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Stability of electron plasmas in stellarators

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It has long been known that the equilibrium of an electron plasma in a stellarator possesses unique properties when compared with other geometries. Previous analyses, both numerical and analytical, as well as experimental results, have indicated that these equilibria are minimum-energy states. Here, it is definitively shown that the equilibrium of an electron plasma on magnetic surfaces with finite rotational transform minimises a constrained physical energy, which has a thermal and an electrostatic contribution. As such, these equilibria are established to be macroscopically stable to all perturbations that do not change the flux-surface average of the density and do not decrease the entropy of the plasma, under the definition of 'formal stability' established by Holm *et al.* (*Phys. Rep.*, vol. 123, no. 1, 1985, 1–116).

Key words: plasma instabilities, plasma confinement

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1. Introduction

Although the development of stellarators has mostly been in the interest of confining an electron–ion plasma for the purposes of terrestrial fusion, they also possess unique properties for confining more 'exotic' plasmas, including those that are non-neutral. The most extreme example of such a non-neutral plasma is a single species plasma, a pure electron plasma for example. Such plasmas consist of a cloud of electrons which are sufficiently cold and dense that the Debye length $\lambda_D = \sqrt{\varepsilon_0 T_e/(e^2 n_e)}$ is much smaller than the macroscopic scale length of the plasma, for example, the minor radius a of a stellarator, giving $a/\lambda_D \gg 1$. Pure electron plasmas which fit this description have been confined in several different geometries, including toroidal field traps (Stoneking *et al.* 2004), dipole fields (Saitoh *et al.* 2007), linear Penning–Malmberg traps (Malmberg & deGrassie 1975) and stellarators (Kremer *et al.* 2006). The latter are unique among these geometries as the only ones to possess magnetic surfaces mapped out by field lines which, if the rotational transform is irrational, touch each part of the surface as they wind around the torus. The existence of magnetic surfaces spanned by a single field line results in interesting consequences for the equilibrium of a pure electron plasma in a stellarator.

Such stellarator equilibria were the primary focus of the Columbia Non-neutral Torus (CNT) in the early 2000s (Pedersen *et al.* 2004) alongside some experiments in the Compact Helical System (CHS) (Himura *et al.* 2007). Recently, this subject has been given

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new relevance by the Electrons and Positrons in an Optimised Stellarator (EPOS) project, which aims to confine an electron–positron plasma in a stellarator (Stoneking *et al.* 2020). Due to the preciousness of positrons, one possible means of operation of EPOS is to first confine a pure electron plasma in the device and then allow the large electrostatic potential of the non-neutral equilibrium to enhance the injection of positrons into the device.

An important question to be addressed is whether an electron plasma can be stably maintained at macroscopic equilibrium in a stellarator for long enough to inject positrons. Here, we show that the macroscopic equilibria of electron plasmas in a stellarator are minimum-energy states, which minimise the physical energy in the laboratory frame, in agreement with numerical (Lefrancois *et al.* 2005) and experimental (Himura *et al.* 2007) expectation, and are thus robustly stable to a class of perturbations that satisfy a set of physical constraints. This is shown by using a variational energy principle (Lundquist 1951; Bernstein *et al.* 1958; Kruskal & Kulsrud 1958). We find that, if certain constraints are satisfied, electron plasmas in stellarators are robustly stable to macroscopic, low-frequency instabilities.

2. Equilibrium

The macroscopic fluid equilibrium whose stability properties we wish to assess arises from the equilibrium condition for an electron plasma in a magnetic field,

$$m_e n_e \boldsymbol{v}_e \boldsymbol{\cdot} \nabla \boldsymbol{v}_e + \nabla(n_e T_e) - e n_e \nabla\phi = -e n_e \boldsymbol{v}_e \times \boldsymbol{B}, \tag{2.1}$$

where n_e is the number density of the electron plasma, T_e is its temperature and $\boldsymbol{v}_e$ is its mean-flow velocity. The non-negligible equilibrium electrostatic potential ϕ is a result of the non-neutrality of the single species plasma and is given by Poisson's equation $\varepsilon_0 \nabla^2\phi = e n_e$. This potential generates an electrostatically repulsive force on the plasma $e n_e \nabla\phi$ which, along with the pressure gradient force $\nabla(n_e T_e)$ and the inertial force $m_e n_e \boldsymbol{v}_e \boldsymbol{\cdot} \nabla \boldsymbol{v}_e$, must be balanced by the Lorentz force of the electron flow in the confining magnetic field $\boldsymbol{B}$, given by the right-hand side of (2.1) (Pedersen & Boozer 2002).

In a stellarator, the confining magnetic field has a toroidal topology and can be represented in contravariant form as $\boldsymbol{B} = \nabla\psi \times \nabla\theta + \iota(\psi)\nabla\varphi \times \nabla\psi$, where $2\pi\psi$ is the toroidal magnetic flux that labels the surfaces of constant toroidal magnetic flux (flux surfaces), θ and φ are 2π-periodic poloidal and toroidal angles, respectively, and $\iota(\psi)$ is the rotational transform which is defined as

$$\iota(\psi) = \frac{\mathrm{d}\theta}{\mathrm{d}\varphi}. \tag{2.2}$$

The stellarator is set apart from other means of confining a single species plasma, like the magnetic dipole or purely toroidal field traps, by its finite rotational transform, which defines the pitch of magnetic field lines on each of the nested flux surfaces ψ. If ι is irrational (or is a sufficiently high-order rational), then the magnetic field lines touch each part of these toroidal surfaces evenly (Helander 2014).

To confine an electron plasma in such a magnetic field, the electron density must satisfy $n_e \ll n_B$, where $n_B = \varepsilon_0 B^2/(2m_e)$ is the Brillouin density (Boozer 2005). In this limit, the inertia of the plasma is negligible and the equilibrium condition reduces to

$$\nabla(n_e T_e) - e n_e \nabla\phi = -e n_e \boldsymbol{v}_e \times \boldsymbol{B}. \tag{2.3}$$

We now consider the force balance along magnetic field lines by taking the scalar product of (2.3) with $\boldsymbol{B}$. In a stellarator with well defined flux surfaces, this amounts

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to considering the force balance over the entire surface ψ. If it is assumed that the flux surfaces are isotherms with $T_e = T_e(\psi)$[1], we find that the quantity $T_e(\psi)\ln n_e - e\phi = H(\psi)$ is constant on each flux surface. Solving this for the electron density yields the Boltzmann relation $n_e = N(\psi)\exp(e\phi/T_e(\psi))$. Inserting this expression for the density into Poisson's equation yields the nonlinear Poisson–Boltzmann equation,

$$\nabla^2\phi = N(\psi)\exp\left(\frac{e\phi}{T_e(\psi)}\right), \tag{2.4}$$

the solution to which gives the equilibrium electrostatic potential $\phi(\psi,\theta,\varphi)$ for a given $N(\psi)$ and temperature $T_e(\psi)$. The function $N(\psi)$, which has remained arbitrary until this point, is related to the flux-surface average of the density by $\langle n_e\rangle = N(\psi)\langle\exp(e\phi/T_e(\psi))\rangle$, with the flux-surface average defined as (Helander 2014)

$$\langle\ldots\rangle = \frac{\displaystyle\int(\ldots)\sqrt{g}\,\mathrm{d}\theta\,\mathrm{d}\varphi}{\displaystyle\int\sqrt{g}\,\mathrm{d}\theta\,\mathrm{d}\varphi} = \frac{1}{V'(\psi)}\int(\ldots)\sqrt{g}\,\mathrm{d}\theta\,\mathrm{d}\varphi, \tag{2.5}$$

where the Jacobian is given by

$$\frac{1}{\sqrt{g}} = (\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\theta)\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi, \tag{2.6}$$

and $V(\psi)$ denotes the volume enclosed by the flux surface ψ.

The nonlinear partial differential equation (2.4) is the central relation for the equilibria of pure electron plasmas in stellarators. It has been demonstrated numerically that the equilibria which satisfy solutions to this equation behave like minimum-energy states, in that the electron plasma density in equilibrium is attracted to the region of most positive potential in the device, thus minimising the total electrostatic energy of the system (Lefrancois *et al.* 2005). This has also been shown to be the case experimentally, as the electron density has been found to increase at the point on each flux surface which is closest to the grounded outer wall of the device (Himura *et al.* 2007).

The stability of this equilibrium has been treated in previous work by Boozer (2004), where similar stability properties were derived. However, in his analysis, the thermal energy of the plasma was not considered, despite the inclusion of a finite electron temperature T_e in the plasma entropy. Without the inclusion of a finite plasma pressure in the variational principle, the equilibrium, which is a balance of a pressure gradient and electrostatic repulsion, cannot be found to be the minimum-energy state. Here, we conclusively show that the equilibria represented by (2.4) in stellarators minimise the physical energy of the system.

3. Stability theorem

We consider a plasma confined on the magnetic surfaces of a stellarator occupying a confinement volume V bounded by a conducting wall ∂V, with an outward-pointing radial

[1]We will later show in our variational analysis that this assumption arises directly from an argument regarding the flux-surface average of the entropy.

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normal vector $\nabla\psi/|\nabla\psi|$. The total physical energy of the electron plasma is given by

$$W = \int_V \left(\frac{1}{2} m_e n_e v_e^2 + \frac{p}{\gamma - 1} + \frac{|\boldsymbol{B}|^2}{2\mu_0} + \varepsilon_0 \frac{|\boldsymbol{E}|^2}{2} \right) \mathrm{d}V, \tag{3.1}$$

where $v_e = |\boldsymbol{v}_e|$ is the magnitude of the flow velocity of the electron fluid, $p = n_e T_e$ is the thermal pressure, γ is the adiabatic index, and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, respectively. We can compare the relative sizes of the contributions to the energy by estimating the equilibrium electric field as $E \sim a e n_e/\varepsilon_0$, where a is the minor radius. We find that the sizes of the electrostatic and thermal energies scale as

$$\frac{\varepsilon_0 |\boldsymbol{E}|^2}{n_e T_e} \sim \frac{a^2}{\lambda_D^2}. \tag{3.2}$$

In order for a collection of electrons to qualify as a plasma, the parameter a/λ_D needs to exceed unity, so that the pressure makes a smaller contribution to the energy than the electric field, but we shall nevertheless treat these energies as comparable, i.e. of the same order in our expansion procedure. We can similarly estimate the ratio of the kinetic energy of the plasma flow to the pressure by using the fact that the electron flow is largely due to the $\boldsymbol{E} \times \boldsymbol{B}$-drift, such that $v_e \sim E/B$. This yields a scaling

$$\frac{m_e n_e v_e^2}{n_e T_e} \sim \frac{1}{2} \frac{a^2}{\lambda_D^2} \frac{\rho_e^2}{\lambda_D^2} = \frac{a^2}{\lambda_D^2} \frac{n_e}{2 n_B}, \tag{3.3}$$

where $\rho_e = \sqrt{(2T_e/m_e)}/\Omega_e$ is the electron gyroradius with $\Omega_e = |e|B/m_e$, and n_B denotes the Brillouin density. It has been shown that in order to confine an electron plasma in a stellarator, it is required that the plasma density be far below the Brillouin density, $n_e/n_B \ll 1$ (Boozer 2005). We can see that this requirement is equivalent to assuming that the Debye length is much larger than the electron gyroradius, such that $\rho_e/\lambda_D \ll 1$. As a result of this necessary criterion for confinement, the inertial contribution to the energy is much smaller than the pressure contribution and can be safely neglected.

Furthermore, while the magnetic energy, $|\boldsymbol{B}|^2/(2\mu_0)$, may be large, it does not have a meaningful impact on the variation of the energy under a perturbation of the plasma, given that, by the estimate $\delta B \sim a \mu_0 e n_e v_e$, we find

$$\frac{\delta B}{B} \sim \left(\frac{v_{Te}}{c} \right)^2 \frac{a^2}{\lambda_D^2} \frac{n_e}{4 n_B}, \tag{3.4}$$

where $v_{Te} = \sqrt{2T_e/m_e}$ is the thermal velocity and c is the speed of light in vacuum. Once more the requirement that $n_e/n_B \ll 1$ makes this ratio small unless a/λ_D is extremely large, and therefore we can safely assume that $\boldsymbol{B}$ is entirely generated by the external coils of the confinement device and can be treated as constant in our analysis.

Thus, in the limit $n_e/n_B \ll 1$, the only terms of relevance to the analysis of the stability of the system are the thermal and electrostatic energies, allowing us to reduce (3.1) to

$$W = \int_V \left(\frac{p}{\gamma - 1} + \varepsilon_0 \frac{|\boldsymbol{E}|^2}{2} \right) \mathrm{d}V. \tag{3.5}$$

This energy can be thought of as a functional, $W = W[n_e, T_e, \boldsymbol{E}]$, such that for any given functions $(n_e, T_e, \boldsymbol{E})$, W gives a real number as an output, the energy of the plasma.

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The treatment of this variational energy principle here is twofold. Firstly, it will be shown that equilibria of the equations of motion correspond to unique extrema of the energy functional W, such that the first-order variation δW vanishes at equilibrium. Secondly, the nature of such extrema will be investigated, and it will be determined whether they are maxima, minima or 'saddle points' in the energy space.

The nature of the extremum, which is determined by the sign of δW to second order, establishes the stability properties of the equilibrium, with positive definiteness indicating a minimum, negative definiteness indicating maximum and an indeterminate sign indicating a saddle point. The stability of an equilibrium, where δW vanishes to first order, is determined by the nature of the corresponding extremum, where so-called 'formal' stability is implied by a unique maximum or minimum (Holm *et al.* 1985). Because W is conserved by an electron plasma under our assumption of $n_e \ll n_B$, an equilibrium which represents a unique maximum or minimum defined by the functions $(n_1, T_1, \boldsymbol{E}_1)$ cannot access any other state $(n_2, T_2, \boldsymbol{E}_2)$ without a change in energy. Of course, in a physical system, some degree of energy dissipation will be present, so in this sense an energy minimum is more desirable as it is additionally resilient in the face of energy dissipation. If the equilibrium is a degenerate extremum or a saddle point in energy space, stability is not guaranteed.

If one naïvely extremises the energy W in an unconstrained way, an uninteresting $n_e = T_e = \boldsymbol{E} = 0$ solution will be found. Thus, an extremum of (3.5) must be found under a set of physical constraints. Our analysis aims to expand upon the treatment of Boozer (2004), by considering the same physical constraints with the addition that here, the temperature is not only allowed to contribute to the entropy of the plasma, but also its thermal energy. We now consider the following constraints.

(i) The electron plasma must obey Gauss' law of electrostatics, such that $\boldsymbol{E}$ satisfies $\nabla \cdot \boldsymbol{E} = -en_e/\varepsilon_0$ everywhere in V.

(ii) The electrostatic potential ϕ vanishes on a surface ∂V surrounding the plasma. This surface thus corresponds to a grounded, electrically conducting wall.

(iii) The number of electrons on each flux surface is conserved by the magnetic confinement, and thus the electron flow cannot cross flux surfaces. This condition implies that $\langle n_e \rangle = G(\psi)$, where $G(\psi)$ is a fixed function of ψ. Since the total charge enclosed by the plasma (or any region bounded by a flux surface within the plasma) is conserved, any variation of the electric field $\delta \boldsymbol{E}$ must then satisfy

$$\int_{\partial V} \delta \boldsymbol{E} \cdot \nabla \psi \frac{\mathrm{d}S}{|\nabla \psi|} = 0, \tag{3.6}$$

according to Gauss's law.

(iv) The average entropy of the plasma on each flux surface

$$S(\psi) = \frac{1}{\gamma - 1} \langle n_e \ln (T_e / n_e^{\gamma - 1}) \rangle, \tag{3.7}$$

must be conserved or increase on each flux surface ψ. This constraint is motivated by the existence of magnetic surfaces which allow for rapid heat conduction along surfaces, but poor conduction between any two flux surfaces.

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These constraints can be included when seeking an extremum of (3.5) through the method of Lagrange multipliers. This gives the constrained energy,

$$\bar{W} = \int_V \left[\frac{\varepsilon_0}{2}|\boldsymbol{E}|^2 + \frac{n_e T_e}{\gamma - 1} - \eta(\boldsymbol{r})(\varepsilon_0 \nabla \cdot \boldsymbol{E} + e n_e) \right.$$
$$\left. - \lambda(\psi)(n_e - G(\psi)) - \mu(\psi)\left(\frac{1}{\gamma - 1} n_e \ln (T_e/n_e^{\gamma-1}) - S(\psi) \right) \right] \mathrm{d}V. \tag{3.8}$$

Here, the Lagrange multipliers η, λ and μ guarantee that any extremum of the above expression satisfies our constraints. For example varying λ gives

$$\delta_\lambda \bar{W} = \int \mathrm{d}\psi \, \delta\lambda(\psi)(\langle n_e \rangle - G(\psi)), \tag{3.9}$$

such that, at an extremum where $\delta\bar{W}$ vanishes to first order, the constraint $\langle n_e \rangle = G(\psi)$ must be satisfied. Note that we have stipulated that $\lambda = \lambda(\psi)$ and $\mu = \mu(\psi)$ are functions of ψ alone, indicating that the corresponding constraints are only enforced on average on each flux surface, and not at every point in the plasma, as is the case with Gauss' law, where $\eta = \eta(\boldsymbol{r})$ has been used to ensure that it is satisfied at every point in space. Now an equilibrium is sought for which $\delta\bar{W}$ vanishes to first order under the individual variation of $\boldsymbol{E}$, T_e and n_e. By computing $\delta_{\boldsymbol{E}}\bar{W}$ to first order we find

$$\delta_{\boldsymbol{E}} \bar{W} = \int_V (\varepsilon_0 \boldsymbol{E} \cdot \delta\boldsymbol{E} - \eta(\boldsymbol{r})(\varepsilon_0 \nabla \cdot \delta\boldsymbol{E})) \, \mathrm{d}V. \tag{3.10}$$

Integrating by parts and equating to zero gives

$$\int_V (\boldsymbol{E} + \nabla\eta) \cdot \delta\boldsymbol{E} \, \mathrm{d}V - \int_{\partial V} \eta(\boldsymbol{r}) \delta\boldsymbol{E} \cdot \nabla\psi \frac{\mathrm{d}S}{|\nabla\psi|} = 0, \tag{3.11}$$

where the divergence theorem has been used to arrive at the surface integral over the boundary of V. It is now required that both integrals vanish for all choices of the function $\delta\boldsymbol{E}$. The volume integral will do so if $\boldsymbol{E} = -\nabla\eta$, implying that, at the extremum, the Lagrange multiplier $\eta(\boldsymbol{r})$ is equal to the electrostatic potential $\phi(\boldsymbol{r})$ within an additive constant. The surface integral over ∂V then also vanishes, which implies $\phi = \text{const.}$ on the boundary, in accordance with the second constraint on our list above.

The first-order variation $\delta_{T_e}\bar{W}$ with respect to the temperature is found to be

$$\delta_{T_e} \bar{W} = \int_V \left[n_e \delta T_e \left(1 - \frac{\mu(\psi)}{T_e} \right) \right] \mathrm{d}V = 0, \tag{3.12}$$

and implies that $T_e = T_e(\psi)$ depends on ψ alone, and that $\mu(\psi) = T_e(\psi)$ at equilibrium.

Finally, the first-order variation with respect to the density $\delta_{n_e}\bar{W}$ then gives

$$\delta_{n_e} \bar{W} = \int_V \delta n_e \left[\frac{T_e(\psi)}{\gamma - 1} - e\phi - \lambda(\psi) \right.$$
$$\left. - \frac{1}{\gamma - 1} T_e(\psi) \ln T_e(\psi) + T_e(\psi) \ln n_e + T_e(\psi) \right] \mathrm{d}V = 0. \tag{3.13}$$

This admits solutions of the form

$$T_e(\psi) \ln(n_e) - e\phi = H(\psi), \tag{3.14}$$

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where the function $H(\psi)$ is given by

$$H(\psi) = \lambda(\psi) - \frac{1}{\gamma - 1}T_e(\psi) + \frac{T_e(\psi)}{\gamma - 1}\ln(T_e(\psi)) - T_e(\psi). \tag{3.15}$$

Here, $H(\psi)$ amounts to some arbitrary flux function, whose arbitrariness is given by the fact that at equilibrium the Lagrange multiplier $\lambda(\psi)$ is unspecified. From this, the equilibrium condition is found to be

$$n_e = N(\psi)\exp\left(e\phi/T_e(\psi)\right). \tag{3.16}$$

Thus, it is clear that the finite-temperature equilibria of interest, given by (2.4), extremise $\bar{W}$. With each equilibrium being uniquely specified by the choice of $N(\psi)$ and $T_e(\psi)$, the uniqueness of the equilibrium is guaranteed thanks to the uniqueness theorem of the nonlinear Poisson–Boltzmann equation on magnetic surfaces (Durand de Gevigney 2011). Now, all that remains is to determine if such an extremum is a maximum, minimum or saddle point in energy space.

In order to facilitate a more direct interpretation of the role of our constraints on the stability of the system, in computing the second variation we abandon the method of Lagrange multipliers in favour of directly enforcing the constraints when necessary. We now consider the perturbed energy on each flux surface ψ of the equilibrium to second order,

$$\delta U(\psi) = \left\langle \varepsilon_0 \nabla\phi \cdot \nabla\delta\phi + \varepsilon_0 \frac{|\nabla\delta\phi|^2}{2} + \frac{1}{\gamma - 1}(n_e\delta T_e + T_e\delta n_e + \delta T_e\delta n_e)\right\rangle, \tag{3.17}$$

where we have used $\boldsymbol{E} = -\nabla\phi$. Under such a general perturbation about the equilibrium, the corresponding change in the flux-surface average of the entropy to second order is

$$\begin{aligned}\delta S(\psi) = \frac{1}{\gamma - 1}\Bigg\langle &(\ln T_e - (\gamma - 1)\ln n_e - (\gamma - 1))\delta n_e + \frac{n_e}{T_e}\delta T_e \\ &- \frac{1}{2}\left(\frac{n_e}{T_e^2}\delta T_e^2 + \frac{(\gamma - 1)}{n_e}\delta n_e^2\right) + \frac{1}{T_e}\delta n_e \delta T_e\Bigg\rangle.\end{aligned} \tag{3.18}$$

By enforcing the constraint that $\langle n_e\rangle = G(\psi)$, such that $\langle \delta n_e\rangle = 0$, and exploiting the fact that the equilibrium temperature is of the form $T_e = T_e(\psi)$, the changes in the energy and entropy of each flux surface, respectively, are found to reduce to

$$\delta U(\psi) = \left\langle \varepsilon_0 \nabla\phi \cdot \nabla\delta\phi + \varepsilon_0 \frac{|\nabla\delta\phi|^2}{2} + \frac{1}{\gamma - 1}(n_e\delta T_e + \delta T_e\delta n_e)\right\rangle, \tag{3.19}$$

and

$$\begin{aligned}\delta S(\psi) = \Bigg\langle &-\ln(n_e)\delta n_e + \frac{n_e\delta T_e}{T_e(\gamma - 1)} \\ &- \frac{1}{2}\left(\frac{n_e\delta T_e^2}{T_e^2(\gamma - 1)} + \frac{\delta n_e^2}{n_e}\right) + \frac{\delta n_e \delta T_e}{T_e(\gamma - 1)}\Bigg\rangle.\end{aligned} \tag{3.20}$$

We now consider the linear combination

$$\delta F(\psi) = \delta U(\psi) - T_e(\psi)\delta S(\psi), \tag{3.21}$$

where the choice of the factor $T_e(\psi)$ is motivated by the optimal Lagrange multiplier for enforcing the entropy constraint identified above. The function $\delta F(\psi)$ can be interpreted

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as the change in the Helmholtz free energy of the electron plasma on each flux surface. If we multiply $\delta F(\psi)$ by $V'(\psi)$ and integrate over all the flux surfaces to arrive at a volume integral, we obtain

$$\int \mathrm{d}\psi V'(\psi)\delta F(\psi) = \int \mathrm{d}\psi V'(\psi)\left\langle \varepsilon_0 \nabla\phi \cdot \nabla\delta\phi + \varepsilon_0 \frac{|\nabla\delta\phi|^2}{2} + T_e(\psi)\ln(n_e)\delta n_e + \frac{T_e(\psi)}{\gamma-1}\left(\frac{n_e}{2T_e^2}\delta T_e^2 + \frac{(\gamma-1)}{2n_e}\delta n_e^2\right)\right\rangle. \tag{3.22}$$

The first term in this expression may be rewritten by integrating by parts, noting that ϕ vanishes on the boundary, and using Poisson's equation $\varepsilon_0\nabla^2\delta\phi = e\delta n_e$. Using the equilibrium condition $n_e = N(\psi)\exp(e\phi/T_e(\psi))$ and once again enforcing $\langle\delta n_e\rangle = 0$, we find

$$\int_V \delta F \mathrm{d}V = \int_V \left(\varepsilon_0\frac{|\nabla\delta\phi|^2}{2} + \frac{n_e}{2(\gamma-1)T_e}\delta T_e^2 + \frac{T_e}{2n_e}\delta n_e^2\right)\mathrm{d}V, \tag{3.23}$$

which is clearly positive definite. Hence, the change in energy,

$$\delta W = \int_V (\delta F + T_e\delta S)\,\mathrm{d}V, \tag{3.24}$$

is positive definite if $\delta S \geq 0$. Thus, the electron plasma equilibrium in a stellarator given by (2.4) minimises the energy W under the constraints we have imposed.

This argument can also be considered from the perspective of equilibrium as a state of maximum entropy, where in this case the average energy on each flux surface is held fixed. Both approaches yield the same result for the equilibrium and stability (an example of the entropy maximisation approach can be seen in Malmberg & O'Neil (1977)). This can be most readily seen in (3.21), where one could instead consider $\delta\bar{S} = -\delta F/T_e(\psi)$ as the entropy change constrained by holding the flux-surface average of the energy fixed. The volume-integrated constrained entropy change,

$$\int_V \delta\bar{S}\mathrm{d}V = -\int_V \frac{\delta F}{T_e(\psi)}\mathrm{d}V = \int_V \left(\delta S - \frac{\delta U}{T_e(\psi)}\right)\mathrm{d}V, \tag{3.25}$$

is then clearly negative definite at equilibrium from (3.23) such that the entropy is maximised.

4. Discussion and conclusions

We have now demonstrated that the equilibria of a pure electron plasma, of arbitrary temperature, confined by toroidal magnetic surfaces, are minimum-energy states, and so, are formally stable in agreement with the analysis of Boozer (2004) and the numerical studies of Pedersen (2003) and Lefrancois *et al.* (2005), under the definition of formal stability described by Holm *et al.* (1985). The stability of the equilibria is guaranteed as long as perturbations do not change the flux-surface average of the electron density and entropy. Note that this result is in contrast to a Penning–Malmberg trap and a purely toroidal field trap, which are stable maximum-energy states (O'Neil & Smith 1992, 1994).

We now consider the limitations of this analysis. We have assumed that the flux-surface averages of $\langle n_e\rangle$ and $\langle n_e \ln(T_e/n_e^{\gamma-1})\rangle$ are conserved quantities in this system. In order to understand the implication of the constraints of $\delta S(\psi) = 0$ and $\delta G(\psi) = 0$, we

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consider the first-order variation of the entropy

$$\delta S(\psi) = \frac{1}{\gamma - 1}\left\langle \delta n_e \ln T_e + \frac{n_e \delta T_e}{T_e} - (\gamma - 1)\delta n_e \ln n_e - (\gamma - 1)\delta n_e \right\rangle. \tag{4.1}$$

If we enforce our equilibrium conditions $T_e = T_e(\psi)$ and $n_e = N(\psi)\exp(e\phi/T_e)$, and add the constraint $\langle \delta n_e \rangle = 0$, this expression reduces to

$$\delta S(\psi) = \frac{1}{T_e(\psi)}\left\langle \frac{n_e \delta T_e}{\gamma - 1} - e\phi\delta n_e \right\rangle = 0. \tag{4.2}$$

Therefore, if these constraints are to be fulfilled, for the thermal energy to increase on a flux surface due to a temperature change δT_e, the electrostatic energy change due to the movement of electrons in the equilibrium electrostatic potential $-e\phi\delta n_e$ must act to decrease the energy on that flux surface in compensation.

The conservation of these averaged quantities is only possible if the communication time between two points on a flux surface required to establish the flux-surface average is faster than the characteristic time scale of the evolution of a perturbation. We can make a conservative estimate for the communication time as the time required for a thermal electron with velocity $v_{Te} = \sqrt{2T_e/m_e}$ to traverse the length of a perturbation $1/k_\parallel$ along the magnetic field. This time must be shorter than the time scale of a typical macroscopic instability, which in electron plasmas is the diocotron instability (Davidson 2001). The diocotron instability is common to many non-neutral plasmas, and arises as a result of shear flow in the plasma due to variation of the $\boldsymbol{E} \times \boldsymbol{B}$ velocity within the plasma volume. If the shear is sufficiently strong, the diocotron modes can become unstable in an analogous way to the Kelvin–Helmholtz instability. We estimate the diocotron frequency as, $\omega_D \sim E/(aB)$ and require that $\omega_D/(v_{Te}k_\parallel) \ll 1$, such that our imposed constraints are valid. If the rotational transform ι is small, the poloidal connection length is the longest scale to equilibrate by parallel dynamics, so we can take $k_\parallel = \iota/R$, where R is the major radius. This gives a lower bound for the required rotational transform of

$$\iota \gg \frac{1}{2}\frac{a}{\lambda_D}A\sqrt{\frac{n_e}{n_B}} = \frac{1}{2}\frac{a}{\lambda_D}A\frac{\rho_e}{\lambda_D}, \tag{4.3}$$

where $A = R/a$, the aspect ratio of the device. In the opposite limit where ι is very large, the toroidal connection length becomes more relevant, and by taking $k_\parallel = 1/(\iota a)$, we find that

$$\iota \ll 2\frac{\lambda_D}{a}\sqrt{\frac{n_B}{n_e}} = 2\frac{\lambda_D}{a}\frac{\lambda_D}{\rho_e}. \tag{4.4}$$

These conservative estimates are easily fulfilled if the temperature is finite as n_e/n_B is necessarily small to ensure confinement of the plasma.

However, these estimates are likely much more conservative than required and actually become impossible to fulfil in the limit $T_e \to 0$. This is ameliorated by the results of Kondoh, Tatsuno & Yoshida (2001), which indicate that the parallel communication time along magnetic field lines if $k_\parallel \neq 0$ is actually the plasma frequency $\omega_{pe} = \sqrt{e^2 n_e/(m_e \varepsilon_0)}$. For this time scale to be faster than the diocotron frequency, the only condition is that $n_e/n_B \ll 1$. If the plasma frequency is indeed the relevant parallel communication time, then the only requirement for this stability theorem to be valid in a realistic system is $k_\parallel \neq 0$, which amounts to requiring that ι be finite. Moreover, $\iota(\psi)$ must be irrational otherwise the diocotron modes can become destabilised on rational surfaces (Hirota *et al.* 2002).

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