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RESEARCH ARTICLE

On almost quotient Yamabe solitons

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Abstract
In this paper, we investigate the structure of certain solutions of the fully nonlinear Yamabe flow, which we call almost quotient Yamabe solitons as they extend quite naturally those already called quotient Yamabe solitons. We present sufficient conditions for a compact almost quotient Yamabe soliton to be either trivial or isometric with an Euclidean sphere. We also characterize noncompact almost gradient quotient Yamabe solitons satisfying certain conditions on both its Ricci tensor and potential function.

1. Introduction and main results

The Yamabe flow

$$\frac{\partial g}{\partial t}(t) = -(R_{g(t)} - r_{g(t)})g(t), \quad g(0) = g_0, \tag{1}$$

where $R_{g(t)}$ is the scalar curvature of $g(t)$ and

$$r_{g(t)} = \frac{\displaystyle\int_M R_{g(t)} dv_{g(t)}}{\displaystyle\int_M dv_{g(t)}},$$

is the mean value of $R_{g(t)}$ along M^n, it was introduced by R. Hamilton [1] and has become one of the standard tools of recent differential geometry. Yamabe solitons arise as self-similar solutions of (1).

Definition 1. *A self-similar solution $g(t)$ of (1) is a Yamabe soliton if there exists a scalar factor $\alpha : [0, \varepsilon) \to (0, \infty)$, $\varepsilon > 0$, and a 1-parameter family $\{\psi_t\}$ of diffeomorphisms of M^n such that*

$$g(t) = \alpha(t)\psi_t^*(g_0), \quad \alpha(0) = 1 \quad \text{and} \quad \psi_0 = id_M.$$

One gets

$$\frac{1}{2}\mathcal{L}_X g = \left(R_g - \lambda\right) g, \tag{2}$$

by substituting $g(t) = \alpha(t)\psi_t^*(g_0)$ into (1) and evaluating the resulting expression at $t = 0$, where $\mathcal{L}_X g$ is the Lie derivative of g with respect to the field X of directions associated with the 1-parameter family $\{\psi_t\}$ and $\lambda = \alpha'(0) + r_g$. Equation (2) is the fundamental equation of Yamabe solitons. Since their beginning, a lot of results were proved on the nature of Yamabe solitons. For example, Chow [2] proved that compact Yamabe solitons have constant scalar curvature (see also [3,4]). Daskalopoulos and Sesum [5] proved that complete locally conformally flat Yamabe solitons with positive sectional curvature are rotationally symmetric and must belong to the conformal class of flat Euclidean space.

A new notion of soliton is born if one replaces the scalar curvature in (1) by functions of the higher order scalar curvatures. As is the case with any generalization, it is hoped that one recovers the old objects as particular instances of the new ones, while open up room for new and exciting phenomena to happen. In what follows, we give formal definitions and even before we state our main results, we examine a few examples. We included a section containing the lemmas that we have used in the text for the convenience of the reader and a separate section with the proofs of our statements can be found right after it.

The Riemann curvature tensor Rm of (M^n, g) admits the following decomposition

$$\mathrm{Rm} = W_g + A_g \owedge g,$$

where W_g and A_g are the tensors of Weyl and Schouten, respectively, and $\owedge$ is the Kulkarni–Nomizu product of (M^n, g). Recall that the Schouten tensor is given by

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\right).$$

The σ_k-curvature of g is defined as the kth elementary symmetric function of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the endomorphism $g^{-1}A_g$, that is,

$$\sigma_k(g) = \sigma_k(g^{-1}A_g) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \lambda_{i_1}\cdots\lambda_{i_k}, \quad 1 \leqslant k \leqslant n.$$

Here, we set $\sigma_0(g) = 1$ for convenience. A simple calculation shows that $\sigma_1(g) = \frac{R_g}{2(n-1)}$, which indicates that the σ_k-curvature is a reasonable substitute for the scalar curvature of (M^n, g) in (1).

Guan and Guofang introduced [6] the fully nonlinear flow

$$\frac{\partial g}{\partial t}(t) = -\left(\log\frac{\sigma_k(g(t))}{\sigma_l(g(t))} - \log r_{k,l}(g(t))\right)g(t), \quad g(0) = g_0, \tag{3}$$

where

$$\log r_{k,l}(g(t)) = \frac{\displaystyle\int_M \sigma_l(g(t))\log\frac{\sigma_k(g(t))}{\sigma_l(g(t))}dv_{g(t)}}{\displaystyle\int_M \sigma_l(g(t))dv_{g(t)}},$$

was defined as to make the flow preserve the quantities

$$\mathcal{E}_l(g(t)) = \begin{cases} \displaystyle\int_M \sigma_l(g(t))dv_{g(t)}, & \text{if } \ l \neq \frac{n}{2}, \\ \displaystyle -\int_0^1 dt \int_M u\sigma_{n/2}(g(t))dv_{g(t)}, & \text{if } \ l = \frac{n}{2}, \end{cases}$$

where $u \in C^\infty(M)$, $g = e^{-2u}g_0$, and $g(t) = e^{-2tu}g_0$. The convergence of the fully nonlinear flow was then proved under certain conditions to be satisfied by the eigenvalues of the Schouten tensor. The authors also provided geometric inequalities such as the Sobolev-type inequality in case $0 \leqslant l < k < \frac{n}{2}$, the conformal quasimass-integral-type inequality for $\frac{n}{2} \leqslant k \leqslant n$, $1 \leqslant l < k$, and the Moser–Trudinger-type inequality for $k = \frac{n}{2}$.

Bo et al. [7] presented quotient Yamabe solitons as self-similar solutions of the flow (3) and stated rigidity results for the existence of such objects on top of locally conformally flat manifolds. For example, it was shown that any compact and locally conformally flat manifold with the structure of a quotient Yamabe soliton, where both $\sigma_k > 0$ and $\sigma_l > 0$, must have constant quotient curvature $\frac{\sigma_k}{\sigma_l}$. Also, for the so-called gradient k-Yamabe soliton ($l = 0$), they proved that, for $k > 1$, any compact gradient k-Yamabe soliton with negative constant scalar curvature has necessarily constant σ_k-curvature. Almost Yamabe solitons were introduced by Barbosa and Ribeiro [8] as generalizations of self-similar solutions of the

Yamabe flow. Essentially, they allowed the parameter λ in (2) to be a function on M. The authors then stated rigidity results for almost Yamabe solitons on compact manifolds. We refer the reader to [8–11] for further information.

Catino et al. [12] proposed the study of conformal solitons. A conformal soliton is a Riemannian manifold (M^n, g) together with a nonconstant function $f \in C^\infty(M)$ satisfying $\nabla^2 f = \lambda g$ for some $\lambda \in \mathbb{R}$. They provided classification results according to the number of critical points of f. It should be noticed that solitons of Yamabe, k-Yamabe, and quotient Yamabe types are examples of conformal solitons.

We introduce almost quotient Yamabe solitons in extension to the quotient Yamabe solitons.

Definition 2. *A solution $g(t)$ of (3) is an almost quotient Yamabe soliton if there exist a scalar factor $\alpha : M \times [0, \varepsilon) \to (0, \infty)$, $\varepsilon > 0$, and a 1-parameter family $\{\psi_t\}$ of diffeomorphisms of M^n such that*

$$g(t) = \alpha(x, t)\psi_t^*(g_0), \quad \alpha(\,\cdot\,, 0) \equiv 1 \text{ on } M^n \quad \text{and} \quad \psi_0 = id_M.$$

Equivalently, (M^n, g) is an almost quotient Yamabe soliton if there exists a pair $X \in \mathfrak{X}(M)$, $\lambda \in C^\infty(M)$ satisfying

$$\frac{1}{2}\mathcal{L}_X g = \left(\log \frac{\sigma_k}{\sigma_l} - \lambda\right) g, \quad \sigma_k \cdot \sigma_l > 0. \tag{4}$$

We will write the soliton in (4) as (M^n, g, X, λ) for the sake of simplicity. Following the terminology already in use with almost Yamabe solitons, a soliton (M^n, g, X, λ) will be called:

a. *expanding* if $\lambda < 0$,
b. *steady* if $\lambda = 0$,
c. *shrinking* if $\lambda > 0$ and, finally,
d. *indefinite* if λ change signs on M^n.

Definition 3. *An almost gradient quotient Yamabe soliton is an almost quotient Yamabe soliton (M^n, g, X, λ) such that $X = \nabla f$ is the gradient field of a function $f \in C^\infty(M)$.*

Since

$$\frac{1}{2}\mathcal{L}_{\nabla f} g = \nabla^2 f,$$

it follows from (4) that an almost gradient quotient Yamabe soliton $(M^n, g, \nabla f, \lambda)$ is characterized by the equation

$$\nabla^2 f = \left(\log \frac{\sigma_k}{\sigma_l} - \lambda\right) g, \quad \sigma_k \cdot \sigma_l > 0. \tag{5}$$

Almost quotient Yamabe solitons, gradient or not, are regarded as *trivial* if their defining equation vanishes identically. Thus, (M^n, g, X, λ) is trivial if $\mathcal{L}_X g = 0$ and $(M^n, g, \nabla f, \lambda)$ if $\nabla^2 f = 0$. In either case, $\log \frac{\sigma_k}{\sigma_l} - \lambda = 0$. Let us take a look at a few examples.

Example 1. *The product manifold $(\mathbb{R} \times \mathbb{S}^n, g = dt^2 + g_{\mathbb{R}^n})$ alongside the function*

$$f : \mathbb{R} \times \mathbb{S}^n \to \mathbb{R}, \quad (t, x) \mapsto f(t, x) = at + b \quad (a, b \in \mathbb{R}),$$

is, for $k = l = 1$, a trivial almost gradient quotient Yamabe soliton with $\lambda = 0$, since $\sigma_1(g^{-1}A_g) = \frac{n}{2}$ and $\nabla^2 f = 0$.

Example 2. *Identities*

$$\mathrm{Ric}_{g_{\mathbb{S}^n}} = (n-1)g_{\mathbb{S}^n}, \quad R_{g_{\mathbb{S}^n}} = n(n-1) \quad \text{and} \quad A_{g_{\mathbb{S}^n}} = \frac{1}{2}g_{\mathbb{S}^n},$$

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stand for the Ricci tensor, scalar curvature, and Schouten tensor, respectively, of the Euclidean sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$*. Therefore, we have that*

$$\sigma_k(g_{\mathbb{S}^n}^{-1}A_{g_{\mathbb{S}^n}}) = \frac{1}{2^k}\binom{n}{k}, \qquad 1 \leqslant k \leqslant n.$$

Consider the height function

$$h_v : \mathbb{S}^n \to \mathbb{R}, \qquad x \mapsto h_v(x) = \langle x, v\rangle,$$

on $\mathbb{S}^n$ *with respect to a given* $v \in \mathbb{S}^n$*. It then follows that*

$$\nabla^2 h_v = -h_v g_{g_{\mathbb{S}^n}},$$

which shows that $(\mathbb{S}^n, g_{\mathbb{S}^n}, \nabla h_v, \lambda)$ *is a compact almost quotient Yamabe soliton with*

$$\lambda : \mathbb{S}^n \to \mathbb{R}, \qquad x \mapsto h_v(x) + \log \frac{\sigma_k}{\sigma_l}.$$

Example 3. *On the hyperbolic space* $(\mathbb{H}^n, g_{\mathbb{H}^n})$*, we consider*

$$\mathrm{Ric}_{g_{\mathbb{H}^n}} = -(n-1)g_{\mathbb{H}^n}, \quad R_{g_{\mathbb{H}^n}} = -n(n-1) \quad \text{and} \quad A_{g_{\mathbb{H}^n}} = -\frac{1}{2}g_{\mathbb{H}^n},$$

to denote the Ricci tensor, scalar curvature, and Schouten tensor, respectively. Therefore, we have that

$$\sigma_k(g_{\mathbb{H}^n}^{-1}A_{g_{\mathbb{H}^n}}) = \frac{(-1)^k}{2^k}\binom{n}{k}, \qquad 1 \leqslant k \leqslant n.$$

We consider the model $\mathbb{H}^n = \{x \in \mathbb{R}^{n,1} : \langle x, x\rangle_0 = -1, x_1 > 0\}$ *of the hyperbolic space, where* $\mathbb{R}^{n,1}$ *denotes the Euclidean space* $\mathbb{R}^{n+1}$ *endowed with Lorentzian inner product* $\langle x, x\rangle_0 = -x_1^2 + x_2^2 + \cdots + x_{n+1}^2$*. As in our previous example, we consider the height function*

$$h_v : \mathbb{H}^n \to \mathbb{R}, \qquad x \mapsto h_v(x) = \langle x, v\rangle_0,$$

on $\mathbb{H}^n$ *with respect to a given* $v \in \mathbb{H}^n$*. Because*

$$\nabla^2 h_v = h_v g_{g_{\mathbb{H}^n}},$$

we conclude that $(\mathbb{H}^n, g_{\mathbb{H}^n}, \nabla h_v, \lambda)$ *is an almost quotient Yamabe soliton with*

$$\lambda : \mathbb{H}^n \to \mathbb{R}, \qquad x \mapsto -h_v(x) + \log \frac{\sigma_k}{\sigma_l},$$

as long as we have $k \equiv l \pmod 2$.

Example 4. *Consider* $\mathbb{R}^n$ *endowed with a metric tensor of the form*

$$g_{ij} = e^{2u_i}\delta_{ij}, \qquad 1 \leqslant i, j \leqslant n,$$

so given in cartesian coordinates $x = (x_1, \ldots, x_n)$ *of* $\mathbb{R}^n$*, where* $u_1, \ldots, u_n \in C^\infty(\mathbb{R}^n)$*. Then, the Ricci tensor of* $(\mathbb{R}^n, g)$ *is given in* [13] *by the formulas*

$$\mathrm{Ric}_g(\partial_j, \partial_k) = \begin{cases} \sum_{l \neq k,j} U_{jk}^l + u_{j,k}u_{l,j}, & \text{if} \quad j \neq k, \\ \sum_{l \neq k} e^{2(u_k - u_l)}U_{ll}^k + U_{kk}^l - \sum_{m \neq k,l} e^{2(u_k - u_m)}u_{k,m}u_{l,m}, & \text{if} \quad j = k, \end{cases}$$

where

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} \quad \text{and} \quad u_{i,j,k} = \frac{\partial^2 u_i}{\partial x_k \partial x_j},$$

and

$$U_{jk}^l = u_{l,k}(u_k - u_l)_j - u_{l,j,k},$$

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for every $1 \leqslant i, j, k, l \leqslant n$*. Assume that* $n \geqslant 4$*. Also, let* τ *be the n-cycle* $(1, 2, 3, \ldots, n)$ *in the symmetric group* S_n *of degree n. It turns out that by choosing functions*

$$u_i(x_1, \ldots, x_n) = \begin{cases} \log\cosh\left(x_{\tau(i)}\right), & \text{if} \quad i \equiv 0 \pmod 2, \\ 0, & \text{if} \quad i \equiv 1 \pmod 2, \end{cases}$$

we simplify the situation quite a little bit as the Ricci tensor of $(\mathbb{R}^n, g)$ *ends up being a constant multiple of the metric,* $\mathrm{Ric}_g = -g$*. Therefore,* $(\mathbb{R}^n, g)$ *is a complete Einstein manifold and, as such,* $A_g = \frac{-1}{2(n-1)} g$*. Then, we have that*

$$\sigma_k(g^{-1}A_g) = \frac{(-1)^k}{2^k(n-1)^k}\binom{n}{k}, \quad 1 \leqslant k \leqslant n.$$

Because $X = (0, 1, \ldots, 0, 1)$ *is a Killing field on* $(\mathbb{R}^n, g)$ *we know that* $(\mathbb{R}^n, g, X, \lambda)$ *is a trivial almost quotient Yamabe soliton whenever* $k \equiv l \pmod 2$*. It should be noticed that X is not a gradient field with respect to the metric g.*

Any smooth vector field X on a compact Riemannian manifold (M^n, g) can be written in the form

$$X = \nabla h + Y, \tag{6}$$

where $Y \in \mathfrak{X}(M)$ is divergence free and $h \in C^\infty(M)$. In fact, by the Hodge-de Rham Theorem [14], we have that

$$X^\flat = d\alpha + \delta\beta + \gamma.$$

Now, take $Y = (\delta\beta + \gamma)^\sharp$, $\nabla h = (d\alpha)^\sharp$ and we are done. The function h is called the Hodge-de Rham potential of X. Our first theorem states the triviality of a compact almost quotient Yamabe soliton under certain integral assumptions.

Theorem 1. *A compact almost quotient Yamabe soliton* (M^n, g, X, λ) *is trivial if one of the following assertions holds:*

a) $\displaystyle\int_M e^\lambda \sigma_l \langle \nabla\lambda, X\rangle dv_g = -\int_M e^\lambda \langle \nabla\sigma_l, X\rangle dv_g$*, plus any of these:*
   i. $\nabla \mathrm{Ric}_g = 0$*;*
   ii. *div* $C_g = 0$*, where* C_g *is the Cotton tensor of* (M^n, g)*;*
   iii. $X = \nabla f$ *is a gradient vector field;*

b) $\displaystyle\int_M \langle \nabla h, X\rangle dv_g \leqslant 0$*, where h is the Hodge-de Rham potential of X.*

The next two corollaries deal with quotient Yamabe solitons (λ is a real constant) and constitute direct applications of Theorem 1. In [7], Bo et al. proved that σ_k/σ_l must be constant on any compact and locally conformally flat quotient Yamabe soliton. We extend Bo's result.

Corollary 1. *Let* (M^n, g, X, λ) *be any compact quotient Yamabe soliton with a vanishing cotton tensor. Then,* σ_k/σ_l *is constant and, as such, the soliton is trivial.*

In [12], Catino et al. proved that any compact gradient k-Yamabe soliton with a nonnegative Ricci tensor is trivial. Bo et al. [7] also proved that any compact gradient k-Yamabe soliton with constant negative scalar curvature is trivial. In [15], it was shown that any compact gradient k-Yamabe soliton must be trivial. We extend all these results at once.

Corollary 2. *Let* $(M^n, g, \nabla f, \lambda)$ *be any compact quotient gradient Yamabe soliton. Then,* σ_k/σ_l *is constant and, as such, the soliton is trivial.*

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Yet another triviality result holds for almost quotient Yamabe solitons if one drops compactity on M^n in favor of a decay condition on the norm of the soliton field X.

Theorem 2. *Let (M^n, g, X, λ) be a complete and noncompact almost quotient Yamabe soliton satisfying*

$$\int_{M^n\setminus B_r(x_0)} \frac{|X|}{d(x,x_0)} dv_g < \infty \quad \text{and} \quad \mathcal{L}_X g \geqslant 0,$$

where d is the distance function with respect to g and $B_r(x_0)$ is the ball of radius $r > 0$ centered at x_0. Then, (M^n, g, X, λ) is trivial.

Next, we give a sufficient condition for a compact almost quotient gradient Yamabe soliton to be isometric with an Euclidean sphere.

Theorem 3. *Let $(M^n, g, \nabla f, \lambda)$ be a nontrivial compact quotient gradient almost Yamabe soliton with constant scalar curvature $R_g = R > 0$. Then (M^n, g) is isometric to the Euclidean sphere $\mathbb{S}^n(\sqrt{r})$, $r = R/n(n-1)$. Moreover, up to a rescaling, the potential f is given by $f = h_v + c$ where h_v is the height function on the sphere and c is a real constant.*

Another situation in which an almost gradient quotient Yamabe soliton must be isometric with an Euclidean sphere is described below.

Theorem 4. *Let $(M^n, g, \nabla f, \lambda)$ be a nontrivial compact quotient gradient almost Yamabe soliton with constant σ_k-curvature, for some $k = 2, \ldots, n$, and $A_g > 0$. Then, (M^n, g) is isometric with an Euclidean sphere $\mathbb{S}^n$.*

Remark 1. *A similar result concerning almost Ricci solitons can be found in* [16].

Finally, we investigate the structure of noncompact almost quotient gradient Yamabe solitons satisfying reasonable conditions on its potential function and both Ricci and scalar curvatures.

Theorem 5. *Let $(M^n, g, \nabla f, \lambda)$ be a nontrivial and noncompact almost quotient gradient Yamabe soliton. Assume that*

$$\mathcal{L}_{\nabla f^2} R \geqslant 0, \quad \mathring{\mathrm{Ric}}_g(\nabla f, \nabla f) \geqslant 0 \quad \text{and} \quad |\mathring{\mathrm{Ric}}(\nabla f^2)| \in L^1(M).$$

Then, (M^n, g) has constant scalar curvature $R_g = R \leqslant 0$ and f has at most one critical point. Moreover, we have that:

- *a) If $R = 0$, then (M^n, g) is isometric with a Riemannian product manifold $(\mathbb{R} \times \mathbb{F}^{n-1}, dt^2 + g_{\mathbb{F}})$;*
- *b) If $R < 0$ and f has no critical points, then (M^n, g) is isometric with a warped product manifold $(\mathbb{R} \times \mathbb{F}^{n-1}, dt^2 + \xi(t)^2 g_{\mathbb{F}})$ such that*

$$\xi'' + \frac{R}{n(n-1)}\xi = 0;$$

- *c) If $R < 0$ and f has only one critical point, then (M^n, g) is isometric with a hyperbolic space.*

Remark 2. *Einstein manifolds satisfy the hypothesis of Theorem* (5) *quite naturally for if*

$$\mathrm{Ric} = \rho g,$$

for some $\rho \in \mathbb{R}$, then R is constant over M and, as such, we have that

$$\mathcal{L}_{\nabla f^2} R \equiv 0.$$

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Furthermore, the traceless Ricci tensor

$$\overset{\circ}{\mathrm{Ric}} = \mathrm{Ric} - \frac{R}{n} g \equiv 0,$$

vanishes identically, thus giving $|\overset{\circ}{Ric}(\nabla f^2)| \in L^1(M)$.

2. Key Lemmas

In this section, we collect some useful lemmas that will used in the proof of the main results.

Lemma 1. ([17,18]). *Let* (M^n, g) *be a compact Riemannian manifold with a possibly empty boundary* ∂M. *Then,*

$$\int_M X(\mathrm{tr}\, T) dv_g = n \int_M \mathrm{div}\, T(X) dv_g + \frac{n}{2} \int_M \langle \overset{\circ}{T}, \mathcal{L}_X g \rangle dv_g - n \int_{\partial M} \overset{\circ}{T}(X, \nu) ds_g,$$

for every symmetric $(0, 2)$*-tensor* T *and every vector field* X *on* M, *where*

$$\mathrm{tr}\, T = g^{ij} T_{ij} \quad \textit{and} \quad \overset{\circ}{T} = T - \frac{\mathrm{tr}\, T}{n} g,$$

and ν *is the outward unit normal field on* ∂M.

Proof. First, notice that integration by parts yields

$$\int_{\partial M} T(X, \nu) dA_g = \int_M \nabla^i (T_{ij} X^j) dv_g,$$

and because

$$\begin{aligned} \nabla^i (T_{ij} X^j) &= \nabla^i T_{ij} X^j + T_{ij} \nabla^i X^j \\ &= \nabla^i T_{ij} X^j + \frac{1}{2} T_{ij} (\nabla^i X^j + \nabla^j X^i) \\ &= \mathrm{div}\, T(X) + \frac{1}{2} \langle T, \mathcal{L}_X g \rangle, \end{aligned}$$

we get that

$$\begin{aligned} \int_{\partial M} T(X, \nu) dA_g &= \int_M \nabla^i (T_{ij} X^j) dv_g \\ &= \int_M \mathrm{div}\, T(X) dv_g + \frac{1}{2} \int_M \langle T, \mathcal{L}_X g \rangle dv_g \\ &= \int_M \mathrm{div}\, T(X) dv_g + \frac{1}{2} \int_M \langle \overset{\circ}{T}, \mathcal{L}_X g \rangle dv_g + \frac{1}{2} \int_M \frac{\mathrm{tr}\, T}{n} \langle g, \mathcal{L}_X g \rangle dv_g \\ &= \int_M \mathrm{div}\, T(X) dv_g + \frac{1}{2} \int_M \langle \overset{\circ}{T}, \mathcal{L}_X g \rangle dv_g + \frac{1}{n} \int_M \mathrm{tr}\, T \cdot \mathrm{div}\, X dv_g. \end{aligned} \tag{7}$$

On the other hand, we have that

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$$\int_M \operatorname{tr} T \cdot \operatorname{div} X dv_g = \int_{\partial M} \operatorname{tr} T \cdot \langle X, \nu \rangle dA_g - \int_M X(\operatorname{tr} T) dv_g. \tag{8}$$

The result now follows from (7) and (8) above. □

We now recall a useful result established in [19].

Lemma 2. ([19]) *Let* (M^n, g) *be a Riemannian manifold and* T *be a symmetric* $(0, 2)$*-tensor field on* M^n*. Then*

$$\operatorname{div}\,(T(\varphi X)) = \varphi(\operatorname{div} T)(X) + \varphi\langle \nabla X, T\rangle + T(\nabla\varphi, X),$$

for any $X \in \mathfrak{X}(M)$ *and* $\varphi \in C^\infty(M)$ *where* $T(X)$ *is the vector field* g*-equivalent to* T.

For locally conformally flat manifolds, a proposition similar to the next one can be found in [20]. Recall that a vector field X on a Riemannian manifold (M^n, g) is a conformal field in case

$$\frac{1}{2}\mathcal{L}_X g = \varphi g,$$

for some $\varphi \in C^\infty(M)$.

Recall that the k-Newton tensor field associated with $g^{-1}A_g$ is defined by

$$T_k(g^{-1}A_g) = \sum_{j=0}^{k} (-1)^j \sigma_{k-j}(g)(g^{-1}A_g)^j, \quad 1 \leqslant k \leqslant n.$$

Among the identities satisfied by $T_k(g^{-1}A_g)$ one finds (see [16])

$$\operatorname{tr} T_k(g^{-1}A_g) = (n-k)\sigma_k(g) \quad \text{and} \quad \operatorname{div} T_k(g^{-1}A_g) = 0,$$

for every $1 \leqslant k \leqslant n$.

Proposition 1. *If* X *is a conformal vector field on a compact Riemannian manifold* (M^n, g) *with null Cotton tensor, then*

$$\int_{M^n} \langle X, \nabla\sigma_k\rangle dv_g = 0,$$

for every $k = 1, 2, \ldots, n$.

Proof. Let $\varphi \in C^\infty(M)$ be such that

$$\frac{1}{2}\mathcal{L}_X g = \varphi g,$$

and take $T_k = T_k(g^{-1}A_g)$ where $k \in \{1, 2, \ldots, n-1\}$. Now a direct application of Lemma 1 yields

$$\int_M X(\operatorname{tr} T_k) dv_g = n \int_M \operatorname{div} T_k(X) dv_g + n \int_M \varphi\langle \mathring{T}_k, g\rangle dv_g. \tag{9}$$

It follows from Corollary 1 of [16] that $\operatorname{div} T_k = 0$ and because

$$\mathring{T}_k = T_k - \frac{\operatorname{tr} T_k}{n} g = T_k - \frac{n-k}{n}\sigma_k g,$$

Equation (9) can rewritten in the simpler form

$$(n-k)\int_M \langle X, \nabla\sigma_k\rangle dv_g = 0,$$

which proves the proposition in case $k \neq n$. As for the remaining case, it follows from [20] that

$$n\langle X, \nabla\sigma_n\rangle = \nabla_a\left[T^a_b \nabla^b(\operatorname{div} X) + 2n\sigma_n X^a\right],$$

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where T^a_b are the components of $T_{n-1}(g^{-1}A_g)$. Therefore, if we go there and write

$$Y^a = T^a_b \nabla^b (\operatorname{div} X) + 2n\sigma_n X^a,$$

we get that

$$n\int_M \langle X, \nabla\sigma_k\rangle dv_g = \int_M \nabla_a Y^a dv_g = 0,$$

which proves the proposition also for $k = n$. □

Our next lemma states some structural equations for almost quotient gradient Yamabe solitons.

Lemma 3. *Let $(M^n, g, \nabla f, \lambda)$ be an almost gradient quotient Yamabe soliton. Then, we have that:*

a) $\Delta f = n\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right)$*;*

b) $(n-1)\nabla\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) + \mathrm{Ric}(\nabla f) = 0$*;*

c) $(n-1)\Delta\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) + \frac{1}{2}\langle\nabla R, \nabla f\rangle + \left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right)R = 0.$

Proof.

a) The first assertion is obtained by tracing (5);
b) Next, we differentiate (5) to get

$$\nabla_j\nabla_r\nabla_i f = \nabla_j\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) g_{ri},$$

from what we see that

$$\nabla_i\nabla_j\nabla_r f + \sum_s R_{rijs}\nabla_s f = \nabla_j\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) g_{ri},$$

with the help of the Ricci identity that can be found in ([21], pg. 4). Now, we only need to contract this equation on the indices j, r in order to get

$$\nabla_i\Delta f + \sum_s Ric_{is}\nabla_s f = \nabla_i\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right),$$

then yielding

$$(n-1)\nabla_i\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) + \sum_s Ric_{is}\nabla_s f = 0, \tag{10}$$

by a), which proves the second assertion;
c) Now, we deal with the third one. We apply the divergence operator on both sides of (10) and use the twice contracted second Bianchi's identity to obtain

$$(n-1)\Delta\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) + \frac{1}{2}\langle\nabla R, \nabla f\rangle + \sum_{sl} Ric_{sl}\nabla_s\nabla_l f = 0,$$

which is equivalent to

$$(n-1)\Delta\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) + \frac{1}{2}\langle\nabla R, \nabla f\rangle + \left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right)R = 0,$$

by using the fundamental equation (5), one concludes the asserted result. □

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3. Proofs of the main results

This section contains proofs for the main results in this paper.

Proof of Theorem 1.

a) Integrating by parts, one sees that

$$\int_M \mathrm{Ric}_{jk}\nabla_i C_{ijk}dv_g = -\int_M \nabla_i \mathrm{Ric}_{jk} C_{ijk}dv_g = 0,$$

if either $\nabla\mathrm{Ric}_g = 0$ or div $C_g = 0$ and because

$$\begin{aligned}\int_M \nabla_i \mathrm{Ric}_{jk} C_{ijk}dv_g &= \\ &= \int_M \left[C_{ijk} + \frac{1}{2(n-1)}\left(g_{jk}\nabla_i R_g - g_{ij}\nabla_j R_g\right)\right]C_{ijk}dv_g \\ &= \int_M |C_g|^2 dv_g + \frac{1}{2(n-1)}\int_M \left(C_{ijk}g_{jk}\nabla_i R_g - C_{ijk}g_{ij}\nabla_j R_g\right)dv_g \\ &= \int_M |C_g|^2 dv_g, \end{aligned} \tag{11}$$

we conclude that $C_g = 0$. Equation (5) implies that X is a conformal field and so we can apply Proposition 1 to conclude that

$$\int_{M^n} \sigma_k\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right)dv_g = -\frac{1}{n}\int_{M^n}\langle\nabla\sigma_k, X\rangle dv_g = 0.$$

Therefore, we have

$$\begin{aligned}\int_{M^n}\frac{\sigma_l}{n}\left(\frac{\sigma_k}{\sigma_l} - e^{\lambda}\right)\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right)dv_g &= \\ &= -\int_{M^n}\frac{e^{\lambda}\sigma_l}{n}\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right)dv_g \\ &= \int_{M^n} e^{\lambda}\sigma_l\langle\nabla\lambda, X\rangle dv_g + \int_{M^n} e^{\lambda}\langle\nabla\sigma_l, X\rangle dv_g = 0, \end{aligned} \tag{12}$$

by our hypothesis on the nullity of the integral at the right hand of (12). Since $\sigma_l \neq 0$ does not change sign on M^n, we then admit that $\sigma_k/\sigma_l = e^{\lambda}$, which proves our assertion in case of parallel Ricci curvature or divergence free Cotton tensor. On the other hand, if $X = \nabla f$, we argue by contradiction to show that f is a constant function. Should f not be constant on M^n, the manifold (M^n, g) could not lie in any conformal class other than that of the Euclidean sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$, by Theorem 1.1 of [12]. So, just as it happens with any locally conformally flat manifold, the Cotton tensor of (M^n, g) would then vanish identically and by what has been said above $(M^n, g, \nabla f, \lambda)$ ought to be trivial. This contradiction shows that f is indeed a constant function, now concluding a);

b) Because the fields $\nabla h, Y$ in the Hodge-de Rham decomposition $X = \nabla h + Y$ of X are orthogonal to one another in $L^2(M)$, we get that

$$\int_{M^n}|\nabla h|^2 dv_g = \int_{M^n}\langle\nabla h, \nabla h + Y\rangle dv_g = \int_{M^n}\langle\nabla h, X\rangle dv_g \leqslant 0,$$

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the inequality is part of the hypothesis. Then, $\nabla h = 0$ and $X = Y$. Since Y is divergence free, we conclude that

$$n\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) = \mathrm{div}\, X = 0,$$

and, as such, the soliton is trivial. So, the proof of the theorem is complete. □

Proof of Theorem 2. As we already know, the fundamental equation

$$\frac{1}{2}\mathcal{L}_X g = \left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) g,$$

leads to

$$\mathrm{div}\, X = n\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right), \tag{13}$$

and because we suppose that $\mathcal{L}_X g \geqslant 0$ we must then admit that $\log\frac{\sigma_k}{\sigma_l} - \lambda \geqslant 0$. So, if we now take a cutoff function $\psi : M \to \mathbb{R}$ satisfying

$$0 \leqslant \psi \leqslant 1 \text{ on } M, \quad \psi \equiv 1 \text{ in } B_r(x_0), \quad \mathrm{supp}(\psi) \subset B_{2r}(x_0) \quad \text{and} \quad |\nabla\psi| \leqslant \frac{K}{r},$$

where $K > 0$ is a real constant, we are in place to conclude that

$$\begin{aligned}
n\int_{B_r(x_0)}\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) dv_g &= \int_{B_r(x_0)} n\psi\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) dv_g \\
&\leqslant \int_{B_{2r}(x_0)} n\psi\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) dv_g \\
&= \int_{B_{2r}(x_0)} \psi \,\mathrm{div}\, X dv_g \\
&= -\int_{B_{2r}(x_0)} g(\nabla\psi, X) dv_g \\
&\leqslant \int_{B_{2r}(x_0)\setminus B_r(x_0)} |-\nabla\psi||X| dv_g \\
&\leqslant K\int_{B_{2r}(x_0)\setminus B_r(x_0)} \frac{|X|}{r} dv_g, \\
&\leqslant 2K\int_{M\setminus B_r(x_0)} \frac{|X|}{d(x, x_0)} dv_g,
\end{aligned}$$

from what it follows that

$$\begin{aligned}
0 \leqslant \int_M\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) dv_g &= \lim_{r\to\infty}\int_{B_r(x_0)}\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) dv_g \\
&\leqslant \frac{2K}{n}\lim_{r\to\infty}\int_{M\setminus B_r(x_0)} \frac{|X|}{d(x, x_0)} dv_g = 0.
\end{aligned}$$

Henceforth, we have that $\mathcal{L}_X g = \log\frac{\sigma_k}{\sigma_l} - \lambda = 0$ which proves the Theorem. □

Proof of Theorem 3. It follows from Lemma 3 (c) that if the scalar curvature of $(M^n, g, \nabla f, \lambda)$ is a constant function on M^n, then

$$\Delta\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) + \frac{R}{n-1}\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) = 0, \tag{14}$$

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and, by the min-max principle, we must have $R > 0$. By using that

$$\Delta f = n\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right), \tag{15}$$

we then get

$$\Delta\left(\log\frac{\sigma_k}{\sigma_l} - \lambda + \frac{R}{n(n-1)}f\right) = 0,$$

and since (M^n, g) is a compact Riemannian manifold, one see that

$$\log\frac{\sigma_k}{\sigma_l} - \lambda + \frac{R}{n(n-1)}f = c \quad \text{on} \quad M^n,$$

for a certain $c \in \mathbb{R}$, by the maximum principle. Hence,

$$\nabla\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) + \frac{R}{n(n-1)}\nabla f = 0,$$

and so

$$\nabla_X\nabla\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right) = -\frac{R}{n(n-1)}\nabla_X\nabla f = -\frac{R}{n(n-1)}\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right)X.$$

We can now apply Obata's theorem ([22], Theorem 1) to conclude that (M^n, g) is isometric with an Euclidean sphere of radius $\sqrt{r}$, $r = R/n(n-1)$. To prove our last claim, we notice that we can assume that $R = n(n-1)$ possibly at the cost of rescaling the metric g. From equations (14) and (15), it is seen that $\frac{\Delta f}{n}$ is an eigenfunction of the Laplacian on $(\mathbb{S}^n, g)$ and so there must exist a $v \in \mathbb{S}^n$ such that $\frac{1}{n}\Delta f = h_v = -\frac{1}{n}\Delta h_v$. Hence, $\Delta(f + h_v) = 0$ but then $f = h_v + c$ for some real c. □

Proof of Theorem 4. By Theorem 1.1 of [12] the only nontrivial compact almost gradient quotient Yamabe solitons reside in the conformal class of the Euclidean sphere and because of that we can assume that

$$M^n = \mathbb{S}^n \quad \text{and} \quad \varphi^{-2}g = g_{\mathbb{S}^n},$$

where $\varphi \in C^\infty(\mathbb{S}^n)$ is strictly positive. Then, the Ricci tensors of g and $g_{\mathbb{S}^n}$ are correlated by the equation [23]

$$\mathrm{Ric}_{\mathbb{S}^n} = \mathrm{Ric}_g + \frac{1}{\varphi^2}\left\{(n-2)\varphi\nabla^2\varphi + [\varphi\Delta\varphi - (n-1)|\nabla\varphi|^2]g\right\},$$

which we algebraically manipulate in order to get the similar equation

$$A_{g_{\mathbb{S}^n}} = A_g + \frac{\nabla^2\varphi}{\varphi} - \frac{1}{2}\frac{|\nabla\varphi|^2}{\varphi^2}g, \tag{16}$$

for the Schouten tensors. But then we have

$$\frac{1}{2}\left(\varphi^2 + \frac{|\nabla\varphi|^2}{\varphi^2}\right)g = A_g + \frac{\nabla^2\varphi}{\varphi},$$

from what it follows that

$$\nabla^2\varphi = \varphi\left[-A_g + \frac{1}{n}\left(\sigma_1(g) + \frac{\Delta\varphi}{\varphi}\right)g\right]. \tag{17}$$

Notice that Lemma 1 applied to $T = T_k(g^{-1}A_g)$ and $X = \nabla\varphi$ gives

$$\int_M \langle T_k(g^{-1}A_g), \nabla^2\varphi\rangle dv_g = 0, \tag{18}$$

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because $\operatorname{tr} T_k(g^{-1}A_g) = (n-k)\sigma_k(g)$ is constant on $\mathbb{S}^n$ by hypothesis and $\operatorname{div} T_k(g^{-1}A_g) = 0$. A combination of (18) and (17) above leads to

$$\begin{aligned} 0 &= \int_M \langle T_k(g^{-1}A_g), -\varphi A_g + \frac{\sigma_1(g)\varphi + \Delta\varphi}{n} g\rangle dv_g = 0 \\ &= \int_M \left[-\varphi \langle T_k(g^{-1}A_g), A_g\rangle + \frac{\sigma_1(g)\varphi + \Delta\varphi}{n} \langle T_k(g^{-1}A_g), g\rangle \right] dv_g \\ &= \int_M \varphi \left[\left(\frac{n-k}{n} \right) \sigma_1(g)\sigma_k(g) - (k+1)\sigma_{k+1}(g) \right] dv_g \end{aligned}$$

where we have used the identity $\operatorname{tr} T_k(g^{-1}A_g \circ A_g) = (k+1)\sigma_{k+1}(g)$ [24]. By Lemma 23 of [25], we conclude that

$$\left(\frac{n-k}{n} \right) \sigma_1\sigma_k = (k+1)\sigma_{k+1},$$

implying that $(\mathbb{S}^n, g)$ is an Einstein manifold. In particular, the scalar curvature of g is constant on $\mathbb{S}^n$ and by Theorem 3 there is even an isometry between $(\mathbb{S}^n, g)$ and $(\mathbb{S}^n, g_{\mathbb{S}^n})$ which proves the Theorem. □

Proof of Theorem 5. Lemma 2 applied to the data $T = \mathring{\mathrm{Ric}}_g$, $X = \nabla f$, and $\varphi = f$ gives

$$\operatorname{div} \mathring{\mathrm{Ric}}_g(f\nabla f) = f(\operatorname{div} \mathring{\mathrm{Ric}}_g)(\nabla f) + f\langle \nabla^2 f, \mathring{\mathrm{Ric}}_g\rangle + \mathring{\mathrm{Ric}}_g(\nabla f, \nabla f), \tag{19}$$

and it then follows from the second contracted Bianchi identity that

$$(\operatorname{div} \mathring{\mathrm{Ric}}_g)(\nabla f) = \frac{n-2}{2n}\langle \nabla f, \nabla R\rangle. \tag{20}$$

A straightforward computation shows that

$$f\langle \nabla^2 f, \mathring{\mathrm{Ric}}_g\rangle = f\left(\log\frac{\sigma_k}{\sigma_l} - \lambda\right)\langle g, \mathring{\mathrm{Ric}}_g\rangle = 0, \tag{21}$$

and equations (19), (20), and (21) together give

$$\frac{1}{2}\operatorname{div} \mathring{\mathrm{Ric}}_g(\nabla f^2) = \frac{n-2}{4n}\langle \nabla R_g, \nabla f^2\rangle + \mathring{\mathrm{Ric}}_g(\nabla f, \nabla f). \tag{22}$$

Proposition 1 of [26] tell us that $\operatorname{div} \mathring{\mathrm{Ric}}_g(\nabla f^2) = 0$ because $|\mathring{\mathrm{Ric}}_g(\nabla f^2)| \in L^1(M)$. Consequently,

$$\langle \nabla R_g, \nabla f^2\rangle = 0 \quad \text{and} \quad \mathring{\mathrm{Ric}}_g(\nabla f, \nabla f) = 0.$$

As $(M^n, g, \nabla f, \lambda)$ is a nontrivial almost quotient gradient Yamabe soliton, any regular level set Σ of the potential function f admits a maximal open neighborhood $U \subset M$ in which g can be written like

$$g = dr \otimes dr + (f'(r))^2 g^{\Sigma}, \tag{23}$$

where g^{Σ} is the restriction of g to Σ (see [12]). Since M is noncompact, f has at most one critical point. As the Ricci tensor of a warped product metric, Ric_g now admits the following decomposition

$$\mathrm{Ric}_g = \mathrm{Ric}^{\Sigma} - (n-1)\frac{f'''}{f'} dr \otimes dr - [(n-2)(f'')^2 + f'f''']g^{\Sigma}, \tag{24}$$

thus giving $\frac{R_g}{n} = -(n-1)\frac{f'''}{f'}$ because $\mathring{\mathrm{Ric}}_g(\nabla f, \nabla f) = 0$. Equation (24) can also be manipulated to show that

$$\mathrm{Ric}_g(\nabla f) = \frac{R_g}{n}\nabla f,$$

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of which

$$\nabla\left(\log\frac{\sigma_k}{\sigma_l}-\lambda\right)+\frac{R_g}{n(n-1)}\nabla f=0, \tag{25}$$

is a consequence by Lemma 3 b). The divergence of equation (25) is

$$\Delta\left(\log\frac{\sigma_k}{\sigma_l}-\lambda\right)+\frac{1}{n(n-1)}\langle\nabla R_g,\nabla f\rangle+\frac{R_g}{n-1}\left(\log\frac{\sigma_k}{\sigma_l}-\lambda\right)=0. \tag{26}$$

which we compare with the expression in Lemma 3 c) to see that $\langle\nabla R_g,\nabla f\rangle=0$. Since R_g only depends on r we get that

$$f'R_g'=f'\langle\nabla R_g,\partial r\rangle=\langle\nabla R_g,\nabla f\rangle=0,$$

implying that the scalar curvature $R_g=R$ is constant. We claim that $R\leqslant 0$. As a matter of fact, if we had $R>0$, then from (25) we would then have that $\log\frac{\sigma_k}{\sigma_l}-\lambda$ is not constant on M^n and satisfies

$$\nabla_X\nabla\left(\log\frac{\sigma_k}{\sigma_l}-\lambda\right)=-\frac{R}{n(n-1)}\nabla_X\nabla f=-\frac{R}{n(n-1)}\left(\log\frac{\sigma_k}{\sigma_l}-\lambda\right)X.$$

From Obata's theorem [22], the manifold M^n would then be compact, which is absurd. Therefore, $R\leqslant 0$.

a) It follows from (25) that $\log\frac{\sigma_k}{\sigma_l}-\lambda=c$ for some $c\in\mathbb{R}$ because we now have $R=0$. By Theorem 2 of [27] (M^n,g) must be isometric with flat Euclidean space $\mathbb{R}^n$ in case $c\neq 0$. Since this would leave us with $\sigma_1(g)=\sigma_2(g)=\cdots=\sigma_n(g)=0$, the function $\log\frac{\sigma_k}{\sigma_l}$ could not be defined. Then, $c=0$ and so $\nabla^2 f=0$ by the fundamental equation (5). Theorem B of Kanai [28] then implies that (M^n,g) is isometric with a Riemannian product manifold $\mathbb{R}\times\mathbb{F}^{n-1}$;

b) If f has no critical points and $R<0$, then once more by (25) we get that $\log\frac{\sigma_k}{\sigma_l}-\lambda$ is not constant on M^n and satisfies

$$\nabla_X\nabla\left(\log\frac{\sigma_k}{\sigma_l}-\lambda\right)=-\frac{R}{n(n-1)}\nabla_X\nabla f=-\frac{R}{n(n-1)}\left(\log\frac{\sigma_k}{\sigma_l}-\lambda\right)X,$$

on M^n for every $X\in\mathfrak{X}(M)$. In virtue of Theorem D in [28], the manifold (M^n,g) is isometric with a warped product manifold $(\mathbb{R}\times\mathbb{F}^{n-1},dr^2+\xi(r)^2g_{\mathbb{F}})$ in which the warping function ξ solves the second-order linear ODE with constant coefficients $\xi''+\frac{R}{n(n-1)}\xi=0$;

c) In our last call to equation (25), we observe that if f has exactly one critical point and $R<0$ then $\log\frac{\sigma_k}{\sigma_l}-\lambda$ is not constant on M^n and must satisfy

$$\nabla_X\nabla\left(\log\frac{\sigma_k}{\sigma_l}-\lambda\right)=-\frac{R}{n(n-1)}\nabla_X\nabla f=-\frac{R}{n(n-1)}\left(\log\frac{\sigma_k}{\sigma_l}-\lambda\right)X,$$

on M^n for every $X\in\mathfrak{X}(M)$. We then apply Theorem C in [28] to conclude that (M^n,g) is isometric with a hyperbolic space. □

Competing interests. The authors declare that there is no competing interests.

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