to apply (3.4), we see that

$$|w(t + 1) - w(t)| \leq |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n|,$$

for $t \in [n, n + 1]$. Therefore, $\lim_{t\to\infty}(w(t + 1) - w(t)) = 0$ and w is an S-asymptotically 1-periodic function.

However, w is not an asymptotically 1-periodic function. To establish this assertion, we assume that $w = \beta + \alpha$, where β is a 1-periodic function and α is a function that vanishes at infinity. In such case, $w(n) = a_n = \beta(n) + \alpha(n) = \beta(0) + \alpha(n) \to \beta(0)$, as $n \to \infty$, which contradicts our selection of the sequence $(a_n)_n$.

In the following proposition we establish conditions under which an S-asymptotically ω-periodic function is asymptotically ω-periodic.

PROPOSITION 3.15. *Let $u \in C_b([0, \infty), X)$ be an S-asymptotically ω-periodic and uniformly continuous function with relatively compact range. Assume that there exists a sequence of natural numbers $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \to \infty$ and a sequence of positive numbers $(\gamma_n)_{n\in\mathbb{N}}$ such that $\|u(t + \omega) - u(t)\| \leq \gamma_n$ for every $t \geq \tau_n$ and $\sum_{j\geq 0}(\tau_{j+1} - \tau_j)\gamma_j < \infty$. Then $u(\cdot)$ is asymptotically ω-periodic.*

PROOF. Let $(m_j)_{j\in\mathbb{N}}$ be a subsequence of $(\tau_n)_{n\in\mathbb{N}}$ with $m_j = \tau_{n_j}$ and $v \in C_\omega([0, \infty), X)$ such that $u_{m_j\omega} \to v$ as $j \to \infty$ uniformly on compact sets. We confirm that $v(t) - u(t) \to 0$ as $t \to \infty$. To establish our claim, for each $\varepsilon > 0$, we choose $n_{j_0} \in \mathbb{N}$ such that $\sum_{j\geq n_{j_0}}(\tau_{j+1} - \tau_j)\gamma_j \leq \varepsilon$ and $\|v(s) - u(s + m_j\omega)\| \leq \varepsilon$ for every $s \in [0, \omega]$ and $j \geq j_0$.

Let $t \geq m_{j_0}\omega$. Then there exists an index $p \in \mathbb{N}$ with $p \geq j_0$ such that $t \in [m_p, m_{p+1}]$. The interval $[m_p, m_{p+1}]$ can contain another point of the original sequence $(\tau_n)_{n\in\mathbb{N}}$, which we describe in the form $\tau_{n_p} < \tau_{n_p+1} < \cdots < \tau_{n_p+q} = \tau_{n_{p+1}}$. Similarly, each interval $[\tau_{n_p+i}, \tau_{n_p+i+1}]$, with $i = 0, \ldots, q - 1$, can contain natural numbers $\tau_{n_p+i} + h$, with $h = 0, \ldots, H(i)$ so that $\tau_{n_p+i} + H(i) = \tau_{n_p+i+1}$. We abbreviate the notation by writing $k(i) = \tau_{n_p+i}$. Moreover, we select $0 \leq s < q$ such that $t \in [\tau_{n_p+s}\omega, \tau_{n_p+s+1}\omega]$ and we decompose $t = \xi(t) + \eta(t)\omega$ with $\xi(t) \in [0, \omega)$ and $\eta(t)$ is a natural number such that $\eta(t) = \tau_{n_p+s} + h(t)$ where

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$0 \le h(t) \le H(s)$. With this notation, we obtain

$$\begin{aligned}
\|v(t) - u(t)\| &= \|v(\xi(t) + \eta(t)\omega) - u(\xi(t) + \eta(t)\omega)\| \\
&\le \|v(\xi(t)) - u(\xi(t) + m_p\omega)\| \\
&\quad + \|u(\xi(t) + m_p\omega) - u(\xi(t) + \eta(t)\omega)\| \\
&\le \varepsilon + \sum_{i=0}^{s-1} \sum_{j=k(i)}^{k(i)+H(i)-1} \|u(\xi(t) + (j+1)\omega) - u(\xi(t) + j\omega)\| \\
&\quad + \sum_{j=k(s)}^{k(s)+h(t)-1} \|u(\xi(t) + (j+1)\omega) - u(\xi(t) + j\omega)\| \\
&\le \varepsilon + \sum_{i=0}^{s-1} \sum_{j=k(i)}^{k(i)+H(i)-1} \gamma_{n_p+i} + \sum_{j=k(s)}^{k(s)+h(t)-1} \gamma_{n_p+s} \\
&\le \varepsilon + \sum_{i=0}^{s} \gamma_{n_p+i} H(i) \\
&= \varepsilon + \sum_{i=0}^{s} \gamma_{n_p+i}(\tau_{n_p+i+1} - \tau_{n_p+i}) \\
&\le \varepsilon + \sum_{i \ge n_p} \gamma_i(\tau_{i+1} - \tau_i) \\
&\le 2\varepsilon,
\end{aligned}$$

which shows that $\|v(t) - u(t)\| \le 2\varepsilon$ for every $t \ge m_{j_0}\omega$. This completes the proof. □

We next study the existence of asymptotically ω-periodic solutions for the neutral systems (1.1) and (1.2).

LEMMA 3.16. *Assume that condition (H_1) holds. Let $u \in SAP_\omega(Y)$ and let $v : [0, \infty) \to X$ be the function defined by (3.1). If $\lim_{t\to\infty} \|u(t) - u(t + n\omega)\|_Y = 0$ uniformly for $n \in \mathbb{N}$, then $\lim_{t\to\infty}(v(t) - v(t + n\omega)) = 0$ uniformly for $n \in \mathbb{N}$.*

PROOF. We can prove this statement by arguing as the proof of Lemma 3.4 substituting $n\omega$ instead of ω. □

The same argument, with Lemma 3.5 instead of Lemma 3.4 allows us to state the following result.

LEMMA 3.17. *Let $u \in SAP_\omega(X)$ be a function such that $\lim_{t\to\infty}(u(t) - u(t + n\omega)) = 0$ uniformly for $n \in \mathbb{N}$. If $v : [0, \infty) \to X$ is the function given by (3.2), then $\lim_{t\to\infty}(v(t) - v(t + n\omega)) = 0$ uniformly for $n \in \mathbb{N}$.*

As consequence of Corollary 3.12, Lemmas 3.16, 3.17 and arguing as in the proof of Theorem 3.6, we can state the following result.

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PROPOSITION 3.18. *Assume that the hypotheses of Theorem 3.6 hold, and for each bounded set $K \subseteq \mathcal{B}$, $\lim_{t\to\infty} \|f(t, \psi) - f(t + n\omega, \psi)\|_Y = 0$ and $\lim_{t\to\infty} \|g(t, \psi) - g(t + n\omega, \psi)\| = 0$ uniformly for $\psi \in K$ and $n \in \mathbb{N}$. Then there exists a unique asymptotically ω-periodic mild solution $u(\cdot)$ of (1.1)–(1.3).*

Similarly, as a consequence of Theorem 3.8, Corollary 3.12 and Lemma 3.17 we can state the following result.

PROPOSITION 3.19. *Assume that the hypotheses of Theorem 3.8 hold, and for each bounded set $K \subseteq \mathcal{B}$, $\lim_{t\to\infty} \|f(t, \psi) - f(t + n\omega, \psi)\| = 0$ and $\lim_{t\to\infty} \|g(t, \psi) - g(t + n\omega, \psi)\| = 0$ uniformly for $\psi \in K$ and $n \in \mathbb{N}$. Then there exists a unique asymptotically ω-periodic mild solution $u(\cdot)$ of (1.2)–(1.3).*

4. Applications

In this section, we apply the previous results to some partial neutral functional differential equations with infinite delay.

4.1. A neutral equation in the theory of heat conduction. In the following, we consider the problem of the existence of S-asymptotically ω-periodic mild solutions for a neutral system of type (1.4)–(1.5) in the one-dimensional case with $c_0 = c_1 = 1$ and $k_1 = k_2$. Specifically, we consider the differential system

$$\begin{aligned}
&\frac{\partial}{\partial t}\bigg[u(t, \xi) + \int_{-\infty}^{t} a(s - t)u(s, \xi)\, ds\bigg] \\
&\quad = \frac{\partial^2}{\partial \xi^2}\bigg[u(t, \xi) + \int_{-\infty}^{t} a(s - t)u(s, \xi)\, ds\bigg] \\
&\qquad + \int_{-\infty}^{t} b(s - t)u(s, \xi)\, ds + c(t)F(u(t, \xi)), \qquad (4.1)\\
&u(t, 0) = u(t, \pi) = 0, \qquad (4.2)\\
&u(\theta, \xi) = \varphi(\theta, \xi), \quad -\infty < \theta \le 0, \qquad (4.3)
\end{aligned}$$

for $t > 0$ and $\xi \in [0, \pi]$.

In what follows we consider the space $X = L^2([0, \pi])$ and $A : D(A) \subseteq X \to X$ is the operator defined by $Ax = x''$ with domain $D(A) = \{x \in X \mid x'' \in X, x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\ge 0}$ on X. Furthermore, the spectrum of A is reduced to a point spectrum with eigenvalues of the form $-n^2$ for $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) = (2/\pi)^{1/2} \sin(n\xi)$. In addition, the following properties hold:

(a) $\{z_n \mid n \in \mathbb{N}\}$ is an orthonormal basis of X;

(b) for $x \in X$, $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, z_n\rangle z_n$. It follows from this representation that $\|T(t)\| \le e^{-t}$ for every $t \ge 0$.

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Moreover, it is possible to define the fractional powers of A (see, for instance, [24]). In particular:

(c) for $x \in X$ and $\alpha \in (0, 1)$, $(-A)^{-\alpha}x = \sum_{n=1}^{\infty}(1/n^{2\alpha})\langle x, z_n\rangle z_n$. Moreover, the operator $(-A)^{\alpha} : D((-A)^{\alpha}) \subseteq X \to X$ is given by

$$(-A)^{\alpha} = \sum_{n=1}^{\infty} n^{2\alpha}\langle x, z_n\rangle z_n \quad \text{for } x \in D((-A)^{\alpha})$$
$$= \left\{ x \in X \;\middle|\; \sum_{n=1}^{\infty} n^{2\alpha}\langle x, z_n\rangle z_n \in X \right\}.$$

As phase space we choose the space $\mathcal{B} = C_0 \times L^2(\rho, X)$ defined in Example 2.3, and we assume that the conditions (g-5)–(g-7) in the nomenclature of [19] are satisfied. We note that, under these conditions, $\mathcal{B}$ is a fading memory space and

$$L = 1 + \left(\int_{-\infty}^{0} \rho(\theta)\, d\theta \right)^{1/2}.$$

To study this system, we assume that the function $\varphi(\theta)(\xi) = \varphi(\theta, \xi)$ belongs to $\mathcal{B}$, the functions $a, b : (-\infty, 0] \to \mathbb{R}$ and $c : [0, \infty) \to \mathbb{R}$ are continuous, the function $c(\cdot)$ is S-asymptotically ω-periodic, $F : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous with Lipschitz constant $L_F > 0$ and

$$L_f = \left(\int_{-\infty}^{0} \frac{a^2(s)}{\rho(s)}\, ds \right)^{1/2} < \infty, \quad L_g^1 = \left(\int_{-\infty}^{0} \frac{b^2(s)}{\rho(s)}\, ds \right)^{1/2} < \infty.$$

Under these conditions, we can define the functions $D, f, g : [0, \infty) \times \mathcal{B} \to X$ by

$$f(t, \psi)(\xi) = -\int_{-\infty}^{0} a(s)\psi(s, \xi)\, ds,$$
$$g(t, \psi)(\xi) = \int_{-\infty}^{0} b(s)\psi(s, \xi)\, ds + c(t)F(\psi(0, \xi)),$$
$$D(t, \psi)(\xi) = \psi(0)(\xi) - f(t, \psi)(\xi).$$

With this notation the system (4.1)–(4.3) is reduced to the abstract form (1.2)–(1.3). Moreover, for every $t \geq 0$, the function $f(t, \cdot)$ is a bounded linear operator with $\| f(t, \cdot)\|_{\mathcal{L}(\mathcal{B},X)} \leq L_f$ and $g(t, \cdot)$ is globally Lipschitz continuous with Lipschitz constant $L_g(t) = L_g^1 + | c(t) |_{\infty} L_F$. The next result is a consequence of Theorem 3.8 and Proposition 3.19.

PROPOSITION 4.1. *Assume that $LL_f < 1$. Then there exists a unique S-asymptotically ω-periodic mild solution $u(\cdot)$ of the problem (4.1)–(4.3). If, in addition, $\lim_{t\to\infty}(c(t + n\omega) - c(t)) = 0$ uniformly for $n \in \mathbb{N}$, then $u(\cdot)$ is asymptotically ω-periodic.*

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4.2. A neutral system in control theory. A method to stabilize lumped control systems is to use a hereditary PID feedback control, see [2, 21].

Consider the linear distributed hereditary system with unbounded delay (1.6) and the PID-hereditary control $u(t)$ defined by (1.7). Let $\mathcal{B} = C_0 \times L^2(\rho, X)$ and ρ satisfying conditions (g-5)–(g-7) in the nomenclature of [19]. Assume that $B : \mathbb{R}^m \to X$ is a bounded linear operator such that $\mathcal{R}(B) \subseteq D((-A)^\beta)$ for some $0 < \beta < 1$, $a(\cdot) \in SAP_\omega(\mathbb{R})$; $\lambda : \mathcal{B} \to X$ is a bounded linear operator; $K_0 : X \to \mathbb{R}^m$ is a bounded linear operator and $K_1,\ K_2 : [0, \infty) \to \mathcal{L}(X; \mathbb{R}^m)$ are strongly continuous operator valued functions such that

$$L_i = \left(\int_{-\infty}^0 \frac{\|K_i(-\theta)\|^2}{\rho(\theta)} \, d\theta \right)^{1/2} < \infty, \quad i = 1, 2.$$

Under these conditions, the operators λ_i defined by

$$\lambda_i \psi := \int_{-\infty}^0 K_i(-\theta)\psi(\theta) \, d\theta, \quad i = 1, 2,$$

are bounded linear operators from $\mathcal{B}$ into $\mathbb{R}^m$ and $\|\lambda_i\| \le L_i$, $i = 1, 2$. The closed system corresponding to the PID-hereditary control (1.7) takes the form

$$\frac{d}{dt}(x(t) + B\lambda_1(x_t)) = (A + BK_0)x(t) + (a(t)\lambda + B\lambda_2)(x_t). \tag{4.4}$$

The operator $A + BK_0$ is the infinitesimal generator of an analytic semigroup and we shall assume that this semigroup is uniformly bounded and $0 \in \rho(A + BK_0)$. Under these conditions it follows from [28, formula (2.6.9)] that $D(-A - BK_0)^\beta = D(-A)^\beta$, which implies that $(-A - BK_0)^\beta B\lambda_1$ is bounded.

The following proposition follows from Theorem 3.6 and Proposition 3.18.

PROPOSITION 4.2. *Assume* $L\|(-A - BK_0)^\beta B\lambda_1\| \ \|\iota\|_{\mathcal{L}(X_\beta, X)} < 1$. *Then there exists a unique S-asymptotically ω-periodic mild solution* $u(\cdot)$ *of problem (4.4). If, in addition,* $\lim_{t\to\infty}[a(t) - a(t + n\omega)] = 0$ *uniformly for* $n \in \mathbb{N}$, *then* $u(\cdot)$ *is asymptotically ω-periodic.*

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