Nagoya Math. J., **229** (2018), 52–98
DOI 10.1017/nmj.2016.62

SPECTRAL ZETA FUNCTIONS FOR THE QUANTUM RABI MODELS

SHINGO SUGIYAMA

Abstract. We introduce the Hurwitz-type spectral zeta functions for the quantum Rabi models, and give their meromorphic continuation to the whole complex plane with only one simple pole at $s = 1$. As an application, we give the Weyl law for the quantum Rabi models. As a byproduct, we also give a rationality of Rabi–Bernoulli polynomials introduced in this paper.

§1. Introduction

The spectrum of a Hamiltonian has been studied in both physics and mathematics, especially in spectral theory. One of the methods to do so is to make use of spectral zeta functions. For a $\mathbb{C}$-Hilbert space V and a densely defined linear operator $A : V \to V$, the multiset of all eigenvalues of A in $\mathbb{C}$ is denoted by $\mathrm{Spec}(A)$. If $\mathrm{Spec}(A)$ is discrete, the spectral zeta function of A is defined as

$$\zeta_A(s) = \sum_{\lambda \in \mathrm{Spec}(A)} \frac{1}{\lambda^s}$$

for $s \in \mathbb{C}$ if it makes sense. For example, it is well known that, for the harmonic oscillator $h = \frac{1}{2}(-\partial_x^2 + x^2)$ densely defined in $L^2(\mathbb{R})$, the set $\mathrm{Spec}(h)$ is given by $\{n + 1/2 \mid n \in \mathbb{Z}_{\geqslant 0}\}$ with multiplicity 1. From this, the spectral zeta function of h is of the form

$$\zeta_h(s) = \sum_{n=0}^{\infty} \frac{1}{(n + 1/2)^s} = (2^s - 1)\zeta(s),$$

where $\zeta(s)$ denotes the Riemann zeta function. This defining series is absolutely convergent for $\mathrm{Re}(s) > 1$, and has a meromorphic continuation to the whole s-plane. Furthermore, the only pole $s = 1$ of $\zeta_h(s)$ is simple, and $\zeta_h(-2n) = 0$ holds for all nonnegative integers n. The points

Received June 13, 2016. Revised November 5, 2016. Accepted November 5, 2016.
2010 Mathematics subject classification. Primary 11M41; Secondary 81Q10, 46N50.
This work was supported by CREST, JST.

$0, -2, -4, -6, \ldots$ are called trivial zeros of $\zeta_h(s)$. The spectral zeta function $\zeta_A(s)$ for an operator A encodes information on $\mathrm{Spec}(A)$ in its analytic properties. For example, as seen in applications to the Weyl law for A, some pole of $\zeta_A(s)$ with real part maximal is related to an asymptotic behavior of the spectral counting function of A:

$$N_A(T) = \#\{\lambda \in \mathrm{Spec}(A) \mid \lambda \leqslant T\}, \quad T > 0$$

(cf. [1], [18, Section 6.4] and [29, Section 14]). In a quite general setting, Robert [27] studied spectral zeta functions for pseudodifferential operators in $\mathbb{R}^n$, and later his result was generalized by Aramaki [1] to some infinite-dimensional situations. As a remarkable example of spectral zeta functions of matrix-valued pseudodifferential operators, we should mention that Ichinose and Wakayama [8] investigated very quantitatively the spectral zeta function $\zeta_{Q_{(\alpha,\beta)}}(s)$ of the noncommutative harmonic oscillator

$$Q_{(\alpha,\beta)} = \frac{-\partial_x^2 + x^2}{2} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} + (x\partial_x + 1/2) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

densely defined in $L^2(\mathbb{R}) \otimes_{\mathbb{C}} \mathbb{C}^2$ with $\alpha > 0$ and $\beta > 0$ such that $\alpha\beta > 1$. In [8], they gave a meromorphic continuation of $\zeta_{Q_{(\alpha,\beta)}}(s)$ to the whole s-plane and established a formula of $\zeta_{Q_{(\alpha,\beta)}}(s)$, which includes information on its poles and trivial zeros. By the formula, $\zeta_{Q_{(\alpha,\beta)}}(s)$ has the only one pole $s = 1$, which is simple, and the following asymptotic formula holds:

$$N_{Q_{(\alpha,\beta)}}(T) \sim \mathrm{Res}_{s=1}\, \zeta_{Q_{(\alpha,\beta)}}(s) T = \frac{\alpha + \beta}{\sqrt{\alpha\beta(\alpha\beta - 1)}} T, \quad (T \to \infty).$$

Later, the formula in [8, Main theorem] was extended by Parmeggiani [18] to the case where $Q_{(\alpha,\beta)}$ is replaced with a general globally elliptic $N \times N$ self-adjoint regular partial differential system with polynomial coefficients (PPDSs) in $\mathbb{R}^n$ of order 2 (cf. [18, Theorem 7.2.1]). For more details for $Q_{(\alpha,\beta)}$, see [19].

In this paper, we explore fine analysis of the Hurwitz-type spectral zeta function for the *quantum Rabi model* with Hamiltonian $H = H_{\mathrm{Rabi}}$, and give a meromorphic continuation to $\mathbb{C}$ and an asymptotic behavior of spectral counting function $N_H(T)$ of H. Here, the quantum Rabi model is a model describing an interaction of light and matter of a two-level atom coupled to a single quantized photon of the electromagnetic field (cf. [3]). The Hamiltonian for the quantum Rabi model, which is called the Rabi

Hamiltonian, is given by

$$H = H_{\mathrm{Rabi}} = \hbar\omega a^\dagger a + \Delta\sigma_z + \hbar g\sigma_x(a^\dagger + a)$$

densely defined in $L^2(\mathbb{R}) \otimes_{\mathbb{C}} \mathbb{C}^2$. Here $\hbar$ is the Dirac constant, a and $a^\dagger$ are the annihilation and creation operators for a Bosonic mode of frequency $\omega > 0$, respectively, the symbols σ_x, σ_y, and σ_z are the Pauli matrices for the two-level system, $2\Delta > 0$ is the difference of the two-level energies, and $g > 0$ is the coupling constant for atom and photon. In [23] and [24], Rabi introduced originally a semiclassical model, and Jaynes and Cummings [9] fully quantized the Rabi model as H. It is known that every $\lambda \in \mathrm{Spec}(H)$ is real and one of the three forms:

(1) $\lambda = x_n^{\pm} - g^2$ with multiplicity 1 (nondegenerate), where $\{x_n^+\}_{n=1}^{\infty}$ and $\{x_n^-\}_{n=1}^{\infty}$, which are contained in $\mathbb{C} - \mathbb{Z}_{\geqslant 0}$, are the zeros of $G_+(x)$ and of $G_-(x)$, respectively;
(2) $\lambda = n - g^2$ for some $n \in \mathbb{Z}_{\geqslant 0}$ with multiplicity 1 (nondegenerate);
(3) $\lambda = n - g^2$ for some $n \in \mathbb{Z}_{\geqslant 1}$ with multiplicity 2 (doubly degenerate);

(cf. [2, 3, 16, 17]). Here $G_{\pm}(x)$ is a meromorphic function with at most simple poles at all $n \in \mathbb{Z}_{\geqslant 0}$, which Braak [2] gave as power series satisfying $\mathrm{Spec}(H) - \{n - g^2 \mid n \in \mathbb{Z}_{\geqslant 0}\} = \{y - g^2 \mid y \in \mathbb{R},\ G_+(y)G_-(y) = 0\}$ by explicitly describing recurrence equations for the coefficients of $G_{\pm}(x)$. The eigenvalues in case (1) are called the regular spectrum, and those in cases (2) and (3) are called the exceptional spectrum. We should mention that Parmeggiani and Wakayama in [20] and [21] described a part of the spectrum of the noncommutative harmonic oscillators $Q_{(\alpha,\beta)}$ with $\alpha\beta > 1$, which is similar to Braak's work recalled as above.

Several mathematicians have studied the Rabi Hamiltonian H and contributed theoretically to the field of quantum optics. As recent works, Hirokawa and Hiroshima [7] proved that the ground state energy for H is nondegenerate (i.e., the smallest eigenvalue of H has multiplicity 1), and that the ground state energy for H has no crossing for all g and Δ. By a representation theoretic approach, Wakayama and Yamasaki [33] captured the doubly degenerate exceptional spectrum of H via finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{R})$. Furthermore, Wakayama [32] pioneered a new relation between the noncommutative harmonic oscillator $Q_{(\alpha,\beta)}$ and the Rabi Hamiltonian H, through a confluence process by Heun's picture. Nevertheless it seems difficult to capture finer properties of the spectrum of H.

1.1 Main results

In what follows, we consider the Hurwitz-type spectral zeta function of H

$$\zeta_H(s;\tau) := \zeta_{H+\tau I}(s) = \sum_{\lambda \in \mathrm{Spec}(H)} \frac{1}{(\lambda+\tau)^s}$$

for $\tau \in \mathbb{C}$. Throughout this paper, we normalize H so that $\hbar = \omega = 1$ without loss of generality and both g and Δ are supposed to be arbitrary nonnegative real numbers. For the mathematical definition of H, see Section 2.2. Then, the defining series converges absolutely for $\mathrm{Re}(s) > 1$ (see Proposition 3.1). We give its meromorphic continuation as follows by using the method of the parametrix of the heat equation investigated in [8].

THEOREM 1.1. *For any $g \geqslant 0$, $\Delta \geqslant 0$, and $\tau \in \mathbb{R}$ such that $\tau > g^2 + \Delta$, we have the following.*

(1) *There exists an explicitly computable sequence $\{C_{H,\tau}(k)\}_{k \in \mathbb{Z}_{\geqslant 0}}$ of complex numbers such that, for any $n \geqslant 2$,*

$$\zeta_H(s;\tau) = \frac{1}{\Gamma(s)} \left\{ \frac{2}{s-1} + \sum_{k=1}^{\infty} \frac{2g^{2k}}{k!} \frac{1}{s+k-1} + \sum_{k=0}^{n-2} \frac{C_{H,\tau}(k)}{s+k} + F_{H,n}(s;\tau) \right\}, \quad \mathrm{Re}(s) > 1,$$

where $F_{H,n}(s;\tau)$ is a holomorphic function on $\mathrm{Re}(s) > -n/2$. In particular, $\zeta_H(s;\tau)$ has a meromorphic continuation to $\mathbb{C}$. Moreover, it is holomorphic on $\mathbb{C}$ except for the only one simple pole $s=1$ with the residue $\mathrm{Res}_{s=1}\, \zeta_H(s;\tau) = 2$.

(2) *We have $C_{H,\tau}(k) \in \mathbb{Q}[g^2, \Delta^2, \tau]$ for any $k \in \mathbb{Z}_{\geqslant 0}$.*

Theorem 1.1(1) solves Wakayama's conjecture on a meromorphic continuation of $\zeta_H(s;\tau)$ (see [31, Section 3]), and is regarded as an analogue of [18, Theorem 7.2.1] and [8, Main theorem]. We remark that [18, Theorem 7.1.1], which is a special case of [27], cannot be applied to the Rabi Hamiltonian H since H is not classical in the sense of [18, Definition 3.2.19] although the noncommutative harmonic oscillator $Q_{(\alpha,\beta)}$ is classical. By a general result [27, Théorèmes (6.3) et (6.4)] by Robert, we can obtain a meromorphicity of $\zeta_H(s;\tau)$ and its poles are contained in $\{1\} \cup \{1/2 - j \mid j \in \mathbb{Z}_{\geqslant 0}\}$. However, Robert's method by the parametrix of the resolvent does not seem to work well in order to eliminate $s = 1/2 - j$ with $j \in \mathbb{Z}_{\geqslant 0}$ from the possible poles.

As for Theorem 1.1(2), we introduce a generalization of Bernoulli polynomials as follows. By our proof of Theorem 1.1(1), we obtain that a polynomial $R_k(g,\Delta;x)\in\mathbb{C}[g,\Delta,x]$ for every $k\in\mathbb{Z}_{\geqslant 1}$ can be defined by the relation

$$\zeta_H(1-k;\tau)=(-1)^{k-1}\left\{\frac{2g^{2k}}{k}+(k-1)!C_{H,\tau}(k-1)\right\}=-\frac{2R_k(g,\Delta;\tau)}{k}. \tag{1.1}$$

Then, Theorem 1.1(2) implies that $R_k(g,\Delta;x)\in\mathbb{Q}[g^2,\Delta^2,x]$. If $g=\Delta=0$, the polynomial $R_k(0,0;x)$ coincides with $B_k(x)$ for every $k\in\mathbb{Z}_{\geqslant 1}$, where $B_k(x)$ is the kth Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k(x)}{k!}t^k.$$

Thus we call $R_k(g,\Delta;x)$ the kth *Rabi–Bernoulli polynomial*. As in the case of $B_k(x)$, the polynomial $R_k(g,\Delta;x)$ is monic, all the coefficients of $R_k(g,\Delta;x)$ are rational, and its degree with respect to x equals exactly k (see Proposition 5.3). Although we can compute explicitly Rabi–Bernoulli polynomials by definition and integration, it seems difficult to give simple formulas of them. We might expect some number theoretic properties of quantum Rabi models, as Kimoto and Wakayama extracted from non-commutative harmonic oscillators $Q_{(\alpha,\beta)}$ via (higher) Apéry-like numbers[1] encoded in special values of $\zeta_{Q_{(\alpha,\beta)}}(s)$ (cf. [10–15]).

Braak in [2] conjectured that the interval $[n-g^2,n+1-g^2]$ for every $n\in\mathbb{Z}_{\geqslant 0}$ contains at most two eigenvalues of H, that the interval $[n,n+1]$ for every $n\in\mathbb{Z}_{\geqslant 0}$ has at most two zeros of $G_+(x)G_-(x)$, that two intervals containing no zeros are not adjacent, and that two intervals containing two zeros are also not adjacent. As an application of Theorem 1.1, we have the Weyl law for H by using Tauberian theorem (cf. [1, Theorem 1.1] and [8, Corollary 2.6]).

COROLLARY 1.2. *We have*

$$N_H(T)\sim 2T,\quad (T\to\infty).$$

This corollary supports Braak's conjecture.

[1] The definition of Apéry-like numbers $J_k(n)$ in [10–13] was renewed in [14, 15].

https://doi.org/10.1017/nmj.2016.62 Published online by Cambridge University Press

Here is a remark on zeta regularized products. The zeta regularized product of Spec(A) for an operator A is defined by

$$\prod_{\lambda \in \mathrm{Spec}(A)} \lambda := \exp\left(-\frac{d}{ds}\zeta_A(s)|_{s=0}\right)$$

if $\zeta_A(s)$ is analytically continued to a function holomorphic around $s = 0$. The zeta regularized product of Spec(A) is applied to the existence of an entire function whose zeros coincide with Spec(A) as a multiset, and plays a pivotal role as a functional determinant $\det(A) := \prod_{\lambda \in \mathrm{Spec}(A)} \lambda$ (cf. [25], [26] and [30]). Wakayama [31, Conjecture 1] conjectured that $\zeta_H(s;\tau)$ would be meromorphic or holomorphic at $s = 0$. Since $\zeta_H(s;\tau)$ is holomorphic at $s = 0$ by Theorem 1.1 (or [27, Théorème (6.4), a)]), we can actually define the zeta regularized product $\prod_{\lambda \in \mathrm{Spec}(H)}(z - \lambda)$ as an entire function by [22, Theorem 1]. A formula of $\prod_{\lambda \in \mathrm{Spec}(H)}(z - \lambda)$ using $F_{H,n}(s;\tau)$ is given in Proposition 5.2. The comparison of $\prod_{\lambda \in \mathrm{Spec}(H)}(z - \lambda)$ with $G_+(z + g^2)G_-(z + g^2)$ may be an interesting problem (cf. [31, Conjecture 1]).

This paper is organized as follows. After fixing our notation, we explain the Rabi Hamiltonians H defined for any $g \geqslant 0$ and $\Delta \geqslant 0$ describing the quantum Rabi models in Section 2, referring mainly to [6], [18] and [29]. In the same section, a lower bound of Spec(H) is given in Lemma 2.2, and the maximal domain of H is given as $B^2(\mathbb{R}) \otimes_{\mathbb{C}} \mathbb{C}^2$, where $B^2(\mathbb{R})$ is a Shubin–Sobolev space originally introduced by Shubin [29, Section 25] (the Russian version of [29] was published in 1978). The convergence of the spectral zeta functions for the quantum Rabi models is discussed in Section 3. In the rest of Sections 3 and 4, we explore the method of the parametrix of the heat equation from [8], by which a meromorphic continuation of $\zeta_H(s;\tau)$ is given in Section 5. The method is a finer analysis of the trace Tr $K(t)$ of the heat operator $K(t) = e^{-t(H+\tau I)}$ for any $t > 0$ and sufficiently large $\tau \in \mathbb{R}$. In Section 3, the heat operator $K(t)$ is decomposed into the finite sum of explicitly computable operators $K_m(t)$ over $m = 1, \ldots, n$ and the residual operator $R_{n+1}(t)$ (see (3.2)). In the same section, $K_1(t)$ is made explicit and Tr $R_{n+1}(t)$ is estimated. In Section 4, we give the asymptotic series expansion of Tr $K_m(t)$ for $m \geqslant 2$ as $t \to +0$ in Theorem 4.1 with the aid of the asymptotic series of $e^{ixy/t}$ as $t \to +0$ in Lemma 4.2. The vanishing results of coefficients of the asymptotic series of Tr $K_m(t)$, Lemmas 4.6–4.9, are key ingredients of proving Theorem 4.1. Finally, we prove Theorem 1.1(1) in

Section 5. Theorem 1.1(2) is also proved in Section 5.2 by a rationality of $C_{H,\tau}(k)$ (see Theorem 5.4). Lemmas 4.6–4.9 on vanishing results are effectively used to prove Theorem 1.1(2). The first, second, and third Rabi–Bernoulli polynomials $R_k(g,\Delta;x)$ for $k\in\{1,2,3\}$ are explicitly computed in Section 6. For treating some matrix-valued exponential functions $t\mapsto \exp(tX)$ for some 2-by-2 square matrices X, the commutativity of two matrices $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ matches our computation from Sections 3–5, while such a treatment is difficult in the case of the noncommutative harmonic oscillator $Q_{(\alpha,\beta)}$ [8], which is described by noncommutative matrices $\left[\begin{smallmatrix}\alpha&0\\0&\beta\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$. However, difficulty in the case of the quantum Rabi models seems to be inherent in the simultaneous use of three matrices $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, among which $\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ are noncommutative.

§2. Preliminaries

2.1 Notation

For any $a\in\mathbb{Z}$, let $\mathbb{Z}_{\geqslant a}$ denote the set of all $n\in\mathbb{Z}$ such that $n\geqslant a$. For complex-valued functions f_1 and f_2 on a set X, we write $f_1(t)=\mathcal{O}(f_2(t))$ if there exists a constant $C>0$ such that $|f_1(t)|\leqslant C|f_2(t)|$ for all $t\in X$. We write $f_1(t)\asymp f_2(t)$ if both $f_1(t)=\mathcal{O}(f_2(t))$ and $f_2(t)=\mathcal{O}(f_1(t))$ hold. Furthermore, if $X=\{t\in\mathbb{R}\mid t>0\}$, we write $f_1(t)\sim f_2(t)(t\to\infty)$ for $\lim_{t\to\infty} f_1(t)/f_2(t)=1$. We also write

$$f_1(t)\sim_0\sum_{j=0}^{\infty}c_jt^j$$

if there exists a positive constant C_N for every $N\in\mathbb{Z}_{\geqslant 0}$ such that

$$\left|f_1(t)-\sum_{j=0}^{N}c_jt^j\right|\leqslant C_Nt^{N+1},\quad (t\to +0).$$

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$ and $\delta(x-a):\mathcal{S}(\mathbb{R})\to\mathbb{C}$ the Dirac delta distribution supported at $a\in\mathbb{R}$. By abuse of notation, $\delta(x-a)$ is used as if it is an integrand.

For a $\mathbb{C}$-Hilbert space V and a densely defined linear operator $A:V\to V$, let $\mathrm{Spec}(A)$ denote the multiset of eigenvalues of A in $\mathbb{C}$. Suppose that $\mathrm{Spec}(A)$ is discrete. Then, the norms $\|A\|_p$ on V for $p\in\{1,2\}$ are defined by

$$\|A\|_p=\left\{\sum_{\lambda\in\mathrm{Spec}(A)}|\lambda|^p\right\}^{1/p}$$

as long as they make sense. For any $N \in \{1, 2\}$, we set $L^2(\mathbb{R}; \mathbb{C}^N) = L^2(\mathbb{R}) \otimes_{\mathbb{C}} \mathbb{C}^N$. The natural L^2-inner product on $L^2(\mathbb{R})$ is denoted by $\langle \cdot, \cdot \rangle_{L^2}$. Then, we endow $L^2(\mathbb{R}; \mathbb{C}^2)$ with the L^2-inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle {}^t(u_1, u_2), {}^t(u_1', u_2') \rangle = \langle u_1, u_1' \rangle_{L^2} + \langle u_2, u_2' \rangle_{L^2} \tag{2.1}$$

for any $u_1, u_2, u_1', u_2' \in L^2(\mathbb{R})$. We note that the Schwartz space $\mathcal{S}(\mathbb{R}; \mathbb{C}^N) = \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} \mathbb{C}^N$ on $\mathbb{R}$ is densely embedded in $L^2(\mathbb{R}; \mathbb{C}^N)$ for any $N \in \{1, 2\}$. If $A : L^2(\mathbb{R}; \mathbb{C}^N) \to L^2(\mathbb{R}; \mathbb{C}^N)$ for $N \in \{1, 2\}$ is a densely defined differential operator, then a dense domain of A is supposed to be the maximal domain $\mathcal{D}(A)$ defined by

$$\mathcal{D}(A) = \{u \in L^2(\mathbb{R}; \mathbb{C}^N) \mid Au \in L^2(\mathbb{R}; \mathbb{C}^N)\},$$

where Au is the derivative of u as a tempered distribution on $\mathbb{R}$.

2.2 Quantum Rabi models

Throughout this paper, we use the following 2-by-2 complex matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We note the relations $IW = WI$, $IL = LI$, and $LW = -WL$.

For real numbers $\hbar > 0$, $\omega > 0$, $g \geqslant 0$, and $\Delta \geqslant 0$, let us define the Rabi Hamiltonian H densely defined in $L^2(\mathbb{R}; \mathbb{C}^2)$ by

$$H = H_{\text{Rabi}} = \hbar\omega a^\dagger a + \Delta\sigma_z + \hbar g \sigma_x (a^\dagger + a),$$

where

$$a = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{\omega}{\hbar}} x + \sqrt{\frac{\hbar}{\omega}} \partial_x \right),$$
$$a^\dagger = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{\omega}{\hbar}} x - \sqrt{\frac{\hbar}{\omega}} \partial_x \right),$$

$\sigma_z = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $\sigma_x = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Throughout this paper, we normalize H so that $\hbar = \omega = 1$ without loss of generality, and use the following expression:

$$H = \frac{-\partial_x^2 + x^2 - 1}{2} I + \Delta L + \sqrt{2} g x W,$$

which is the Weyl quantization of

$$H(x,\xi)=\frac{\xi^2+x^2-1}{2}I+\Delta L+\sqrt{2}gxW.$$

Then the operator $H: L^2(\mathbb{R};\mathbb{C}^2)\to L^2(\mathbb{R};\mathbb{C}^2)$ is unbounded, closed, and symmetric (cf. [18, pp. 8–9]). Besides, H is a global pseudodifferential operator of order 2 and elliptic in the sense of [18, Definition 3.2.19], that is, $|\det H(x,\xi)|\sim\sqrt{1+x^2+\xi^2}^4$, $(x^2+\xi^2\to\infty)$. We remark that H is not classical but semiregular classical in the sense of [18, Definition 3.2.19 and Remark 3.2.4], and that H is actually classical in the sense of [6, Définition 1.5.1]. Here, when we use "classical" in the sense of [6, Définition 1.5.1], we need to generalize notions for scalar-valued pseudodifferential operators in [6, Chapitre 1] to matrix-valued ones. However, this procedure is easy to perform by referring to [18, Chapter 3].

PROPOSITION 2.1. *The operator H is self-adjoint.*

Proof. Since H is formally self-adjoint, the assertion follows from [18, Proposition 3.3.10]. □

PROPOSITION 2.2. *All eigenvalues λ of H satisfy $\lambda\geqslant -g^2-\Delta$. In particular, for any $\tau\in\mathbb{R}$ such that $\tau>g^2+\Delta$, the operator $H+\tau I$ is positive.*

Proof. First, we see

$$\left\langle\frac{-\partial_x^2+x^2}{2}v,v\right\rangle_{L^2}\geqslant\frac{1}{2}\langle v,v\rangle_{L^2} \tag{2.2}$$

for any $v\in\mathcal{S}(\mathbb{R})$ by the proof of [18, Theorem 2.2.1].

Let us take any $\lambda\in\operatorname{Spec}(H)$. Then λ is real by the self-adjointness of H. Set $H'=UHU^{-1}$ with

$$U=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}.$$

Then,

$$H'=\begin{bmatrix}\frac{-\partial_x^2+x^2-1}{2}+\sqrt{2}gx & -\Delta\\ -\Delta & \frac{-\partial_x^2+x^2-1}{2}-\sqrt{2}gx\end{bmatrix} \tag{2.3}$$

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holds by noting $ULU^{-1} = -W$ and $UWU^{-1} = L$ (cf. [7, (3.1)]). Since $\mathrm{Spec}(H)$ coincides with $\mathrm{Spec}(H')$ as a multiset, λ is contained in $\mathrm{Spec}(H')$. Put

$$(2.4)\quad D_\pm = \frac{-\partial_x^2 + x^2 - 1}{2} \pm \sqrt{2}gx = \frac{-\partial_x^2 + (x \pm \sqrt{2}g)^2}{2} - \frac{1}{2} - g^2.$$

Let $u = {}^t(u_1, u_2)$ be a fixed eigenvector with $H'u = \lambda u$. We note that u is taken as an element of $\mathcal{S}(\mathbb{R}; \mathbb{C}^2)$ by [6, (1.9.2)]. By virtue of the inequality $\langle D_\pm u, u\rangle \geqslant -g^2\langle u, u\rangle$ by (2.2), a direct computation gives us

$$\begin{aligned}
\lambda\langle u, u\rangle &= \left\langle \begin{bmatrix} D_+ & -\Delta \\ -\Delta & D_- \end{bmatrix} u, u \right\rangle \\
&= \langle D_+u_1, u_1\rangle_{L^2} + \langle D_-u_2, u_2\rangle_{L^2} - \Delta\langle u_1, u_2\rangle_{L^2} - \Delta\langle u_2, u_1\rangle_{L^2} \\
&\geqslant -g^2\langle u_1, u_1\rangle_{L^2} - g^2\langle u_2, u_2\rangle_{L^2} - \Delta(\|u_1 + u_2\|_{L^2}^2 - \|u_1\|_{L^2}^2 - \|u_2\|_{L^2}^2) \\
&\geqslant -g^2(\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2) - \Delta(\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2) = (-g^2 - \Delta)\langle u, u\rangle,
\end{aligned}$$

where $||\cdot||_{L^2}$ is the L^2-norm on $L^2(\mathbb{R})$ induced from $\langle\cdot,\cdot\rangle_{L^2}$. This completes the proof. ∎

Set $\Lambda^2 = 1 + x^2 - \partial_x^2$ and $B^2(\mathbb{R}) = \{u \in \mathcal{S}'(\mathbb{R}) \mid \Lambda^2 u \in L^2(\mathbb{R})\}$, where $\mathcal{S}'(\mathbb{R})$ denotes the space of tempered distributions on $\mathbb{R}$ and $\Lambda^2 u$ is the differential of u as a tempered distribution. Then $B^2(\mathbb{R})$ is called a Shubin–Sobolev space (cf. [29, Section 25]). The Shubin–Sobolev space $B^2(\mathbb{R})$ contains $\mathcal{S}(\mathbb{R})$ obviously and has a Hilbert space structure with inner product $(u_1, u_2)_{B^2} = (\Lambda^2 u_1, \Lambda^2 u_2)_{L^2}$ for any $u_1, u_2 \in B^2(\mathbb{R})$. The space $B^2(\mathbb{R})$ is dense and compactly embedded in $L^2(\mathbb{R})$ by [29, Proposition 25.4] (see also [6, Proposition 1.6.11] and [18, Proposition 3.2.26]).

PROPOSITION 2.3. *We have $\mathcal{D}(H) = B^2(\mathbb{R}) \otimes_{\mathbb{C}} \mathbb{C}^2$.*

Proof. As we see that H is a globally elliptic pseudodifferential operator of order 2 and that H is classical in the sense of [6, Définition 1.5.1], we obtain the assertion by [6, Théorème 1.6.4] (see also [18, Lemma 3.3.9]). ∎

Remark. By Proposition 2.3, the operator H has a compact resolvent, and hence the spectrum of H coincides with the set $\mathrm{Spec}(H)$ of the eigenvalues of H as a multiset, that is, the continuous and the residual spectra of H are empty (cf. [28, Proposition 2.11] or [29, Theorem 26.3]). In particular, $\mathrm{Spec}(H)$ is discrete. Such a discreteness also follows from the location of zeros of $G_+(x)G_-(x)$ constructed in [2].

LEMMA 2.4. *Let* τ *be a real number such that* $\tau > g^2 + \Delta$ *and let* $0 < \lambda_1' \leqslant \lambda_2' \leqslant \lambda_3' \leqslant \cdots \leqslant \lambda_n' \leqslant \cdots$ *be the sequence of all eigenvalues of* $H + \tau I$. *Then, we have* $\lambda_n' \asymp n$.

Proof. Let H' denote the Hamiltonian given by (2.3). By $\mathrm{Spec}(H) = \mathrm{Spec}(H')$ as a multiset, we may consider H' instead of H. Put

$$B = H' + \Delta W = \left(\frac{-\partial_x^2 + x^2 - 1}{2} + \tau \right) I + \sqrt{2} g x L$$

as an operator in $L^2(\mathbb{R}; \mathbb{C}^2)$ whose domain is $\mathcal{D}(H')$. Then,

$$B = \begin{bmatrix} \frac{-\partial_x^2 + (x + \sqrt{2} g)^2}{2} - 1/2 - g^2 + \tau & 0 \\ 0 & \frac{-\partial_x^2 + (x - \sqrt{2} g)^2}{2} - 1/2 - g^2 + \tau \end{bmatrix}$$

is a positive self-adjoint operator and its nth eigenvalue $\lambda_n(B)$ satisfies $\lambda_n(B) \asymp n$ as $n \to \infty$. Since the canonical injection $\mathcal{D}(H') = \mathcal{D}(H) \hookrightarrow L^2(\mathbb{R}; \mathbb{C}^2)$ is compact by Proposition 2.3, both H' and B have compact resolvents. By $H' = B - \Delta W$, we easily have

$$\|H'u\|^2 \leqslant 2(\|Bu\|^2 + \|\Delta W u\|^2) \leqslant 2(1 + \Delta^2)(\|Bu\|^2 + \|u\|^2)$$

for all $u \in \mathcal{S}(\mathbb{R}; \mathbb{C}^2)$, where $\|\cdot\|$ is the L^2-norm on $L^2(\mathbb{R}; \mathbb{C}^2)$ induced from $\langle \cdot, \cdot \rangle$. Similarly we have also $\|Bu\|^2 \leqslant 2(1 + \Delta^2)(\|H'u\|^2 + \|u\|^2)$ for all $u \in \mathcal{S}(\mathbb{R}; \mathbb{C}^2)$. Therefore, by applying [18, Proposition 4.2.2] to H' and B, we obtain the desired assertion. □

§3. Spectral zeta functions

For any $\tau \in \mathbb{C}$, the Hurwitz-type spectral zeta function of the Rabi Hamiltonian H is given as the formal series

$$\zeta_H(s; \tau) = \sum_{\lambda \in \mathrm{Spec}(H)} \frac{1}{(\lambda + \tau)^s}, \quad s \in \mathbb{C}.$$

We check the convergence for some suitable choices of τ and s. The following is a consequence from Lemma 2.4.

PROPOSITION 3.1. *For any fixed* $\tau \in \mathbb{C} - \mathrm{Spec}(-H)$, *the series* $\zeta_H(s; \tau)$ *converges absolutely for* $\mathrm{Re}(s) > 1$. *Furthermore, the series* $\zeta(1; \tau)$ *is divergent.*

For the Rabi Hamiltonian H and $\tau \in \mathbb{C}$, we consider the heat operator $K(t) = e^{-t(H+\tau I)}$ on $t > 0$. From now on, we fix $\tau \in \mathbb{R}$ such that $\tau > g^2 + \Delta$. By Propositions 2.1 and 2.2, $H + \tau I$ is a self-adjoint operator and all its eigenvalues are positive. Hence the Hurwitz-type spectral zeta function of H has an integral expression

$$\zeta_H(s;\tau) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr} K(t)\, dt$$

as long as the integral on the left-hand side is absolutely convergent. Set

$$Z_0(s) = \int_0^1 t^{s-1} \operatorname{Tr} K(t)\, dt, \qquad Z_\infty(s) = \int_1^\infty t^{s-1} \operatorname{Tr} K(t)\, dt.$$

Lemma 3.2. *The integral $Z_0(s)$ converges absolutely for $\operatorname{Re}(s) > 1$, and the integral $Z_\infty(s)$ converges absolutely for all $s \in \mathbb{C}$. In particular, the function $Z_\infty(s)$ has an analytic continuation to $\mathbb{C}$. Furthermore, the integral $\int_0^\infty t^{s-1} \operatorname{Tr} K(t)\, dt$ converges absolutely for $\operatorname{Re}(s) > 1$ and we have*

$$\zeta_H(s;\tau) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr} K(t)\, dt = \frac{1}{\Gamma(s)}(Z_0(s) + Z_\infty(s)), \quad \operatorname{Re}(s) > 1. \tag{3.1}$$

Proof. We follow the method given in the proof of [8, Proposition 2.1]. We remark that

$$\begin{aligned}\operatorname{Tr} K(t) &= \sum_{\lambda \in \operatorname{Spec}(H)} e^{-(\lambda+\tau)t} \leqslant \sum_{\lambda \in \operatorname{Spec}(H)} \frac{\{(1+\epsilon)/e\}^{1+\epsilon}}{\{(\lambda+\tau)t\}^{1+\epsilon}} \\ &= \{(1+\epsilon)/e\}^{1+\epsilon} \zeta_H(1+\epsilon;\tau) t^{-1-\epsilon} < \infty\end{aligned}$$

for any $\epsilon > 0$, where we use Proposition 3.1 and the inequality $e^{-b} \leqslant (a/e)^a b^{-a}$ for all $a, b > 0$. Thus it is obvious that $Z_0(s)$ converges absolutely for $\operatorname{Re}(s) > 1$. Put $\sigma = \operatorname{Re}(s)$ and take $a \in \mathbb{R}$ such that $a > \max(\sigma, 1)$. Then,

$$\begin{aligned}\int_1^\infty |t^{s-1} \operatorname{Tr} K(t)|\, dt &\leqslant \int_1^\infty \sum_{\lambda \in \operatorname{Spec}(H)} \frac{(a/e)^a}{(\lambda+\tau)^a} t^{-a} \times t^{\sigma-1}\, dt \\ &= \zeta_H(a;\tau) \int_1^\infty t^{\sigma-a-1}\, dt < \infty.\end{aligned}$$

Equalities (3.1) follow from

$$\Gamma(s) \sum_{\lambda \in \operatorname{Spec}(H)} (\lambda+\tau)^{-s} = \sum_{\lambda \in \operatorname{Spec}(H)} \int_0^\infty e^{-t} \left(\frac{t}{\lambda+\tau}\right)^s \frac{dt}{t}$$
$$= \int_0^\infty \sum_{\lambda \in \operatorname{Spec}(H)} e^{-(\tau+\lambda)t} t^s \frac{dt}{t} = \int_0^\infty t^{s-1} \operatorname{Tr} K(t)\, dt.$$

The change of integrals and series is justified when $\operatorname{Re}(s) > 1$. ∎

For the operator $H+\tau I$, we define an operator $K_1(t)$ and its kernel $K_1(t,x,y)$ by

$$\begin{aligned}(K_1(t)f)(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty e^{i(x-y)\xi} \exp\left[-t\left(\frac{\xi^2+y^2}{2}I + \sqrt{2}gyW\right)\right] f(y)\, dy\, d\xi \\ &= \int_{-\infty}^\infty K_1(t,x,y) f(y)\, dy\end{aligned}$$

for any $f \in \mathcal{S}(\mathbb{R};\mathbb{C}^2)$. We set $R_2(t) = K(t) - K_1(t)$. Then the equation $(\partial_t + H + \tau I)K_1(t) + (\partial_t + H + \tau I)R_2(t) = 0$ holds. Furthermore, we set $F(t,x,y) = (\partial_t + H_x + \tau I)R_2(t,x,y) = -(\partial_t + H_x + \tau I)K_1(t,x,y)$, where H_x is the operator H acting on the x-variables. By $K_1(t,x,y) \to \delta(x-y)I$ as $t \to +0$, we have easily $R_2(t) \to 0I$ as $t \to +0$. Therefore, by Duhamel's principle (cf. [4, pp. 202–204]), we have the following expression

$$R_2(t) = \int_0^t e^{-(t-u)(H+\tau I)} F(u)\, du,$$

where we put $(F(u)f)(x) = \int_{-\infty}^\infty F(u,x,y) f(y)\, dy$ for any $f \in \mathcal{S}(\mathbb{R};\mathbb{C}^2)$.

LEMMA 3.3. *We have*

$$\begin{aligned}&F(t,x,y) \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty e^{i(x-y)\xi} \left[\frac{y^2-x^2}{2}I + \sqrt{2}g(y-x)W\right] \\ &\quad\times \exp\left[-t\left(\frac{\xi^2+y^2}{2}I + \sqrt{2}gyW\right)\right] d\xi - \frac{1}{2\pi}\int_{-\infty}^\infty e^{i(x-y)\xi} \\ &\quad\times \left[\left(\tau - \frac{1}{2}\right)I + \Delta L\right] \exp\left[-t\left(\frac{\xi^2+y^2}{2}I + \sqrt{2}gyW\right)\right] d\xi.\end{aligned}$$

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Proof. It follows from the definition of $K_1(t,x,y)$ and the expression $\int_{-\infty}^{\infty} F(t,x,y)f(y)\,dy = \int_{-\infty}^{\infty}(-\partial_t - H - \tau I)K_1(t,x,y)f(y)\,dy$ for any $f \in \mathcal{S}(\mathbb{R};\mathbb{C}^2)$. ∎

The function $\mathrm{Tr}\,K_1(t)$ is analyzed as follows.

LEMMA 3.4. *We have the following formulas:*

$$\mathrm{Tr}\,K_1(t) = \frac{2e^{g^2t}}{t}, \quad t>0,$$

$$\int_0^1 t^{s-1}\,\mathrm{Tr}\,K_1(t)\,dt = \frac{2}{s-1} + \sum_{k=1}^{\infty}\frac{2g^{2k}}{k!}\frac{1}{s+k-1}, \quad \mathrm{Re}(s)>1.$$

Proof. Since I and W are commutative, by [5, 3.323, 2.[10]], the function $K_1(t,x,y)$ can be described as

$$\begin{aligned} &K_1(t,x,y)\\ &= \frac{1}{2\pi}\int_{\xi\in\mathbb{R}} e^{i(x-y)\xi}\exp\left(-\frac{t\xi^2}{2}I\right)\,d\xi \times \exp\left(-\frac{ty^2}{2}I\right)\exp(-\sqrt{2}gtyW)\\ &= \frac{1}{\sqrt{2\pi t}}e^{-(x-y)^2/2t-ty^2/2}\exp(-\sqrt{2}gtyW). \end{aligned}$$

Hence we obtain

$$\begin{aligned} \mathrm{Tr}\,K_1(t) &= \int_{-\infty}^{\infty}\mathrm{tr}K_1(t,x,x)\,dx\\ &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-tx^2/2}2\cosh(\sqrt{2}gtx)\,dx = \frac{2e^{g^2t}}{t}, \end{aligned}$$

where we use [5, 3.546.2]. This completes the proof. ∎

3.1 Estimates of residual operators

Set

$$\begin{aligned} &F_1(t,x,y)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-y)\xi}\frac{y^2-x^2}{2}I\exp\left[-t\left(\frac{\xi^2+y^2}{2}I+\sqrt{2}gyW\right)\right]d\xi,\\ &F_2(t,x,y)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-y)\xi}\sqrt{2}g(y-x)W\exp\left[-t\left(\frac{\xi^2+y^2}{2}I+\sqrt{2}gyW\right)\right]d\xi, \end{aligned}$$

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$$F_3(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-y)\xi}(-\Delta)L\exp\left[-t\left(\frac{\xi^2+y^2}{2}I+\sqrt{2}gyW\right)\right]d\xi,$$

$$F_4(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-y)\xi}\left(\frac{1}{2}-\tau\right)I\exp\left[-t\left(\frac{\xi^2+y^2}{2}I+\sqrt{2}gyW\right)\right]d\xi,$$

and $F_j(t) = \int_{-\infty}^{\infty} F_j(t,x,y)f(y)\,dy$ for any $f\in\mathcal{S}(\mathbb{R};\mathbb{C}^2)$ and any $j\in\{1,2,3,4\}$. Then we see

$$F(t,x,y) = F_1(t,x,y)+F_2(t,x,y)+F_3(t,x,y)+F_4(t,x,y)$$

and

$$F(t) = F_1(t)+F_2(t)+F_3(t)+F_4(t).$$

We have the following by the same computation as in Lemma 3.4.

LEMMA 3.5. *We have the following explicit formulas:*

$$F_1(t,x,y) = \frac{1}{\sqrt{2\pi t}}\frac{y^2-x^2}{2}Ie^{-(x-y)^2/2t-ty^2/2}\exp(-\sqrt{2}gtyW),$$

$$F_2(t,x,y) = \frac{1}{\sqrt{2\pi t}}\sqrt{2}g(y-x)We^{-(x-y)^2/2t-ty^2/2}\exp(-\sqrt{2}gtyW),$$

$$F_3(t,x,y) = \frac{1}{\sqrt{2\pi t}}(-\Delta)Le^{-(x-y)^2/2t-ty^2/2}\exp(-\sqrt{2}gtyW),$$

$$F_4(t,x,y) = \frac{1}{\sqrt{2\pi t}}\left(\frac{1}{2}-\tau\right)Ie^{-(x-y)^2/2t-ty^2/2}\exp(-\sqrt{2}gtyW).$$

LEMMA 3.6. *For $0<t<1$, we have $\|F_1(t)+F_2(t)\|_2 = \mathcal{O}(t^{-1/2})$ and $\|F_3(t)+F_4(t)\|_2 = \mathcal{O}(t^{-1/2})$.*

Proof. Set $\tau' = \tau - 1/2$, $F_-(t,x,y) = F_3(t,x,y)+F_4(t,x,y)$, and $F_-(t) = F_3(t)+F_4(t)$. Since the adjoint operator $F_-(t)^*$ of $F_-(t)$ is given by $(F_-(t)^*f)(x) = \int_{-\infty}^{\infty}\overline{F_-(t,y,x)}f(y)\,dy$, by noting $WL=-LW$ and Lemma 3.5, we obtain

$$\|F_-(t)\|_2^2 = \mathrm{Tr}(F_-(t)^*F_-(t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{tr}F_-(t,x,y)\overline{F_-(t,x,y)}\,dy\,dx$$

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$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{tr}\left[\frac{1}{2\pi t}(\tau' I + \Delta L)e^{-(x-y)^2/2t-ty^2/2}\exp(-\sqrt{2}gtyW)\right.$$
$$\left.\times\ (\tau' I + \Delta L)e^{-(x-y)^2/2t-ty^2/2}\exp(-\sqrt{2}gtyW)\right]\,dy\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi t}e^{-(x-y)^2/t-ty^2}\operatorname{tr}\left[(\tau' I + \Delta L)\,(\tau'\exp(-\sqrt{2}gtyW)\right.$$
$$\left.+\,\Delta L\exp(\sqrt{2}gtyW))\exp(-\sqrt{2}gtyW)\right]\,dy\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi t}e^{-(x-y)^2/t-ty^2}$$
$$\times\operatorname{tr}[(\tau' I + \Delta L)(\tau'\exp(-2\sqrt{2}gtyW) + \Delta L)]\,dy\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi t}e^{-(x-y)^2/t-ty^2} \times 2\{\tau'^2\cosh(2\sqrt{2}gty) + \Delta^2\}\,dy\,dx$$
$$= \frac{\tau'^2}{\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x-y)^2/t-ty^2}\cosh(2\sqrt{2}gty)\,dy\,dx$$
$$+\,\frac{\Delta^2}{\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x-y)^2/t-ty^2}\,dy\,dx,$$

and hence we have $\|F_-(t)\|_2^2 = t^{-1}(\tau'^2 e^{2g^2t} + \Delta^2)$ by the formulas

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x-y)^2/t-ty^2}\,dy\,dx = \pi,$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x-y)^2/t-ty^2}e^{\pm 2\sqrt{2}gty}\,dy\,dx = \pi e^{2g^2t}.$$

Next we set $F_+(t) = F_1(t) + F_2(t)$ and $F_+(t,x,y) = F_1(t,x,y) + F_2(t,x,y)$. Combining $IW = WI$ with Lemma 3.5, we obtain

$$\|F_+(t)\|_2^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{tr}F_+(t,x,y)\overline{F_+(t,x,y)}\,dy\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi t}\operatorname{tr}\left[\left\{\frac{y^2-x^2}{2}I + \sqrt{2}g(y-x)W\right\}^2\right.$$
$$\left.\times\ e^{-(x-y)^2/t-ty^2}\exp(-2\sqrt{2}gtyW)\right]\,dy\,dx$$

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$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi t} e^{-(x-y)^2/t - ty^2} \times \left[\left\{ \frac{y^2 - x^2}{2} + \sqrt{2} g(y-x) \right\}^2 e^{-2\sqrt{2}gty} \right.$$

$$\left. + \left\{ \frac{y^2 - x^2}{2} - \sqrt{2} g(y-x) \right\}^2 e^{2\sqrt{2}gty} \right] dy\, dx.$$

Hence $\|F_+(t)\|_2^2 = \mathcal{O}(t^{-1})$ follows from

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi t} e^{-(x-y)^2/t - ty^2} \left\{ \frac{y^2 - x^2}{2} \pm \sqrt{2} g(x-y) \right\}^2 e^{\mp 2\sqrt{2}gty}\, dy\, dx$$

$$= \frac{1}{2\pi t} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2 - v^2} \frac{u^2}{4} (tu + 2v \pm 2\sqrt{2} g\sqrt{t})^2 e^{\pm 2\sqrt{2} g \sqrt{t} v}\, du\, dv$$

$$= \mathcal{O}(t^{-1}). \qquad \square$$

Set

$$K_m(t) = \int_0^t \int_0^{t-u_1} \cdots \int_0^{t-u_1-\cdots-u_{m-2}} \times K_1(t - u_1 - \cdots u_{m-1}) F(u_{m-1}) F(u_{m-2}) \cdots F(u_1)\, du_{m-1} \cdots du_1$$

for $m \geqslant 2$ and

$$R_{n+1}(t) = \int_0^t \int_0^{t-u_1} \cdots \int_0^{t-u_1-\cdots-u_{n-1}} K(t - u_1 - \cdots u_n) \times F(u_n) F(u_{n-1}) \cdots F(u_1)\, du_n\, du_{n-1} \cdots du_1$$

for $n \geqslant 2$, respectively. Then, by the same argument in [8, pp. 704–705], we decompose $K(t)$ as

$$K(t) = K_1(t) + \sum_{m=2}^{n} K_m(t) + R_{n+1}(t), \quad n \in \mathbb{Z}_{\geqslant 1}. \tag{3.2}$$

LEMMA 3.7. *For any $\epsilon \in (0, 1/2)$, there exists a positive constant $C = C(g, \Delta, \tau, \epsilon)$ such that*

$$|\mathrm{Tr}\, R_2(t)| \leqslant C t^{-\epsilon},$$

$$|\mathrm{Tr}\, R_{n+1}(t)| \leqslant C^n \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} t^{n/2}$$

for any $t \in (0, 1)$ and any $n \in \mathbb{Z}_{\geqslant 2}$.

Proof. We follow the method in [8, Proposition 2.3]. For any $t > 0$, $u \in (0, t)$, and $\epsilon > 0$, we have

$$\begin{aligned}
&\|e^{-(t-u)(H+\tau I)}\|_2 \\
&\quad = \left(\sum_{\lambda\in\mathrm{Spec}(H)} e^{-2(t-u)(\lambda+\tau)}\right)^{1/2} \\
&\quad \leqslant \left(\frac{1+2\epsilon}{e}\right)^{(1+2\epsilon)/2}\left(\sum_{\lambda\in\mathrm{Spec}(H)}\frac{1}{(\lambda+\tau)^{1+2\epsilon}}\right)^{1/2} 2^{-(1+2\epsilon)/2}(t-u)^{-(1+2\epsilon)/2} \\
&\quad = \left(\frac{1+2\epsilon}{e}\right)^{(1+2\epsilon)/2}\zeta_H(1+2\epsilon;\tau)^{1/2}2^{-(1+2\epsilon)/2}(t-u)^{-(1+2\epsilon)/2}.
\end{aligned}$$

We note that $\zeta_H(1+2\epsilon;\tau)$ is convergent by Proposition 3.1. By virtue of Lemma 3.6, we have $\|F(t)\|_2 \leqslant \|F_1(t)+F_2(t)\|_2 + \|F_3(t)+F_4(t)\|_2 = \mathcal{O}(t^{-1/2})$. Then, we estimate $\mathrm{Tr}\, R_2(t) = \int_0^t \mathrm{Tr}(e^{-(t-u)(H+\tau I)}F(u))\, du$ as

$$\begin{aligned}
|\mathrm{Tr}\, R_2(t)| &= \int_0^t |\mathrm{Tr}(e^{-(t-u)(H+\tau I)}F(u))|\, du \leqslant \int_0^t \|e^{-(t-u)(H+\tau I)}F(u)\|_1\, du \\
&\leqslant \int_0^t \|e^{-(t-u)(H+\tau I)}\|_2\|F(u)\|_2\, du \leqslant C\int_0^t (t-u)^{-1/2-\epsilon}u^{-1/2}\, du \\
&= Ct^{-\epsilon}B(1/2-\epsilon, 1/2),
\end{aligned}$$

where $C = C(g,\Delta,\tau,\epsilon)$ is a positive constant depending only on g,Δ,τ, and ϵ. From this, $|\mathrm{Tr}\, R_{n+1}(t)|$ for $n \geqslant 2$ is majorized as $|\mathrm{Tr}\, R_{n+1}(t)| \leqslant C^n t^{n/2}\frac{\pi^{n/2}}{\Gamma(n/2+1)}$ in the same way as [8, Proposition 2.3]. ∎

§4. Asymptotic expansions

In this section, we give the asymptotic series of $\mathrm{Tr}\, K_m(t)$ as $t \to +0$, which is used in Section 5. Recall that $\tau \in \mathbb{R}$ is fixed so that $\tau > g^2 + \Delta$ as in Section 3. The main result in this section is the following.

THEOREM 4.1. *For any $m \in \mathbb{Z}_{\geqslant 2}$, there exists a sequence $\{c_q^{(m)}\}_{q\in\mathbb{Z}_{\geqslant 0}}$ of complex numbers such that*

$$\mathrm{Tr}\, K_m(t) \sim_0 \sum_{q=0}^{\infty} c_q^{(m)} t^q.$$

Moreover, we have $c_q^{(m)} = 0$ if $q < m-2$.

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For $m \in \mathbb{Z}_{\geqslant 2}$ and $\epsilon = (\epsilon_j)_{j=1,\ldots,m-1} \in \{1,2,3,4\}^{m-1}$, we set

$$K_{m,\epsilon}(t) = \int_0^t \int_0^{t-u_1} \cdots \int_0^{t-u_1-\cdots-u_{m-2}} K_1(t-u_1-\cdots-u_{m-1}) \cdot F_{\epsilon_{m-1}}(u_{m-1}) F_{\epsilon_{m-2}}(u_{m-2}) \cdots F_{\epsilon_2}(u_2) F_{\epsilon_1}(u_1)\, du_{m-1} \cdots du_1.$$

Then $K_m(t)$ is decomposed as

$$K_m(t) = \sum_{\epsilon \in \{1,2,3,4\}^{m-1}} K_{m,\epsilon}(t).$$

Therefore, we only have to consider an asymptotic behavior of $\operatorname{Tr} K_{m,\epsilon}(t)$ for each $\epsilon \in \{1,2,3,4\}^{m-1}$. By the change of variables $u_j = tu'_j$, it holds that

$$K_{m,\epsilon}(t) = t^{m-1} \int_0^1 \int_0^{1-u'_1} \cdots \int_0^{1-u'_1-\cdots-u'_{m-2}} K_1(t(1-u'_1-\cdots-u'_{m-1})) \cdot F_{\epsilon_{m-1}}(tu'_{m-1}) F_{\epsilon_{m-2}}(tu'_{m-2}) \cdots F_{\epsilon_2}(tu'_2) F_{\epsilon_1}(tu'_1)\, du'_{m-1} \cdots du'_1.$$

By putting $D_{m-1} = \{u \in \mathbb{R}^{m-1} \mid u_j \geqslant 0 (\forall j = 1, \ldots, m-1), \sum_{j=1}^{m-1} u_j \leqslant 1\}$, we have

$$\begin{aligned}\operatorname{Tr} K_{m,\epsilon}(t) = t^{m-1} \int_{D_{m-1}} du \int_{z_0 \in \mathbb{R}} \operatorname{tr} \int_{(z_1,\ldots,z_{m-1}) \in \mathbb{R}^{m-1}} &\times K_1(t(1-u_1-\cdots-u_{m-1}), z_0, z_{m-1}) \\ &\cdot F_{\epsilon_{m-1}}(tu_{m-1}, z_{m-1}, z_{m-2}) \cdots F_{\epsilon_2}(tu_2, z_2, z_1) \\ &\times F_{\epsilon_1}(tu_1, z_1, z_0)\, dz_0\, dz_1 \cdots dz_{m-1}.\end{aligned}$$

By the definition of $F_\epsilon(t,x,y)$ for $\epsilon \in \{1,2,3,4\}$ and by the change of variables $\sqrt{t} z_j \leftrightarrow z_j$ and $\sqrt{t}\xi_j \leftrightarrow \xi_j$, the integral above is transformed to

$$\begin{aligned}&\operatorname{Tr} K_{m,\epsilon}(t) \\ &= t^{-1} \int_{D_{m-1}} du \int_{(\xi_1,\ldots,\xi_m) \in \mathbb{R}^m} \prod_{j=1}^m d\xi_j \int_{(z_0,\ldots,z_{m-1}) \in \mathbb{R}^m} \prod_{j=1}^{m-1} dz_j \frac{1}{(2\pi)^m} \\ &\quad \times e^{i[(z_0-z_{m-1})\xi_m + (z_{m-1}-z_{m-2})\xi_{m-1} + \cdots + (z_1-z_0)\xi_1]/t} \\ &\quad \times \operatorname{tr} \left[e^{-(1-u_1-\cdots-u_{m-1})\left[((\xi_m^2+z_{m-1}^2)/2)I + \sqrt{2} g \sqrt{t} z_{m-1} W\right]} \right.\end{aligned}$$

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$$\times \overleftarrow{\prod_{j=1}^{m-1}} T_{\epsilon_j}(z_{j-1}/\sqrt{t}, z_j/\sqrt{t}) e^{-u_j\left[((\xi_j^2+z_{j-1}^2)/2)I+\sqrt{2}g\sqrt{t}z_{j-1}W\right]}\Bigg]$$

$$= t^{-1-r_1(\epsilon)-(1/2)r_2(\epsilon)} \int_{D_{m-1}} du \int_{(\xi_1,\ldots,\xi_m)\in\mathbb{R}^m} \prod_{j=1}^{m} d\xi_j \int_{(z_0,\ldots,z_{m-1})\in\mathbb{R}^m}$$

$$\times \prod_{j=1}^{m-1} dz_j \frac{1}{(2\pi)^m}$$

$$\times \prod_{j=1}^{m-1} e^{i(z_j-z_{j-1})(\xi_j-\xi_m)/t}$$

$$\times \operatorname{tr}\Bigg[e^{-(1-u_1-\cdots-u_{m-1})\left[((\xi_m^2+z_{m-1}^2)/2)I+\sqrt{2}g\sqrt{t}z_{m-1}W\right]}$$

$$\times \overleftarrow{\prod_{j=1}^{m-1}} T_{\epsilon_j}(z_{j-1}, z_j) e^{-u_j\left[((\xi_j^2+z_{j-1}^2)/2)I+\sqrt{2}g\sqrt{t}z_{j-1}W\right]}\Bigg]. \tag{4.1}$$

Here we put $r_j(\epsilon) = \#\{j \in \{1, \ldots, m-1\} \mid \epsilon_j = k\}$ for each $k \in \{1, 2, 3, 4\}$, $T_1(x, y) = \frac{x^2-y^2}{2} I$, $T_2(x, y) = \sqrt{2}g(x-y)W$, $T_3(x, y) = -\Delta L$, $T_4(x, y) = (1/2-\tau)I$, and

$$\overleftarrow{\prod_{j=1}^{m-1}} A_j = A_{m-1} \cdots A_1$$

for any 2-by-2 matrices $A_1, \ldots, A_{m-1}$. In order to expand integral (4.1), we use the following given in [8, Lemma 3.3].

LEMMA 4.2. *The function $(x, y) \mapsto e^{ixy/t}$ has the asymptotic series*

$$e^{ixy/t} \sim_0 2\pi \sum_{k=0}^{\infty} i^k \frac{\partial_x^k \delta(x) \partial_y^k \delta(y)}{k!} t^{k+1}$$

as a tempered distribution in $\mathbb{R}^2$.

From this lemma (or [8, (4.7)]), by integration by parts, (4.1) is expanded as

$$
\begin{aligned}
&\operatorname{Tr} K_{m,\epsilon}(t)\\
&\sim_0 \frac{1}{2\pi}\sum_{l_1=0}^{\infty}\cdots\sum_{l_{m-1}=0}^{\infty}\frac{i^{l_1+\cdots+l_{m-1}}}{l_1!\cdots l_{m-1}!}t^{l_1+\cdots+l_{m-1}+m-2-r_1(\epsilon)-(1/2)r_2(\epsilon)}\\
&\quad\times\int_{D_{m-1}}du\int_{(\xi_1,\ldots,\xi_m)\in\mathbb{R}^m}\prod_{j=1}^{m}d\xi_j\int_{(z_0,\ldots,z_{m-1})\in\mathbb{R}^m}\prod_{j=0}^{m-1}dz_j\\
&\quad\times\left\{\prod_{j=1}^{m-1}(-1)^{l_j}\partial_{z_{j-1}}^{l_j}\delta(z_{j-1}-z_j)\right\}\\
&\quad\times\left\{\prod_{j=1}^{m-1}(-1)^{l_j}\delta(\xi_j-\xi_m)\right\}\\
&\quad\times\operatorname{tr}\left[e^{-(1-u_1-\cdots-u_{m-1})((\xi_m^2+z_{m-1}^2)/2)I}\left(\prod_{j=1}^{m-1}\partial_{\xi_j}^{l_j}e^{-u_j((\xi_j^2+z_{j-1}^2)/2)I}\right)\right.\\
&\quad\left.\times e^{-(1-u_1-\cdots-u_{m-1})\sqrt{2}g\sqrt{t}z_{m-1}W}\overleftarrow{\prod_{j=1}^{m-1}}T_{\epsilon_j}(z_{j-1},z_j)e^{-u_j\sqrt{2}g\sqrt{t}z_{j-1}W}\right].
\end{aligned}
\tag{4.2}
$$

Here, we remark that the symbol $\delta(z_j - z_{j-1})$ is always replaced with $\delta(z_{j-1} - z_j)$ throughout this paper when $\partial_{z_j}^{l_j}$ is transformed to $(-1)^{l_j}\partial_{z_{j-1}}^{l_j}$ by integration by parts. (The symbol $\delta(z_j - z_{j-1})$ used in [8, (4.8), (4.17b), (4.18), (4.19), (4.21), the first (4.22), and (4,23)] should be replaced with $\delta(z_{j-1} - z_j)$ if it is regarded as a tempered distribution supported at z_j.)

Let us further analyze some factors in the integrand above. The following is obvious.

LEMMA 4.3. *For any $l \in \mathbb{Z}_{\geqslant 0}$ and $u \in (0, 1)$, we have*

$$\partial_\xi^l e^{-u\xi^2/2} = (-1)^l (u/2)^{l/2} H_l(\sqrt{u/2}\,\xi) e^{-u\xi^2/2},$$

where $H_l(x)$ is the lth Hermite polynomial defined by $H_l(x) = (-1)^l e^{x^2}(\partial_x^l e^{-x^2})$.

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Let us consider a transformation of the ordered product

$$\overleftarrow{\prod_{j=1}^{m-1}} T_{\epsilon_j}(z_{j-1}, z_j) e^{-u_j \sqrt{2} g \sqrt{t} z_{j-1} W}.$$

By the definition of $T_{\epsilon_j}(x,y)$, the product as above is described as

$$\left\{ \prod_{\substack{j=1\\ \epsilon_j=1}}^{m-1} \frac{z_{j-1}^2 - z_j^2}{2} \right\} \left\{ \prod_{\substack{j=1\\ \epsilon_j=2}}^{m-1} \sqrt{2} g (z_{j-1} - z_j) \right\} (-\Delta)^{r_3(\epsilon)} \left(-\tau + \frac{1}{2} \right)^{r_4(\epsilon)}$$
$$\times \operatorname{tr} \Bigg[\exp(-(1 - u_1 - \cdots - u_{m-1}) \sqrt{2} g \sqrt{t} z_{m-1} W)$$
$$\times \overleftarrow{\prod_{j=1}^{m-1}} A(\epsilon_j) e^{-\sqrt{2} g \sqrt{t} u_j z_{j-1} W} \Bigg]$$

with $A(1) = A(4) = I$, $A(2) = W$, and $A(3) = L$. In the oriented product, we cannot shift all terms of the form e^{aW} $(a \in \mathbb{R})$ into the left because of the noncommutativity of L and W. However, we can define a mapping $\omega_\epsilon : \{1, \dots, m-1\} \to \{0, 1\}$ by

$$\overleftarrow{\prod_{j=1}^{m-1}} A(\epsilon_j) \exp[-\sqrt{2} g \sqrt{t} u_j z_{j-1} W]$$
$$= \exp\left[-\sum_{j=1}^{m-1} (-1)^{\omega_\epsilon(j)} \sqrt{2} g \sqrt{t} u_j z_{j-1} W \right] \overleftarrow{\prod_{j=1}^{m-1}} A(\epsilon_j) \tag{4.3}$$

because of $WL = -LW$. Set

$$A(\epsilon) = \overleftarrow{\prod_{j=1}^{m-1}} A(\epsilon_j).$$

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LEMMA 4.4. *We have*

$$A(\epsilon) \in \begin{cases} \{\pm I\} & (\text{if } r_2(\epsilon) \text{ is even and } r_3(\epsilon) \text{ is even}), \\ \{\pm W\} & (\text{if } r_2(\epsilon) \text{ is odd and } r_3(\epsilon) \text{ is even}), \\ \{\pm L\} & (\text{if } r_2(\epsilon) \text{ is even and } r_3(\epsilon) \text{ is odd}), \\ \{\pm LW\} & (\text{if } r_2(\epsilon) \text{ is odd and } r_3(\epsilon) \text{ is odd}). \end{cases}$$

Proof. It follows immediately from the relations $W^2 = L^2 = I$ and $WL = -LW$. ∎

By using the mapping ω_ϵ and the Maclaurin expansion

$$e^{a\sqrt{t}W} = \sum_{k=0}^{\infty} \frac{a^k}{k!} t^{k/2} W^k$$

for $a \in \mathbb{R}$ and $t > 0$, the right-hand side of (4.2) is rewritten as

$$\begin{aligned} &\frac{1}{2\pi} \sum_{l_1=0}^{\infty} \cdots \sum_{l_{m-1}=0}^{\infty} \frac{i^{l_1+\cdots+l_{m-1}}}{l_1! \cdots l_{m-1}!} t^{l_1+\cdots+l_{m-1}+m-2-r_1(\epsilon)-(1/2)r_2(\epsilon)} \\ &\quad \times \int_{D_{m-1}} du \int_{(\xi_1,\ldots,\xi_m)\in\mathbb{R}^m} \prod_{j=1}^{m} d\xi_j \int_{(z_0,\ldots,z_{m-1})\in\mathbb{R}^m} \prod_{j=0}^{m-1} dz_j \\ &\quad \times \left\{ \prod_{j=1}^{m-1} (-1)^{l_j} \partial_{z_{j-1}}^{l_j} \delta(z_{j-1} - z_j) \right\} \\ &\quad \times \left\{ \prod_{j=1}^{m-1} (-1)^{l_j} \delta(\xi_j - \xi_m) \right\} \\ &\quad \times \operatorname{tr} \left[e^{-(1-u_1-\cdots-u_{m-1})(\xi_m^2+z_{m-1}^2)/2} \left(\prod_{j=1}^{m-1} \partial_{\xi_j}^{l_j} e^{-u_j\xi_j^2/2} e^{-u_j z_{j-1}^2/2} \right) \right. \\ &\quad \times e^{-(1-u_1-\cdots-u_{m-1})\sqrt{2}g\sqrt{t}z_{m-1}W} \prod_{j=1}^{m-1} e^{-u_j\sqrt{2}g\sqrt{t}(-1)^{\omega_\epsilon(j)} z_{j-1} W} \\ &\quad \left. \times \overleftarrow{\prod_{j=1}^{m-1}} T_{\epsilon_j}(z_{j-1}, z_j) \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi} \sum_{l_1=0}^{\infty} \cdots \sum_{l_{m-1}=0}^{\infty} \frac{i^{l_1+\cdots+l_{m-1}}}{l_1! \cdots l_{m-1}!} t^{l_1+\cdots+l_{m-1}+m-2-r_1(\epsilon)-(1/2)r_2(\epsilon)} \\
&\times \int_{D_{m-1}} du \int_{(\xi_1,\ldots,\xi_m)\in\mathbb{R}^m} \prod_{j=1}^{m} d\xi_j \int_{(z_0,\ldots,z_{m-1})\in\mathbb{R}^m} \prod_{j=0}^{m-1} dz_j \\
&\times \left\{ \prod_{j=1}^{m-1} (-1)^{l_j} \partial_{z_{j-1}}^{l_j} \delta(z_{j-1}-z_j) \right\} \\
&\times \left\{ \prod_{j=1}^{m-1} \delta(\xi_j - \xi_m) \right\} e^{-(1-u_1-\cdots-u_{m-1})\xi_m^2/2} e^{-(1-u_1-\cdots-u_{m-1})z_{m-1}^2/2} \\
&\times \left(\prod_{j=1}^{m-1} e^{-u_j z_{j-1}^2/2} \right) \\
&\times \left\{ \prod_{j=1}^{m-1} (u_j/2)^{l_j/2} H_{l_j}(\sqrt{u_j/2}\,\xi_j) e^{-u_j \xi_j^2/2} \right\} \\
&\times \operatorname{tr} \left[\sum_{k_m=0}^{\infty} \frac{\{-(1-u_1-\cdots-u_{m-1})\sqrt{2}g z_{m-1}\}^{k_m}}{k_m!} \sqrt{t}^{k_m} W^{k_m} \right. \\
&\times \left\{ \prod_{j=1}^{m-1} \sum_{k_j=0}^{\infty} \frac{(-u_j\sqrt{2}g(-1)^{\omega_j(\epsilon)} z_{j-1})^{k_j}}{k_j!} t^{k_j/2} W^{k_j} \right\} \\
&\left. \times \overleftarrow{\prod_{j=1}^{m-1}} T_{\epsilon_j}(z_{j-1}, z_j) \right]. \qquad (4.4)
\end{aligned}
$$

As a consequence, by noting

$$
\begin{aligned}
&\int_{(\xi_1,\ldots,\xi_m)\in\mathbb{R}^m} \prod_{j=1}^{m} d\xi_j \left\{ \prod_{j=1}^{m-1} \delta(\xi_j - \xi_m) \right\} e^{-(1-u_1-\cdots-u_{m-1})\xi_m^2/2} \\
&\quad \times \left\{ \prod_{j=1}^{m-1} (u_j/2)^{l_j/2} H_{l_j}(\sqrt{u_j/2}\,\xi_j) e^{-u_j \xi_j^2/2} \right\}
\end{aligned}
$$

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$$= \int_{\xi\in\mathbb{R}} e^{-\xi^2/2}\left\{\prod_{j=1}^{m-1}(u_j/2)^{l_j/2}H_{l_j}(\sqrt{u_j/2}\xi)\right\}\,d\xi$$

and the analysis made so far, we have the following asymptotic series.

THEOREM 4.5. *For any $m\in\mathbb{Z}_{\geqslant 2}$ and $\epsilon\in\{1,2,3,4\}^{m-1}$, we have*

$$\begin{aligned}
&\operatorname{Tr} K_{m,\epsilon}(t)\\
&\sim_0 \sum_{(l_1,\ldots,l_{m-1})\in\mathbb{Z}_{\geqslant 0}^{m-1}}\ \sum_{(k_1,\ldots,k_m)\in\mathbb{Z}_{\geqslant 0}^{m}} \frac{i^{l_1+\cdots+l_{m-1}}}{2\pi l_1!\cdots l_{m-1}!k_1!\cdots k_m!}\\
&\quad\times t^{l_1+\cdots+l_{m-1}+(1/2)(k_1+\cdots+k_m)+m-2-r_1(\epsilon)-(1/2)r_2(\epsilon)}\\
&\quad\times \int_{D_{m-1}} du \int_{(z_0,\ldots,z_{m-1})\in\mathbb{R}^m}\prod_{j=0}^{m-1}dz_j\left\{\prod_{j=1}^{m-1}(-1)^{l_j}\partial_{z_{j-1}}^{l_j}\delta(z_{j-1}-z_j)\right\}\\
&\quad\times e^{-(1-u_1-\cdots-u_{m-1})z_{m-1}^2/2}\left(\prod_{j=1}^{m-1}e^{-u_jz_{j-1}^2/2}\right)\\
&\quad\times\left(\prod_{j=1}^{m} z_{j-1}^{k_j}\right)\{-\sqrt{2}g(1-u_1-\cdots-u_{m-1})\}^{k_m}\\
&\quad\times\prod_{j=1}^{m-1}\{-\sqrt{2}gu_j(-1)^{\omega_\epsilon(j)}\}^{k_j}\int_{\xi\in\mathbb{R}}e^{-\xi^2/2}\\
&\quad\times\left\{\prod_{j=1}^{m-1}(u_j/2)^{l_j/2}H_{l_j}\left(\sqrt{u_j/2}\xi\right)\right\}d\xi\\
&\quad\times\left\{\prod_{\substack{j=1\\ \epsilon_j=1}}^{m-1}(z_{j-1}^2-z_j^2)/2\right\}\left\{\prod_{\substack{j=1\\ \epsilon_j=2}}^{m-1}\sqrt{2}g(z_{j-1}-z_j)\right\}
\end{aligned}$$

$$\times(-\Delta)^{r_3(\epsilon)}(1/2-\tau)^{r_4(\epsilon)}\operatorname{tr}[W^{k_1+\cdots+k_m}A(\epsilon)]. \tag{4.5}$$

Here $r_k(\epsilon)$ is the cardinality of $\{j\in\{1,\ldots,m-1\}\mid \epsilon_j=k\}$ for each $k\in\{1,2,3,4\}$, a mapping $\omega_\epsilon:\{1,\ldots,m-1\}\to\{0,1\}$ is defined by (4.3), the function $H_{l_j}(x)$ is the l_jth Hermite polynomial defined in Lemma 4.3, *and we set*

$$A(\epsilon) = \overleftarrow{\prod_{j=1}^{m-1}} A(\epsilon_j)$$

with $A(1) = A(4) = I$, $A(2) = W$, *and* $A(3) = L$.

Let $c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)}$ be the coefficient of

$$t^{l_1+\cdots+l_{m-1}+(1/2)(k_1+\cdots+k_m)+m-2-r_1(\epsilon)-(1/2)r_2(\epsilon)}$$

in (4.5). Then, we obtain the following series of vanishing results.

LEMMA 4.6. *The coefficient* $c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)}$ *vanishes if* $l_1 + \cdots + l_{m-1}$ *is odd.*

Proof. If $l_1 + \cdots + l_{m-1}$ is odd, by $H_{l_j}(-x) = (-1)^{l_j} H_{l_j}(x)$ the product $\prod_{j=1}^{m-1} H_{l_j}(x)$ is an odd function in x, and hence we have

$$\int_{\xi\in\mathbb{R}} e^{-\xi^2/2} \prod_{j=1}^{m-1} H_{l_j}(\sqrt{u_j/2}\xi)\, d\xi = 0.$$

This completes the proof. □

LEMMA 4.7. *The coefficient* $c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)}$ *vanishes if* $k_1 + \cdots + k_m - r_2(\epsilon)$ *is odd. In particular, any coefficients of* $\sqrt{t}^{2k+1}$ *of* $\operatorname{Tr} K_{m,\epsilon}(t)$ *for any* $k \in \mathbb{Z}_{\geqslant 0}$ *vanish.*

Proof. If $k_1 + \cdots + k_m - r_2(\epsilon)$ is odd, Lemma 4.4 yields that $W^{k_1+\cdots+k_m} A(\epsilon)$ is equal to $\pm W$ or $\pm LW$, whose trace is zero. This completes the proof. □

LEMMA 4.8. *We have* $\operatorname{Tr} K_{m,\epsilon}(t) = 0$ *if* $r_3(\epsilon)$ *is odd.*

Proof. If $r_3(\epsilon)$ is odd, for any $(k_1, \ldots, k_m) \in \mathbb{Z}^m_{\geqslant 0}$, by Lemma 4.4, $W^{k_1+\cdots+k_m} A(\epsilon)$ is equal to $\pm L$ or $\pm LW$, whose trace is zero. From this and (4.5), we have the desired assertion. □

LEMMA 4.9. *We have* $c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)} = 0$ *if there exists* $j \in \{1.\ldots, m-1\}$ *such that* $\epsilon_j \in \{1, 2\}$ *and* $l_j = 0$. *In particular, we have* $c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)} = 0$ *if* $l_1 + \cdots + l_{m-1} < r_1(\epsilon) + r_2(\epsilon)$.

Proof. We obtain the assertion by noting the factor $\delta(z_{j-1}-z_j)(z_{j-1}-z_j)$. ∎

By Lemmas 4.6–4.8, we have the following.

LEMMA 4.10. *For any* $m \in \mathbb{Z}_{\geqslant 2}$, $\epsilon \in \{1,2,3,4\}^{m-1}$, $(l_1,\ldots,l_{m-1},k_1,\ldots,k_m) \in \mathbb{Z}_{\geqslant 0}^{2m-1}$, *we have*

$$\begin{aligned}
&c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)}\\
&\quad = \frac{i^{l_1+\cdots+l_{m-1}}}{2\pi l_1!\cdots l_{m-1}!k_1!\cdots k_m!}\int_{D_{m-1}} du \int_{(z_0,\ldots,z_{m-1})\in\mathbb{R}^m}\\
&\qquad \times \prod_{j=0}^{m-1} dz_j \left\{\prod_{j=1}^{m-1} \partial_{z_{j-1}}^{l_j}\delta(z_{j-1}-z_j)\right\}\\
&\qquad \times e^{-(1-u_1-\cdots-u_{m-1})z_{m-1}^2/2}\left(\prod_{j=1}^{m-1} e^{-u_j z_{j-1}^2/2}\right)\\
&\qquad \times \left(\prod_{j=1}^{m} z_{j-1}^{k_j}\right)\{-\sqrt{2}g(1-u_1-\cdots-u_{m-1})\}^{k_m}\\
&\qquad \times \prod_{j=1}^{m-1}\{-\sqrt{2}gu_j(-1)^{\omega_\epsilon(j)}\}^{k_j}\int_{\xi\in\mathbb{R}} e^{-\xi^2/2}\\
&\qquad \times \left\{\prod_{j=1}^{m-1}(u_j/2)^{l_j/2}H_{l_j}(\sqrt{u_j/2}\xi)\right\} d\xi\\
&\qquad \times \left\{\prod_{\substack{j=1\\ \epsilon_j=1}}^{m-1}(z_{j-1}^2-z_j^2)/2\right\}\left\{\prod_{\substack{j=1\\ \epsilon_j=2}}^{m-1}\sqrt{2}g(z_{j-1}-z_j)\right\}\\
&\qquad \times \Delta^{r_3(\epsilon)}(1/2-\tau)^{r_4(\epsilon)}\mathrm{tr}[W^{k_1+\cdots+k_m}A(\epsilon)].
\end{aligned}$$

Moreover, it is an element of $\mathbb{R}[g^2,\Delta^2,\tau]$.

Proof. By Lemmas 4.6–4.8, we may assume that $l_1+\cdots+l_{m-1}$, $k_1+\cdots+k_m-r_2(\epsilon)$ and $r_3(\epsilon)$ are all even. Then, the assertion follows from $\prod_{j=1}^{m-1}(-1)^{l_j}=1$, $i^{l_1+\cdots+l_{m-1}}\in\{\pm 1\}$ and $(-\Delta)^{r_3(\epsilon)}=\Delta^{r_3(\epsilon)}$. ∎

Proof of Theorem 4.1. Let us take any $m \in \mathbb{Z}_{\geqslant 2}$, $\epsilon \in \{1, 2, 3, 4\}$, and $q \in \frac{1}{2}\mathbb{Z}$, and set

$$c_q^{(m,\epsilon)} = \sum_{\substack{(l_1,\ldots,l_{m-1},k_1,\ldots,k_m)\in\mathbb{Z}_{\geqslant 0}^{2m-1} \\ l_1+\cdots+l_{m-1}+(1/2)(k_1+\cdots+k_m)+m-2-r_1(\epsilon)-(1/2)r_2(\epsilon)=q}} \times c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)} \tag{4.6}$$

and

$$c_q^{(m)} = \sum_{\epsilon\in\{1,2,3,4\}^{m-1}} c_q^{(m,\epsilon)}. \tag{4.7}$$

By virtue of Lemma 4.7, we have $c_q^{(m,\epsilon)} = 0$ unless $q \in \mathbb{Z}$. Moreover, if $l_1 + \cdots + l_{m-1} + \frac{1}{2}(k_1 + \cdots + k_m) + m - 2 - r_1(\epsilon) - \frac{1}{2}r_2(\epsilon) < m - 2$, we have $c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)} = 0$ with the aid of Lemma 4.9. Thus we obtain $c_q^{(m,\epsilon)} = 0$ if $q < m - 2$. This completes the proof. □

§5. Meromorphic continuations

In this section, we prove Theorem 1.1. Recall that τ is any fixed real number such that $\tau > g^2 + \Delta$ as in Section 3. Theorem 1.1(1) is a consequence of the following theorem.

THEOREM 5.1. *There exists a sequence $\{C_{H,\tau}(k)\}_{k\in\mathbb{Z}_{\geqslant 0}}$ of complex numbers such that*

$$\zeta_H(s;\tau) = \frac{1}{\Gamma(s)}\left\{\frac{2}{s-1} + \sum_{k=1}^{\infty}\frac{2g^{2k}}{k!}\frac{1}{s+k-1} + \sum_{k=0}^{n-2}\frac{C_{H,\tau}(k)}{s+k} + h_1(s) + h_2(s) + Z_\infty(s)\right\}, \quad \mathrm{Re}(s) > 1$$

for any $n \in \mathbb{Z}_{\geqslant 2}$. Here, $Z_\infty(s)$ is the entire function treated in Lemma 3.2, $h_1(s)$ is a holomorphic function on $\mathrm{Re}(s) > -n$ such that $h_1(s) = \mathcal{O}(1/(\mathrm{Re}(s) + n))$ on the region $\mathrm{Re}(s) > -n$, and $h_2(s)$ is a holomorphic function on $\mathrm{Re}(s) > -n/2$ such that $h_2(s) = \mathcal{O}(1/(\mathrm{Re}(s) + n/2))$ on the region $\mathrm{Re}(s) > -n/2$.

In particular, $\zeta_H(s;\tau)$ has a meromorphic continuation to $\mathbb{C}$ and is holomorphic on $\mathbb{C} - \{1\}$. Furthermore, $s = 1$ is a simple pole with $\mathrm{Res}_{s=1}\,\zeta_H(s;\tau) = 2$.

Proof. Recall $\zeta_H(s;\tau) = (1/\Gamma(s)) \int_0^\infty t^{s-1} \operatorname{Tr} K(t)\, dt$. Since $Z_\infty(t) = \int_1^\infty t^{s-1} \operatorname{Tr} K(t)\, dt$ is entire by Lemma 3.2, we only have to consider $\int_0^1 t^{s-1} \operatorname{Tr} K(t)\, dt$. As for the first term of (3.2), Lemma 3.4 yields that

$$\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \operatorname{Tr} K_1(t)\, dt = \frac{1}{\Gamma(s)} \left(\frac{2}{s-1} + \sum_{k=1}^{\infty} \frac{2g^{2k}}{k!} \frac{1}{s+k-1} \right)$$

is entire. As for the third term of (3.2), by Lemma 3.7, the integral $\int_0^1 |t^{s-1} \operatorname{Tr} R_{n+1}(t)|\, dt$ for $\operatorname{Re}(s) > -n/2$ is majorized by

$$\int_0^1 t^{\operatorname{Re}(s)-1+n/2}\, dt = \frac{1}{\operatorname{Re}(s)+n/2}$$

up to a positive constant, and hence the function $h_2(s) = \int_0^1 t^{s-1} \operatorname{Tr} R_{n+1}(t)\, dt$ is holomorphic on $\operatorname{Re}(s) > -n/2$. By setting

$$C_{H,\tau}(k) = \sum_{m=2}^{k+2} c_k^{(m)}, \tag{5.1}$$

Theorem 4.1 gives us

$$\begin{aligned} \sum_{m=2}^{n} \operatorname{Tr} K_m(t) &= \sum_{m=2}^{n} \sum_{q=m-2}^{n-2} c_q^{(m)} t^q + \mathcal{O}(t^{n-1}) \\ &= \sum_{k=0}^{n-2} C_{H,\tau}(k) t^k + \mathcal{O}(t^{n-1}), \quad (t \to +0). \end{aligned}$$

Then the integral of the second term of (3.2) is evaluated as

$$\begin{aligned} \int_0^1 t^{s-1} \sum_{m=2}^{n} \operatorname{Tr} K_m(t)\, dt &= \int_0^1 t^{s-1} \sum_{k=0}^{n-2} C_{H,\tau}(k) t^k\, dt \\ &\quad + \int_0^1 t^{s-1} \left\{ \sum_{m=2}^{n} \operatorname{Tr} K_m(t) - \sum_{k=0}^{n-2} C_{H,\tau}(k) t^k \right\} dt. \end{aligned}$$

As the first term is evaluated as $\sum_{k=0}^{n-2} (C_{H,\tau}(k)/(s+k))$ and the second term as above is majorized by

$$\int_0^1 t^{\operatorname{Re}(s)-1+n}\, dt = \frac{1}{\operatorname{Re}(s)+n}$$

up to a positive constant when $\mathrm{Re}(s) > -n$, the integral

$$h_1(s) = \int_0^1 t^{s-1} \left\{ \sum_{m=2}^{n} \mathrm{Tr}\, K_m(t) - \sum_{k=0}^{n-2} C_{H,\tau}(k) t^k \right\} dt$$

is holomorphic on $\mathrm{Re}(s) > -n$. As a consequence, we have the theorem. ∎

By Theorem 5.1, the function $\zeta_H(s;\tau)$ is holomorphic at $s=0$ for any $\tau \in \mathbb{R}$ such that $\tau > g^2 + \Delta$. Thus, $\prod_{\lambda \in \mathrm{Spec}(H)} (\tau + \lambda)$ can be defined for any $\tau \in \mathbb{C}$ by [22, Theorem 1]. Here is a formula of the zeta regularized product of $\mathrm{Spec}(H + \tau I)$.

Proposition 5.2. *For any $n \in \mathbb{Z}_{\geqslant 2}$ and any real number τ such that $\tau > g^2 + \Delta$, we have*

$$\prod_{\lambda \in \mathrm{Spec}(H)} (\tau + \lambda) = \exp\left(-2 + \sum_{k=2}^{\infty} \frac{2g^{2k}}{k!(k-1)} + \gamma(2g^2 + 1 - 2\tau) + \sum_{k=1}^{n-2} \frac{C_{H,\tau}(k)}{k} \right) \times e^{F_{H,n}(0;\tau)},$$

where $\gamma = -\Gamma'(1)$ is Euler's constant.

Proof. By using Theorem 1.1(1), a direct computation gives us

$$\frac{d}{ds} \zeta_H(s;\tau)|_{s=0} = -2 + \sum_{k=2}^{\infty} \frac{2g^{2k}}{k!(k-1)} + \gamma\{2g^2 + C_{H,\tau}(0)\} + \sum_{k=1}^{n-2} \frac{C_{H,\tau}(k)}{k} + F_{H,n}(0;\tau).$$

We shall compute $C_{H,\tau}(0)$. By (4.6), (4.7), and (5.1), $C_{H,\tau}(0)$ is expressed as $C_{H,\tau}(0) = c_0^{(2)} = \sum_{\epsilon=1}^{4} c_0^{(2,\epsilon)}$ with

$$c_0^{(2,\epsilon)} = \sum_{\substack{(l,k_1,k_2) \in \mathbb{Z}_{\geqslant 0}^3 \\ l + (1/2)(k_1+k_2) - r_1(\epsilon) - (1/2) r_2(\epsilon) = 0}} c_{l,(k_1,k_2)}^{(2,\epsilon)}.$$

For $\epsilon \in \{1,2\}$, we have $c_0^{(2,1)} = c_0^{(2,2)} = 0$ by Lemma 6.1 below. For $\epsilon = 3$, Lemma 4.8 yields $c_0^{(2,3)} = 0$. For $\epsilon = 4$, we have $c_0^{(2,4)} = 1 - 2\tau$ by the proof of Proposition 5.3 below. Hence, we obtain $C_{H,\tau}(0) = 1 - 2\tau$. ∎

https://doi.org/10.1017/nmj.2016.62 Published online by Cambridge University Press

5.1 Simple examples

We give an example of $\zeta_H(s;\tau)$ in terms of the Hurwitz zeta function $\zeta(s;a)=\sum_{n=0}^{\infty}(n+a)^{-s}$ for $a>0$.

The case $\Delta=0$: In the case $\Delta=0$, we consider H' defined in (2.3) instead of H. We have $H'=B-\tau I$, where B is the operator defined in the proof of Lemma 2.4. Hence we have $\mathrm{Spec}(H)=\mathrm{Spec}(B-\tau I)=\{n-g^2 \mid n\in\mathbb{Z}_{\geqslant 0}\}$ with multiplicity 2 and

$$\zeta_H(s;\tau)=2\sum_{n=0}^{\infty}\frac{1}{(n-g^2+\tau)^s}=2\zeta(s;\tau-g^2).$$

From this, the kth Rabi–Bernoulli polynomial, which is defined by (1.1), is given by $R_k(g,0;x)=B_k(x-g^2)$ for any $k\in\mathbb{Z}_{\geqslant 1}$, where $B_k(x)$ is the kth Bernoulli polynomial as in the Introduction.

The case $g=0$: Another simple example is the case $g=0$. Assume $\Delta>0$. In this case, the expression

$$H'=\frac{-\partial_x^2+x^2-1}{2}I+\Delta L$$

gives us $\mathrm{Spec}(H)=\{n+\Delta \mid n\in\mathbb{Z}_{\geqslant 0}\}\cup\{n-\Delta \mid n\in\mathbb{Z}_{\geqslant 0}\}$ with multiplicity 1. As a result, we obtain

$$\begin{aligned}\zeta_H(s;\tau)&=\sum_{n=0}^{\infty}\frac{1}{(n+\Delta+\tau)^s}+\sum_{n=0}^{\infty}\frac{1}{(n-\Delta+\tau)^s}\\&=\zeta(s;\tau+\Delta)+\zeta(s;\tau-\Delta)\end{aligned}$$

and $R_k(0,\Delta;x)=\frac{1}{2}(B_k(x+\Delta)+B_k(x-\Delta))$ for any $k\in\mathbb{Z}_{\geqslant 1}$.

As we see as above, $R_k(g,\Delta;x)$ is monic and its degree is equal to k as a polynomial in x when $g=0$ or $\Delta=0$. We have the following for general $g\geqslant 0$ and $\Delta\geqslant 0$.

PROPOSITION 5.3. *For any $k\in\mathbb{Z}_{\geqslant 1}$, the degree of $R_k(g,\Delta;x)$ with respect to x is equal to k. Furthermore, $R_k(g,\Delta;x)$ is monic as a polynomial in x.*

Proof. By (5.1), it suffices to analyze $C_{H,\tau}(k)=\sum_{m=2}^{k+2}c_k^{(m)}$. With the aid of (4.6) and (4.7), it suffices to study the term for $\epsilon=(4,\ldots,4)\in\{1,2,3,4\}^{k+1}$ appearing in $c_k^{(k+2)}$. Put $\mathbf{0}_{m-1}=(0,\ldots,0)\in\mathbb{Z}_{\geqslant 0}^{m-1}$ and

$\mathbf{4}_{m-1} = (4, \ldots, 4) \in \{1, 2, 3, 4\}^{m-1}$ for any $m \in \mathbb{Z}_{\geqslant 2}$. By Lemma 4.10, we easily obtain

$$c_k^{(k+2,\mathbf{4}_{k+1})} = c_{\mathbf{0}_{k+1},\mathbf{0}_{k+2}}^{(k+2,\mathbf{4}_{k+1})} = \int_{D_{k+1}} du(1/2-\tau)^{k+1} \times 2 = \frac{2}{(k+1)!}(1/2-\tau)^{k+1},$$

and thus we are done. ∎

5.2 Rationality of coefficients

In this subsection, we shall prove Theorem 1.1(2) by combining (4.6), (4.7), and (5.1) with the following theorem.

THEOREM 5.4. *For any $m \in \mathbb{Z}_{\geqslant 2}$, $\epsilon \in \{1, 2, 3, 4\}^{m-1}$, and $(l_1, \ldots, l_{m-1}, k_1, \ldots, k_m) \in \mathbb{Z}_{\geqslant 0}^{2m-1}$, we have*

$$c_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)}^{(m,\epsilon)} \in \mathbb{Q}[g^2, \Delta^2, \tau].$$

We remark that $C_{H,\tau}(k) \in \mathbb{R}[g^2, \Delta^2, \tau]$ is obvious from combining (4.6), (4.7), and (5.1) with Lemma 4.10. The following two lemmas will be used later in order to prove Theorem 5.4.

LEMMA 5.5. *For any polynomial $P(u_1, \ldots, u_{m-1}, \xi) \in \mathbb{Q}[u_1, \ldots, u_{m-1}, \xi]$, we have*

$$\int_{\xi\in\mathbb{R}} e^{-\xi^2/2} P(u_1, \ldots, u_{m-1}, \xi)\, d\xi \in \sqrt{2\pi}\mathbb{Q}[u_1, \ldots, u_{m-1}].$$

Proof. By the formula

$$\begin{aligned} \int_{\xi\in\mathbb{R}} e^{-\xi^2/2}\xi^{2n}\, d\xi &= 2^{n+1/2}\int_0^\infty e^{-x}x^{n-1/2}\, dx \\ &= 2^{n+1/2}\Gamma(n+1/2) = (2n-1)!!\sqrt{2\pi} \end{aligned} \tag{5.2}$$

for any $n \in \mathbb{Z}_{\geqslant 0}$, we obtain the desired assertion. ∎

LEMMA 5.6. *For any polynomial $P(u_1, \ldots, u_{m-1}) \in \mathbb{Q}[u_1, \ldots, u_{m-1}]$, we have*

$$\int_{D_{m-1}} P(u_1, \ldots, u_{m-1})\, du_{m-1} \cdots du_1 \in \mathbb{Q},$$

where D_{m-1} is the subset of $\mathbb{R}^{m-1}$ defined in Section 4.

Proof. The assertion follows from the formula

$$\int_{D_{m-1}} \left(\prod_{j=1}^{m-1} u_j^{a_j} \right) du_{m-1} \cdots du_1 = \frac{\prod_{j=1}^{m-1} a_j!}{(\sum_{j=1}^{m-1} a_j + m - 1)!} \in \mathbb{Q}$$

for any $(a_1, \ldots, a_{m-1}) \in \mathbb{Z}_{\geqslant 0}^{m-1}$ (cf. [5, 4.634[8]]). ∎

Let us take any $m \in \mathbb{Z}_{\geqslant 2}$, $\epsilon \in \{1, 2, 3, 4\}^{m-1}$, and $(l_1, \ldots, l_{m-1}, k_1, \ldots, k_m) \in \mathbb{Z}_{\geqslant 0}^{2m-1}$. We may assume that $l_1 + \cdots + l_{m-1}$, $k_1 + \cdots + k_m + r_2(\epsilon)$, and $r_3(\epsilon)$ are all even by Lemmas 4.6–4.8.

We show a refined formula of $c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)}$. Set $D_1(\epsilon) = \delta_{\epsilon,1}$, $D_2(\epsilon) = \delta_{\epsilon,2}$, and $D_{34}(\epsilon) = \delta_{\epsilon,3} + \delta_{\epsilon,4}$ for any $\epsilon \in \{1, 2, 3, 4\}$, where $\delta_{a,b}$ is the Kronecker delta. For our purpose, we calculate the following integral appearing in the formula of $c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)}$ in Lemma 4.10:

$$\begin{aligned}
&\int_{(z_0,\ldots,z_{m-1})\in\mathbb{R}^m} \left(\prod_{j=0}^{m-1} dz_j \right) e^{-(1-u_1-\cdots-u_{m-1})z_{m-1}^2/2} z_{m-1}^{k_m} \\
&\quad \times \prod_{j=1}^{m-1} (-1)^{l_j} \partial_{z_{j-1}}^{l_j} \delta(z_{j-1} - z_j) \prod_{j=1}^{m-1} e^{-u_j z_{j-1}^2/2} z_{j-1}^{k_j} \\
&\quad \times \{D_1(\epsilon_j)(z_{j-1}^2 - z_j^2)/2 + D_2(\epsilon_j)(z_{j-1} - z_j) + D_{34}(\epsilon_j)\}.
\end{aligned} \tag{5.3}$$

First, let us consider the z_0-integral. By a direct computation, we have

$$\begin{aligned}
&\int_{z_0\in\mathbb{R}} dz_0 \delta(z_0 - z_1) \partial_{z_0}^{l_1} \\
&\quad \times [e^{-u_1 z_0^2/2} z_0^{k_1} \{D_1(\epsilon_1)(z_0^2 - z_1^2)/2 + D_2(\epsilon_1)(z_0 - z_1) + D_{34}(\epsilon_1)\}] \\
&= \int_{z_0\in\mathbb{R}} \delta(z_0 - z_1) e^{-u_1 z_0^2/2} P^{(\epsilon_1)}_{l_1,k_1}(u_1; z_0)\, dz_0 = e^{-u_1 z_1^2/2} P^{(\epsilon_1)}_{l_1,k_1}(u_1; z_1),
\end{aligned}$$

where we set

$$\begin{aligned}
&P^{(\epsilon_1)}_{l_1,k_1}(u_1; z_1) \\
&= \sum_{a=0}^{l_1} \binom{l_1}{a} (-1)^{l_1-a} (u_1/2)^{(l_1-a)/2} H_{l_1-a}(\sqrt{u_1/2} z_0) \\
&\quad \times \left[\frac{k_1!}{(k_1-a)!} z_0^{k_1-a} \{D_1(\epsilon_1)(z_0^2 - z_1^2)/2 + D_2(\epsilon_1)(z_0 - z_1) + D_{34}(\epsilon_1)\} \right.
\end{aligned}$$

$$+\binom{a}{1}\frac{k_1!}{(k_1-a+1)!}z_0^{k_1-a+1}\{D_1(\epsilon_1)z_0+D_2(\epsilon_1)\}$$

$$+\binom{a}{2}\frac{k_1!}{(k-a+2)!}z_0^{k_1-a+2}D_1(\epsilon_1)\Bigg]\Bigg|_{z_0=z_1}. \tag{5.4}$$

Then, $P^{(\epsilon_1)}_{l_1,k_1}(u_1;z_1)\in\mathbb{Q}[u_1,z_1]$ follows easily. Thus, (5.3) is evaluated as

$$\int_{(z_1,\dots,z_{m-1})\in\mathbb{R}^{m-1}}\prod_{j=1}^{m-1}dz_j$$

$$\times\left\{\prod_{j=2}^{m-1}(-1)^{l_j}\partial^{l_j}_{z_{j-1}}\delta(z_{j-1}-z_j)\right\}e^{-(1-u_1-\cdots-u_{m-1})z^2_{m-1}/2}z^{k_m}_{m-1}$$

$$\times\Bigg[\prod_{j=2}^{m-1}e^{-u_jz^2_{j-1}/2}z^{k_j}_{j-1}\{D_1(\epsilon_j)(z^2_{j-1}-z_j^2)/2$$

$$+D_2(\epsilon_j)(z_{j-1}-z_j)+D_{34}(\epsilon_j)\}\Bigg]e^{-u_1z_1^2/2}P^{\epsilon_1}_{l_1,k_1}(u_1;z_1). \tag{5.5}$$

Next let us consider the z_1-integral

$$\int_{z_1\in\mathbb{R}}\delta(z_2-z_1)\partial^{l_2}_{z_1}[e^{-(u_1+u_2)z_1^2/2}z_1^{k_2}$$

$$\times\{D_1(\epsilon_2)(z_1^2-z_2^2)/2+D_2(\epsilon_2)(z_1-z_2)+D_{34}(\epsilon_2)\}P^{(\epsilon_1)}_{l_1,k_1}(u_1;z_1)]\ dz_1.$$

We set

$$P^{(\epsilon_1,\epsilon_2)}_{(l_1,l_2,)(k_1,k_2)}(u_1,u_2;z_2)$$

$$=e^{(u_1+u_2)z_1^2/2}\partial^{l_2}_{z_1}[e^{-(u_1+u_2)z_1^2/2}z_1^{k_2}\{D_1(\epsilon_2)(z_1^2-z_2^2)/2$$

$$+D_2(\epsilon_2)(z_1-z_2)+D_{34}(\epsilon_2)\}P^{(\epsilon_1)}_{l_1,k_1}(u_1;z_1)]\,|_{z_1=z_2}. \tag{5.6}$$

Then, $P^{(\epsilon_1,\epsilon_2)}_{(l_1,l_2,)(k_1,k_2)}(u_1,u_2;z_2)$ is contained in $\mathbb{Q}[u_1,u_2,z_2]$. Thus (5.5) is transformed to

$$\int_{(z_2,\dots,z_{m-1})\in\mathbb{R}^{m-2}}\prod_{j=2}^{m-1}dz_j\left\{\prod_{j=3}^{m-1}\partial^{l_j}_{z_{j-1}}\delta(z_{j-1}-z_j)\right\}$$

$$
\begin{aligned}
&\times e^{-(1-u_1-\cdots-u_{m-1})z_{m-1}^2/2} z_{m-1}^{k_m} \\
&\times \left[\prod_{j=3}^{m-1} e^{-u_j z_{j-1}^2/2} z_{j-1}^{k_j} \{D_1(\epsilon_j)(z_{j-1}^2 - z_j^2)/2\right. \\
&\qquad \left. + D_2(\epsilon_j)(z_{j-1}-z_j) + D_{34}(\epsilon_j)\}\right] \\
&\times e^{-(u_1+u_2)z_2^2/2} P^{(\epsilon_1,\epsilon_2)}_{(l_1,l_2),(k_1,k_2)}(u_1,u_2;z_2). \qquad (5.7)
\end{aligned}
$$

In general, we define a polynomial $P^{(\epsilon_1,\ldots,\epsilon_j)}_{(l_1,\ldots,l_j),(k_1,\ldots,k_j)}(u_1,\ldots,u_j;z_j) \in \mathbb{Q}[u_1,\ldots,u_j,z_j]$ by (5.4) and the recurrence relation

$$
\begin{aligned}
&P^{(\epsilon_1,\ldots,\epsilon_j)}_{(l_1,\ldots,l_j),(k_1,\ldots,k_j)}(u_1,\ldots,u_j;z_j) \\
&\quad = \Big\{ e^{(u_1+\cdots+u_j)z_{j-1}^2/2} (\partial_{z_{j-1}}^{l_{j-1}}) \Big[e^{-(u_1+\cdots+u_j)z_{j-1}^2/2} z_{j-1}^{k_j} \\
&\qquad \times (D_1(\epsilon_j)(z_{j-1}^2 - z_j^2)/2 + D_2(\epsilon_j)(z_{j-1}-z_j) + D_{34}(\epsilon_2)) \\
&\qquad \times P^{(\epsilon_1,\ldots,\epsilon_{j-1})}_{(l_1,\ldots,l_{j-1}),(k_1,\ldots,k_{j-1})}(u_1,\ldots,u_{j-1};z_{j-1}) \Big] \Big\} \Big|_{z_{j-1}=z_j}. \qquad (5.8)
\end{aligned}
$$

In a similar fashion to computations for (5.5) and (5.7), integral (5.3) can be described as

$$
\begin{aligned}
&\int_{z_{m-1}\in\mathbb{R}} e^{-(1-u_1-\cdots-u_{m-1})z_{m-1}^2/2} z_{m-1}^{k_m} \\
&\qquad \times e^{-(u_1+\cdots+u_{m-1})z_{m-1}^2/2} P^{(\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_{m-1})} \\
&\qquad \times (u_1,\ldots,u_{m-1};z_{m-1})\, dz_{m-1} \\
&= \int_{z\in\mathbb{R}} e^{-z^2/2} z^{k_m} P^{(\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_{m-1})}(u_1,\ldots,u_{m-1};z)\, dz. \qquad (5.9)
\end{aligned}
$$

Hence we obtain the following refined expression of Lemma 4.10.

Lemma 5.7. *For any* $m \in \mathbb{Z}_{\geqslant 2}$, $\epsilon \in \{1,2,3,4\}^{m-1}$, *and* $(l_1,\ldots,l_{m-1}, k_1,\ldots,k_m) \in \mathbb{Z}_{\geqslant 0}^{2m-1}$, *we have*

$$
\begin{aligned}
&c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)} \\
&\quad = \frac{i^{l_1+\cdots+l_{m-1}}}{2\pi l_1!\cdots l_{m-1}! k_1!\cdots k_m!}
\end{aligned}
$$

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$$
\begin{aligned}
&\times \int_{D_{m-1}} du(1-u_1-\cdots-u_{m-1})^{k_m} \left(\prod_{j=1}^{m-1} u_j^{k_j}\right) \\
&\times \int_{z\in\mathbb{R}} e^{-z^2/2} z^{k_m} P^{(\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_{m-1})}(u_1,\ldots,u_{m-1};z)\, dz \\
&\times \int_{\xi\in\mathbb{R}} e^{-\xi^2/2} \left\{\prod_{j=1}^{m-1} (u_j/2)^{l_j/2} H_{l_j}(\sqrt{u_j/2}\xi)\right\} d\xi \\
&\times \left\{\prod_{j=1}^{m-1} (-1)^{\omega_\epsilon(j)k_j}\right\} (-1)^{k_1+\cdots+k_m} \\
&\times (\sqrt{2}g)^{k_1+\cdots+k_m+r_2(\epsilon)} \Delta^{r_3(\epsilon)} (1/2-\tau)^{r_4(\epsilon)} \mathrm{tr}[W^{k_1+\cdots+k_m} A(\epsilon)].
\end{aligned}
$$

Here $P^{(\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_{m-1})}(u_1,\ldots,u_{m-1};z) \in \mathbb{Q}[u_1,\ldots,u_{m-1},z]$ *is the polynomial determined by (5.4) and (5.8).*

Proof of Theorem 5.4. It is obvious that

$$
\prod_{j=1}^{m-1} (u_j/2)^{l_j/2} H_{l_j}(\sqrt{u_j/2}\xi) \in \mathbb{Q}[u_1,\ldots,u_{m-1},\xi]
$$

from the fact that H_{l_j} is an even (resp. odd) function if l_j is even (resp. odd). Combining this with Lemma 5.5, there exists a polynomial $Q_1(u_1,\ldots,u_{m-1})$ such that

$$
\begin{aligned}
&\int_{\xi\in\mathbb{R}} e^{-\xi^2/2} \prod_{j=1}^{m-1} (u_j/2)^{l_j/2} H_{l_j}(\sqrt{u_j/2}\xi)\, d\xi \\
&\qquad = \sqrt{2\pi} Q_1(u_1,\ldots,u_{m-1}) \in \sqrt{2\pi}\mathbb{Q}[u_1,\ldots,u_{m-1}].
\end{aligned}
$$

Thus, by Lemma 5.7, the coefficient $c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)}$ is contained in

$$
\begin{aligned}
&(2\pi)^{-1} \int_{D_{m-1}} du(1-u_1-\cdots-u_{m-1})^{k_m} \left(\prod_{j=1}^{m-1} u_j^{k_j}\right) \\
&\times \int_{z\in\mathbb{R}} e^{-z^2/2} z^{k_m} P^{(\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_{m-1})}(u_1,\ldots,u_{m-1};z)\, dz \\
&\times \sqrt{2\pi} Q_1(u_1,\ldots,u_{m-1}) \times (\sqrt{2}g)^{k_1+\cdots+k_m+r_2(\epsilon)} \Delta^{r_3(\epsilon)} (1/2-\tau)^{r_4(\epsilon)} \mathbb{Q}.
\end{aligned}
$$

Moreover, by Lemma 5.5, there exists a polynomial $Q_2(u_1, \ldots, u_{m-1})$ such that

$$\int_{z\in\mathbb{R}} e^{-z^2/2} z^{k_m} P^{(\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_{m-1})}(u_1, \ldots, u_{m-1}; z)\, dz \\ = \sqrt{2\pi} Q_2(u_1, \ldots, u_{m-1}) \in \sqrt{2\pi}\mathbb{Q}[u_1, \ldots, u_{m-1}].$$

Finally, $c^{(m,\epsilon)}_{(l_1,\ldots,l_{m-1}),(k_1,\ldots,k_m)}$ is contained in

$$(2\pi)^{-1}\sqrt{2\pi}^2 \int_{D_{m-1}} (1 - u_1 - \cdots - u_{m-1})^{k_m} \left(\prod_{j=1}^{m-1} u_j^{k_j}\right) \\ \times \left\{\prod_{j=1}^{2} Q_j(u_1, \ldots, u_{m-1})\right\} du_{m-1} \cdots du_1 \\ \times (\sqrt{2}g)^{k_1+\cdots+k_m+r_2(\epsilon)} \Delta^{r_3(\epsilon)} (1/2 - \tau)^{r_4(\epsilon)} \mathbb{Q} \\ \subset (\sqrt{2}g)^{k_1+\cdots+k_m+r_2(\epsilon)} \Delta^{r_3(\epsilon)} (1/2 - \tau)^{r_4(\epsilon)} \mathbb{Q}$$

by virtue of Lemma 5.6. Consequently, we obtain Theorem 5.4. □

§6. Examples of Rabi–Bernoulli polynomials

It seems difficult to give a simple formula of $R_k(g, \Delta; x)$ for a general $k \in \mathbb{Z}_{\geqslant 1}$, although we can explicitly compute it for any fixed k by definition. In this section, we give simple formulas of $R_k(g, \Delta; x)$ for $k \in \{1, 2, 3\}$. By Proposition 5.2, the first Rabi–Bernoulli polynomial is given by $R_1(g, \Delta; x) = x - 1/2 - g^2 = B_1(x - g^2)$.

For preparation, we give another vanishing result on $\operatorname{Tr} K_{2,\epsilon}(t)$ in addition to Lemmas 4.6–4.9. By Lemma 4.8, we have $\operatorname{Tr} K_{2,3}(t) = 0$ for $\epsilon = 3$, that is, $c^{(2,3)}_q = 0$ for all $q \in \mathbb{Z}_{\geqslant 0}$. Such a vanishing is still true for $\epsilon \in \{1, 2\}$.

Lemma 6.1. *We have* $\operatorname{Tr} K_{2,1}(t) = 0$ *and* $\operatorname{Tr} K_{2,2}(t) = 0$. *In particular, we have* $c^{(2,1)}_q = c^{(2,2)}_q = 0$ *for any* $q \in \mathbb{Z}_{\geqslant 0}$. *In particular,* $\operatorname{Tr} K_2(t) = \operatorname{Tr} K_{2,4}(t)$ *holds.*

Proof. We give a proof only in the case $\epsilon = 1$. The case $\epsilon = 2$ is proved in a similar fashion. By using Lemmas 3.4 and 3.5, the trace of $K_{2,1}(t)$ is evaluated as

$$\begin{aligned}\operatorname{Tr} K_{2,1}(t) = \int_0^t du \int_{(z_0,z_1)\in\mathbb{R}^2} \\ \times \operatorname{tr}\Bigg[\frac{1}{\sqrt{2\pi(t-u)}} e^{-(z_0-z_1)^2/2(t-u)-(t-u)z_1^2/2} \exp(-\sqrt{2}g(t-u)z_1 W) \\ \times \frac{1}{\sqrt{2\pi u}}\frac{z_0^2-z_1^2}{2} e^{-(z_1-z_0)^2/2u - uz_0^2/2}\exp(-\sqrt{2}guz_0W)\Bigg] dz_0\, dz_1.\end{aligned}$$

By the change of variables $(z_0, z_1) \mapsto (z_1, z_0)$ and $u \mapsto t-u$, we obtain $\operatorname{Tr} K_{2,1}(t) = -\operatorname{Tr} K_{2,1}(t)$. This completes the proof. □

A simple formula for $k = 2$ is given by a direct computation as follows.

PROPOSITION 6.2. *We have*

$$R_2(g, \Delta; x) = x^2 - (1+2g^2)x + 1/6 + g^2 + g^4 + \Delta^2 = B_2(x - g^2) + \Delta^2.$$

Proof. We shall compute $C_{H,\tau}(1) = c_1^{(2)} + c_1^{(3)}$. First, we observe the first term $c_1^{(2)} = \sum_{\epsilon=1}^4 c_1^{(2,\epsilon)}$. By (4.7), we have

$$c_1^{(2,\epsilon)} = \sum_{\substack{(l_1,k_1,k_2)\in\mathbb{Z}^3_{\geqslant 0} \\ l_1+(1/2)(k_1+k_2)-r_1(\epsilon)-(1/2)r_2(\epsilon)=1}} c^{(2,\epsilon)}_{l_1,(k_1,k_2)}.$$

In the case of $\epsilon \in \{1, 2\}$, both values $c_1^{(2,1)}$ and $c_1^{(2,2)}$ vanish by Lemma 6.1. For $\epsilon = 3$, we have $c^{(2,3)}_{l_1,(k_1,k_2)} = 0$ by Lemma 4.8. For $\epsilon = 4$, the integer l_1 in $c^{(2,4)}_{l_1,(k_1,k_2)}$ satisfies $l_1 \in \{0, 1\}$. In the case of $l_1 = 1$, the value $c^{(2,4)}_{1,(k_1,k_2)}$ vanishes by Lemma 4.6. When $l_1 = 0$, by noting $k_1 + k_2 = 2$, we have

$$\begin{aligned} c^{(2,4)}_{0,(k_1,k_2)} &= \frac{1}{2\pi k_1!k_2!}\int_0^1 du \int_{z_0\in\mathbb{R}} e^{-z_0^2/2} z_0^2\, dz_0 \{-\sqrt{2}g(1-u)\}^{k_2}(-\sqrt{2}gu)^{k_1} \\ &\quad \times \int_{\xi\in\mathbb{R}} e^{-\xi^2/2}\, d\xi (1/2-\tau)\operatorname{tr}(I) \\ &= \frac{1}{k_1!k_2!} B(k_1+1, k_2+1) 2g^2(1-2\tau) \\ &= \frac{1}{(k_1+k_2+1)!} 2g^2(1-2\tau) = \frac{1}{3} g^2(1-2\tau), \end{aligned}$$

which leads us to $c_1^{(2,4)} = c^{(2,4)}_{0,(0,2)} + c^{(2,4)}_{0,(1,1)} + c^{(2,4)}_{0,(2,0)} = g^2(1-2\tau)$. By the argument as above, we finally obtain

$$c_1^{(2)} = g^2(1-2\tau). \tag{6.1}$$

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Next let us consider the case $m=3$. In this case, with the aid of Lemmas 4.6–4.8 and (4.7), only the following cases survive among all $c^{(3,\epsilon)}_{(l_1,l_2),(k_1,k_2,k_3)}$ such that $l_1+l_2+\frac{k_1+k_2+k_3}{2}-r_1(\epsilon)-\frac{1}{2}r_2(\epsilon)=0$:

(1) $\epsilon\in\{(3,3),(4,4)\}$ and $l_1=l_2=k_1=k_2=k_3=0$.
(2) $\epsilon=(1,1)$, $l_1=l_2=1$ and $k_1=k_2=k_3=0$.

In case (1), the coefficient involved is evaluated as

$$\begin{aligned} c^{(3,\epsilon)}_{(0,0),(0,0,0)} &= (2\pi)^{-1}\int_0^1 du_1\int_0^{1-u_1} du_2\int_{z_0\in\mathbb{R}} e^{-z_0^2/2} \\ &\quad\times\int_{\xi\in\mathbb{R}} e^{-\xi^2/2}\,d\xi\Delta^{r_3(\epsilon)}(1/2-\tau)^{r_4(\epsilon)}\mathrm{tr}(I) \\ &= \Delta^{r_3(\epsilon)}(1/2-\tau)^{r_4(\epsilon)}. \end{aligned}$$

In case (2), we have $P^{3,(1,1)}_{(1,1),(0,0)}(u_1,u_2;z)=z^2$ by (5.4) and (5.6). Thus, Lemma 5.7 yields

$$\begin{aligned} c^{(3,(1,1))}_{(1,1),(0,0,0)} &= (2\pi)^{-1}i^2\int_0^1 du_1\int_0^{1-u_1} du_2\int_{z\in\mathbb{R}} e^{-z^2/2}z^2\,dz \\ &\quad\times\int_{\xi\in\mathbb{R}} e^{-\xi^2/2}u_1u_2\xi^2\,d\xi\mathrm{tr}(I) = -1/12 \end{aligned}$$

with the aid of (5.2). Hence,

$$c^{(3)}_1 = \Delta^2+(1/2-\tau)^2-1/12 \tag{6.2}$$

holds. As a consequence, we obtain the desired assertion by (6.1) and (6.2). □

Next let us compute the third Rabi–Bernoulli polynomial $R_3(g,\Delta;x)$. By definition, $R_3(g,\Delta;\tau)=-g^6-3C_{H,\tau}(2)$ and $C_{H,\tau}(2)=c^{(2)}_2+c^{(3)}_2+c^{(4)}_2$ hold with

$$c^{(m)}_2=\sum_{\epsilon\in\{1,2,3,4\}^{m-1}} c^{(m,\epsilon)}_2$$

and

$$c^{(m,\epsilon)}_2=\sum_{\substack{l_1,\dots,l_{m-1},k_1,\dots,k_m\in\mathbb{Z}_{\geqslant 0}\\ l_1+\cdots+l_{m-1}+(k_1+\cdots+k_m)/2+m-2-r_1(\epsilon)-(1/2)r_2(\epsilon)=2}} c^{(m,\epsilon)}_{(l_1,\dots,l_{m-1}),(k_1,\dots,k_m)}.$$

LEMMA 6.3. *We have*

$$c_2^{(2)} = \left(g^4 - \frac{1}{30}\right)\left(\frac{1}{2} - \tau\right).$$

Proof. By Lemmas 4.8 and 6.1, we easily have $c_2^{(2,\epsilon)} = 0$ for any $\epsilon \in \{1, 2, 3\}$. When $\epsilon = 4$, by Lemma 4.6, (l_1, k_1, k_2) in $c^{(2,4)}_{l_1,(k_1,k_2)} \neq 0$ satisfies $l_1 \in \{0, 2\}$ and $k_1 + k_2 = 4 - 2l_1$. If $l_1 = 0$ and $k_1 + k_2 = 4$, we have

$$\begin{aligned} c^{(2,4)}_{0,(k_1,k_2)} &= \frac{1}{2\pi k_1!k_2!}\int_0^1 du(1-u)^{k_2}u^{k_1}\int_{z\in\mathbb{R}} e^{-z^2/2}z^4\,dz \\ &\quad\times \int_{\xi\in\mathbb{R}} e^{-\xi^2/2}\,d\xi(\sqrt{2}g)^4(-1)^4(1/2-\tau)\mathrm{tr}(I) \\ &= \frac{g^4}{5}\left(\frac{1}{2}-\tau\right). \end{aligned}$$

If $l_1 = 2$ and $k_1 = k_2 = 0$, we have

$$\begin{aligned} c^{(2,4)}_{2,(0,0)} &= \frac{i^2}{2\pi 2!}\int_0^1 du\int_{z\in\mathbb{R}} e^{-z^2/2}P^{(4)}_{2,0}(u;z)\,dz \\ &\quad\times \int_{\xi\in\mathbb{R}} e^{-\xi^2/2}(u/2)H_2(\sqrt{u/2}\xi)\,d\xi(1/2-\tau)\mathrm{tr}(I) \\ &= -\frac{1}{30}\left(\frac{1}{2}-\tau\right), \end{aligned}$$

where we use $P^{(4)}_{2,0}(u;z) = (u/2)H_2(\sqrt{u/2}z) = u(uz^2-1)$. Hence we obtain

$$c_2^{(2,4)} = \sum_{\substack{k_1,k_2\in\mathbb{Z}_{\geqslant 0}\\ k_1+k_2=4}} \frac{g^4}{5}\left(\frac{1}{2}-\tau\right) - \frac{1}{30}\left(\frac{1}{2}-\tau\right) = \left(g^4 - \frac{1}{30}\right)\left(\frac{1}{2}-\tau\right). \quad \square$$

LEMMA 6.4. *We have*

$$c_2^{(3)} = -\frac{1}{12}g^2 + g^2\left(\frac{1}{2}-\tau\right)^2 + \frac{1}{3}g^2\Delta^2.$$

Proof. We give a proof by computing $c_2^{(3,\epsilon)}$ for all ϵ. By Lemma 4.8, we only have to consider the cases $\epsilon = (1, 1)$, $(1, 2)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, $(2, 4)$, $(3, 3)$, $(4, 1)$, $(4, 2)$, $(4, 4)$. We shall give expressions of all $c^{(3,\epsilon)}_{(l_1,l_2),(k_1,k_2,k_3)}$ such that $l_1 + l_2 + \frac{k_1+k_2+k_3}{2} + 1 - r_1(\epsilon) - \frac{1}{2}r_2(\epsilon) = 2$ in the following way.

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For $\epsilon = (1,1)$, we may assume $l_1 = l_2 = 1$ and $k_1 + k_2 + k_3 = 2$ by Lemmas 4.6 and 4.9, and thus we compute

$$
\begin{aligned}
&c^{(3,(1,1))}_{(1,1),(k_1,k_2,k_3)}\\
&\quad = \frac{i^2}{2\pi k_1!k_2!k_3!}\int_{D_2} du(1-u_1-u_2)^{k_3}u_1^{k_1}u_2^{k_2}\int_{z\in\mathbb{R}} e^{-z^2/2}z^4\,dz\\
&\qquad \times \int_{\xi\in\mathbb{R}} e^{-\xi^2}u_1u_2\xi^2\,d\xi(\sqrt{2}g)^2\mathrm{tr}(I)\\
&\quad = -\frac{(k_1+1)(k_2+1)}{60}g^2.
\end{aligned}
$$

Hence we have

$$
c^{(3,(1,1))}_2 = \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}_{\geqslant 0}\\ k_1+k_2+k_3=2}} \frac{-(k_1+1)(k_2+1)}{60}g^2 = -\frac{1}{4}g^2.
$$

For $\epsilon = (1,2)$, the numbers (l_1,l_2) and (k_1,k_2,k_3) satisfy $l_1 = l_2 = 1$ and $k_1 + k_2 + k_3 = 1$ by Lemmas 4.6 and 4.9. Then, we have

$$
\begin{aligned}
&c^{(3,(1,2))}_{(1,1),(k_1,k_2,k_3)}\\
&\quad = (2\pi)^{-1}i^2\int_{D_2} du(1-u_1-u_2)^{k_3}u_1^{k_1}u_2^{k_2}\int_{z\in\mathbb{R}} e^{-z^2/2}z^2\,dz\\
&\qquad \times \int_{\xi\in\mathbb{R}} e^{-\xi^2}u_1u_2\xi^2\,d\xi(-1)^1(\sqrt{2}g)^2\mathrm{tr}(I)\\
&\quad = \frac{(k_1+1)(k_2+1)}{30}g^2
\end{aligned}
$$

with the aid of $P^{(2,1)}_{(1,1),(k_1,k_2)}(u_1,u_2;z_2) = z_2^{k_1+k_2+1}$. Hence we obtain

$$
c^{(3,(1,2))}_2 = \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}_{\geqslant 0}\\ k_1+k_2+k_3=1}} \frac{(k_1+1)(k_2+1)}{30}g^2 = \frac{1}{6}g^2.
$$

For $\epsilon = (1,4)$, we may assume (l_1,l_2) and (k_1,k_2,k_3) satisfy $l_1 = l_2 = 1$ and $(k_1,k_2,k_3) = \mathbf{0}_3$ by Lemmas 4.6 and 4.9. By noting $P^{(1,4)}_{(1,1),\mathbf{0}_2}(u_1,u_2;z_2) = 1-(u_1+u_2)z_2^2$ and $P^{(1,4)}_{(2,0),\mathbf{0}_2}(u_1,u_2;z_2) = 1-2u_1z_2^2$, we have

$$\begin{aligned}
&c^{(3,(1,4))}_{(1,1),\mathbf{0}_3}\\
&\quad=\frac{i^2}{2\pi}\int_{D_2}du\int_{z\in\mathbb{R}}e^{-z^2/2}(1-u_1z^2-u_2z^2)\,dz\\
&\qquad\times\int_{\xi\in\mathbb{R}}e^{-\xi^2}u_1u_2\xi^2\,d\xi(1/2-\tau)\mathrm{tr}(I)=-\frac{1}{60}\left(\frac{1}{2}-\tau\right)
\end{aligned}$$

and

$$\begin{aligned}
&c^{(3,(1,4))}_{(2,0),\mathbf{0}_3}\\
&\quad=\frac{i^2}{2\pi 2!}\int_{D_2}du\int_{z\in\mathbb{R}}e^{-z^2/2}(1-2u_1z^2)\,dz\\
&\qquad\times\int_{\xi\in\mathbb{R}}e^{-\xi^2}u_1u_2\xi^2\,d\xi(1/2-\tau)\mathrm{tr}(I)=\frac{1}{60}\left(\frac{1}{2}-\tau\right).
\end{aligned}$$

Hence we obtain $c_2^{(3,(1,4))}=0$.

For $\epsilon=(2,1)$, we may assume $l_1=l_2=1$ and $k_1+k_2+k_3=1$ by Lemmas 4.6 and 4.9. By a direct computation, we have

$$\begin{aligned}
c^{(3,(2,1))}_{(1,1),(k_1,k_2,k_3)}&=(2\pi)^{-1}i^2\int_{D_2}du(1-u_1-u_2)^{k_3}u_1^{k_1}u_2^{k_2}\int_{z\in\mathbb{R}}e^{-z^2/2}z^2\,dz\\
&\qquad\times\int_{\xi\in\mathbb{R}}e^{-\xi^2}u_1u_2\xi^2\,d\xi(-1)(\sqrt{2}g)^2\mathrm{tr}(I)\\
&=\frac{(k_1+1)(k_2+1)}{30}g^2,
\end{aligned}$$

and hence we obtain

$$c_2^{3,(2,1)}=\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}_{\geqslant 0}\\k_1+k_2+k_3=1}}\frac{(k_1+1)(k_2+1)}{30}g^2=\frac{1}{6}g^2.$$

For $\epsilon=(2,2)$, we may assume $l_1=l_2=1$ and $k_1=k_2=k_3=0$ by Lemmas 4.6 and 4.9. Noting $P^{(2,2)}_{(1,1),(0,0)}(u_1,u_2;z)=1$, we obtain

$$\begin{aligned}
c^{(3,(2,2))}_{(1,1),\mathbf{0}_3}&=\frac{i^2}{2\pi}\int_{D_2}du\int_{z\in\mathbb{R}}e^{-z^2/2}\,dz\int_{\xi\in\mathbb{R}}e^{-\xi^2/2}u_1u_2\xi^2\,d\xi(\sqrt{2}g)^2\mathrm{tr}(I)\\
&=-\frac{1}{6}g^2.
\end{aligned}$$

For $\epsilon=(2,4)$, all $c^{(3,(2,4))}_{(l_1,l_2),(k_1,k_2,k_3)}$ concerned vanish by Lemmas 4.6 and 4.9.

For $\epsilon = (3,3)$, we may assume $l_1 = l_2 = 0$ and $k_1 + k_2 + k_3 = 2$. Then we have

$$\begin{aligned} c^{(3,(3,3))}_{(0,0),(k_1,k_2,k_3)} &= \frac{1}{2\pi k_1!k_2!k_3!} \int_{D_2} du(1-u_1-u_2)^{k_3} u_1^{k_1} u_2^{k_2} \int_{z\in\mathbb{R}} e^{-z^2/2} z^2 \, dz \\ &\quad \times \int_{\xi\in\mathbb{R}} e^{-\xi^2/2} \, d\xi \left\{ \prod_{j=1}^{2} (-1)^{\omega_{(3,3)}(j)k_j} \right\} (\sqrt{2}g)^2 \Delta^2 \mathrm{tr}(I) \\ &= \frac{1}{6} g^2 \Delta^2 (-1)^{k_2}, \end{aligned}$$

and hence we obtain

$$c_2^{(3,(3,3))} = \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}_{\geqslant 0} \\ k_1+k_2+k_3=2}} \frac{1}{6}(-1)^{k_2} g^2 \Delta^2 = \frac{1}{3} g^2 \Delta^2.$$

For $\epsilon = (4,1)$, we may assume $l_1 + l_2 = 2$ and $(k_1,k_2,k_3) = \mathbf{0}_3$ by Lemma 4.6. Furthermore, we may assume $l_2 \neq 0$ by Lemma 4.9. Then, a direct computation gives us

$$\begin{aligned} c^{(3,(4,1))}_{(1,1),\mathbf{0}_3} &= \frac{i^2}{2\pi} \int_{D_2} du \int_{z\in\mathbb{R}} e^{-z^2/2}(-u_1 z^2) \, dz \\ &\quad \times \int_{\xi\in\mathbb{R}} e^{-\xi^2/2} u_1 u_2 \xi^2 \, d\xi (1/2-\tau)\mathrm{tr}(I) = \frac{1}{30}(1/2-\tau) \end{aligned}$$

and

$$\begin{aligned} c^{(3,(4,1))}_{(0,2),\mathbf{0}_3} &= \frac{i^2}{2\pi 2!} \int_{D_2} du \int_{z\in\mathbb{R}} e^{-z^2/2}\{1-2(u_1+u_2)z^2\} \, dz \\ &\quad \times \int_{\xi\in\mathbb{R}} (u_2/2) H_2(\sqrt{u_2/2}\,\xi) \, d\xi (1/2-\tau)\mathrm{tr}(I) \\ &= -\frac{1}{30}(1/2-\tau), \end{aligned}$$

and hence we obtain $c_2^{(3,(4,1))} = 0$. For $\epsilon = (4,2)$, all $c^{(3,(4,2))}_{(l_1,l_2),(k_1,k_2,k_3)}$ concerned vanish in the same way as in the case $\epsilon = (2,4)$. For $\epsilon = (4,4)$, we may assume $l_1 = l_2 = 0$ and $k_1 + k_2 + k_3 = 2$ by Lemma 4.6, and a direct computation gives us

$$\begin{aligned}
&c^{(3,(4,4))}_{(0,0),(k_1,k_2,k_3)}\\
&= \frac{1}{2\pi k_1!k_2!k_3!}\int_{D_2} du(1-u_1-u_2)^{k_3}u_1^{k_1}u_2^{k_2}\int_{z\in\mathbb{R}} e^{-z^2/2}z^2\,dz\\
&\quad\times\int_{\xi\in\mathbb{R}} e^{-\xi^2/2}\,d\xi(\sqrt{2}g)^2(1/2-\tau)^2\mathrm{tr}(I)\\
&= \frac{1}{6}g^2\left(\frac{1}{2}-\tau\right)^2.
\end{aligned}$$

Thus we have $c_2^{(3,(4,4))} = g^2(1/2-\tau)^2$.

Finally, by the consideration as above, we obtain the formula as desired. ∎

LEMMA 6.5. *We have*

$$c_2^{(4)} = \frac{1}{3}\left(\frac{1}{2}-\tau\right)^3 + \left(\Delta^2-\frac{1}{20}\right)\left(\frac{1}{2}-\tau\right).$$

Proof. Consider $c^{(4,\epsilon)}_{(l_1,l_2,l_3),(k_1,k_2,k_3,k_4)}$. We may assume $r_1(\epsilon)+r_2(\epsilon)=0$, by which $\epsilon\in\{3,4\}^3$ holds, and also that $r_3(\epsilon)$ is even by Lemma 4.8. A direct computation gives us

$$\begin{aligned}
c^{(4,\epsilon)}_{\mathbf{0}_3,\mathbf{0}_4} &= \frac{1}{2\pi}\int_{D_3} du\int_{z\in\mathbb{R}} e^{-z^2/2}\,dz\int_{\xi\in\mathbb{R}} e^{-\xi^2/2}\,d\xi\Delta^{r_3(\epsilon)}(1/2-\tau)^{r_4(\epsilon)}\mathrm{tr}A(\epsilon)\\
&= \frac{1}{3}\Delta^{r_3(\epsilon)}\left(\frac{1}{2}-\tau\right)^{r_4(\epsilon)}.
\end{aligned}$$

Thus we obtain $c^{(4,(4,4,4))}_{\mathbf{0}_3,\mathbf{0}_4} = \frac{1}{3}(\frac{1}{2}-\tau)^3$ and $c^{(4,(4,3,3))}_{\mathbf{0}_3,\mathbf{0}_4} = c^{(4,(3,4,3))}_{\mathbf{0}_3,\mathbf{0}_4} = c^{(4,(3,3,4))}_{\mathbf{0}_3,\mathbf{0}_4}$ $= \frac{1}{3}\Delta^2(\frac{1}{2}-\tau)$. If $r_1(\epsilon)+r_2(\epsilon)\in\{1,3\}$, then all $c^{(4,\epsilon)}_{(l_1,l_2,l_3),(k_1,k_2,k_3,k_4)}$ vanish by Lemmas 4.6 and 4.9. If $r_1(\epsilon)+r_2(\epsilon)=2$, the only case $r_1(\epsilon)=2$, $r_2(\epsilon)=0$ survives, and in such a case, by Lemmas 4.6 and 4.9, it is sufficient to consider the only case $((l_1,l_2,l_3),\epsilon)\in\{((1,1,0),(1,1,4)),((1,0,1),(1,4,1)),((0,1,1),(4,1,1))\}$ and $(k_1,k_2,k_3,k_4)=\mathbf{0}_4$. A direct computation gives us

$$\begin{aligned}
c^{(4,(1,1,4))}_{(1,1,0),\mathbf{0}_4} &= \frac{i^2}{2\pi}\int_{D_3} du\int_{z\in\mathbb{R}} e^{-z^2/2}z^2\,dz\int_{\xi\in\mathbb{R}} e^{-\xi^2/2}u_1u_2\xi^2\,d\xi(1/2-\tau)\mathrm{tr}(I)\\
&= -\frac{1}{60}\left(\frac{1}{2}-\tau\right)
\end{aligned}$$

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with the aid of $P^{(1,1,4)}_{(1,1,0),\mathbf{0}_3}(u_1,u_2,u_3;z)=z^2$, and in a similar fashion, we obtain $c^{(4,(1,4,1))}_{(1,0,1),\mathbf{0}_4}=c^{(4,(4,1,1))}_{(0,1,1),\mathbf{0}_4}=-\frac{1}{60}(\frac{1}{2}-\tau)$. Finally, we have the formula as desired. ∎

Proposition 6.6. *We have*

$$R_3(g,\Delta;x)=x^3-\left(3g^2+\frac{3}{2}\right)x^2+\left(3g^4+3g^2+3\Delta^2+\frac{1}{2}\right)x-g^6 -\frac{3}{2}g^4-\frac{1}{2}g^2-\frac{3}{2}\Delta^2-g^2\Delta^2 = B_3(x-g^2)+3\Delta^2B_1(x-g^2)+2g^2\Delta^2.$$

Proof. It follows immediately from Lemmas 6.3–6.5. ∎

Acknowledgments. The author would like to thank Professor Masato Wakayama for giving him the interest in spectral zeta functions and the quantum Rabi models, and a lot of useful comments and suggestions. He would also like to thank Professor Kazufumi Kimoto for fruitful discussion and for informing him about papers [11] and [14], and preprints [10] and [15]. Thanks are also due to Professor Yoshinori Yamasaki for useful comments and careful reading of the earlier draft. The author would like to thank Cid Reyes-Bustos for pointing out grammatical errors and Professor Fumio Hiroshima for useful discussion.

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Institute of Mathematics for Industry
Kyushu University
744, Motooka
Nishi-ku
Fukuoka 819-0395
Japan
s-sugiyama@imi.kyushu-u.ac.jp