Glasgow Math. J. **60** (2018) 63–96. © Glasgow Mathematical Journal Trust 2017.
doi:10.1017/S0017089516000574.

ASYMPTOTIC TRIANGULATIONS AND COXETER TRANSFORMATIONS OF THE ANNULUS

HANNAH VOGEL
NAWI Graz, Institute for Mathematics and Scientific Computing, University of Graz, Heinrichstraße 36, 8010 Graz, Austria
e-mail: hannah.vogel@uni-graz.at

and with appendix B by ANNA FELIKSON and PAVEL TUMARKIN
Department of Mathematical Sciences, Durham University, Science Laboratories, South Road, Durham, DH1 3LE, United Kingdom
e-mails: anna.felikson@durham.ac.uk, pavel.tumarkin@durham.ac.uk

(Received 8 October 2015; revised 8 March 2016; accepted 12 July 2016; first published online 27 February 2017)

Abstract. Asymptotic triangulations can be viewed as limits of triangulations under the action of the mapping class group. In the case of the annulus, such triangulations have been introduced in K. Baur and G. Dupont (Compactifying exchange graphs: Annuli and tubes, *Ann. Comb.* **3**(18) (2014), 797–839). We construct an alternative method of obtaining these asymptotic triangulations using Coxeter transformations. This provides us with an algebraic and combinatorial framework for studying these limits via the associated quivers.

2010 *Mathematics Subject Classification.* 16G20, 20F55, 13F60.

1. Introduction. Asymptotic triangulations were introduced by Baur and Dupont in [**2**], with respect to unpunctured marked surfaces. These asymptotic triangulations can be mutated as usual triangulations, and they provide a natural way to compactify the usual exchange graph of the triangulations of an annulus. Asymptotic triangulations are defined by the presence of strictly asymptotic arcs. There are two types of asymptotic arcs: Prüfer and adic. The names of these arcs come from the Prüfer and adic modules. These are infinite-dimensional modules in the representation theory of finite-dimensional algebras.

We can associate a quiver to a triangulation. This quiver can be mutated as defined by Fomin and Zelevinsky [**5**]. Quivers can be associated to asymptotic triangulations in the same way as for finite triangulations. However, these quivers may have loops and two-cycles, and therefore cannot be mutated in the usual manner. This paper provides a way to mutate these associated cyclic quivers using quivers with potentials, and presents an alternative quiver model.

Asymptotic triangulations can be reached from finite triangulations through the Dehn twist. The Dehn twist is a topological move that can be visualized in the triangulation. In this paper, we introduce an equivalent, combinatorial move, namely the Coxeter transformation, which can be visualized on the quiver.

Coxeter transformation are important in the study of representations of algebras, quivers, partially ordered sets, and lattices. In this paper, we describe how Coxeter transformations act on triangulations, and quivers associated to triangulations. First, we briefly introduce Coxeter transformations.

Let Γ be an oriented graph with vertex set Γ_0, $|\Gamma_0| = n$, and edge set Γ_1. An arrow $\alpha \in \Gamma_1$, $\alpha : i \to j$, *starts* at $s(\alpha) = i$, and *terminates* at $t(\alpha) = j$.

To Γ, we can associate its *Euler form*, a bilinear form on $\mathbb{Z}^n$:

$$\langle -, - \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z} \quad \text{with} \quad \langle x, y \rangle = \sum_{i \in \Gamma_0} x_i y_i - \sum_{\alpha \in \Gamma_1} x_{s(\alpha)} y_{t(\alpha)}.$$

We obtain the following symmetric bilinear form on $\mathbb{Z}^n$:

$$(x, y) = \langle x, y \rangle + \langle y, x \rangle.$$

If Γ has no loops, we can define the *reflection map* with respect to a vertex i:

$$\sigma_i : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \quad \text{with} \quad \sigma_i(x) = x - \frac{2(x, e_i)}{(e_i, e_i)} e_i,$$

where e_i is the ith coordinate vector. The σ_i are automorphisms of $\mathbb{Z}^n$ of order two that preserve the bilinear form $(-, -)$.

A vertex i of Γ is called a *source* (resp. *sink*) if there is no arrow in Γ ending (resp. starting) at i. If i is a source or a sink, the graph $\sigma_i \Gamma$ is obtained from Γ by reversing all arrows which start or end at i.

DEFINITION 1.1. An ordering $i_1, \dots, i_n$ of the vertices of Γ is called *source-admissible* if for each p the vertex i_p is a source for $\sigma_{i_{p-1}} \dots \sigma_{i_1} \Gamma$.

In this case, we have that

$$\sigma_{i_n} \sigma_{i_{n-1}} \dots \sigma_{i_2} \sigma_{i_1} \Gamma = \Gamma.$$

Now if Γ is an acyclic graph, and $i_1, \dots, i_n$ is an admissible ordering of its vertices, then the automorphism

$$c : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \quad \text{with} \quad c(x) = \sigma_{i_n} \dots \sigma_{i_1}(x)$$

is called a *Coxeter transformation*.

For oriented trees, there always exists an admissible ordering $\underline{i}$. To every such sequence, we assign a Coxeter transformation depending on the order of the vertices in $\underline{i}$:

$$c = \sigma_{i_n} \sigma_{i_{n-1}} \dots \sigma_{i_2} \sigma_{i_1}.$$

For every orientation of a given simply laced Dynkin diagram, every admissible ordering gives rise to the same Coxeter transformation [**7**]. If the underlying graph is not a tree, we need to consider the orientations of the arrows in the graph before we can assign a Coxeter transformation to the graph.

In this paper, we will focus on triangulations of annuli. It is known that such triangulations give rise to cluster algebras of extended Dynkin type $\tilde{A}_n$. We introduce triangulations and quivers in Section 2, and define asymptotic triangulations and their associated quivers in Section 3. In Sections 4–6, we discuss sequences of flips in triangulations that correspond to Coxeter transformations on the associated quiver, and we describe what happens in the limit of these transformations. Appendix A gives

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

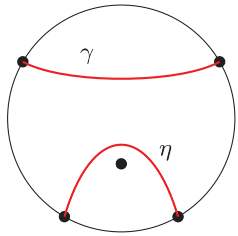


Figure 1. (Colour online) Two arcs γ, η of a marked surface S.

an alternative way to perform quiver mutation for quivers associated to asymptotic triangulations by using potentials, and Appendix B introduces an alternative cluster structure on asymptotic triangulations, and gives a geometric interpretation of these structures.

2. Definitions and notation.

2.1. Triangulations. Let S be a connected, oriented Riemann surface with boundary, and let M be a finite set of marked points in the closure of S. We assume that M is non-empty, and there is at least one marked point on each boundary component. We choose a counter-clockwise orientation of S and label the marked points on each boundary component in a counter-clockwise order.

DEFINITION 2.1. An *arc* γ of a marked surface S is a curve whose endpoints are marked points of S, and which does not intersect itself in the interior of S. The interior of the arc is disjoint from the boundary of S and it does not cut out an unpunctured monogon or digon.

We consider arcs in (S, M) up to isotopy. We write $\gamma = [i, j]$ to denote the arc with endpoints $i, j \in M$. Note that depending on the surface, there may be multiple (non-isotopic) arcs with the same endpoints. See Figure 1 for an example of two arcs in a marked surface.

DEFINITION 2.2. An *ideal triangulation* T of a surface S is a maximal collection of pairwise non-intersecting arcs of S.

Throughout this paper, all triangulations will be ideal. The arcs of a triangulation T cut S into (*ideal*) *triangles*. Triangles are three-sided regions, and self-folded triangles (interior triangles that contain exactly two arcs) may occur.

DEFINITION 2.3. A *flip* μ of an arc in a triangulation is a move that replaces an arc of any given quadrilateral with the other arc in the quadrilateral (cf. Figure 3). We sometimes use μ_k to indicate a flip at the arc d_k.

Given a marked surface (S, M), there may exist many different triangulations of S (cf. Figure 2). By a theorem of **[6]**, any two triangulations of a surface S are related by a sequence of flips. For an example, see Figure 4.

For the rest of this paper, we will be considering triangulations of the annulus (a region bounded by two concentric circles).

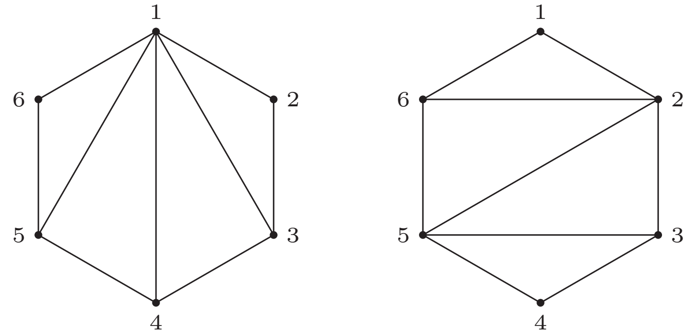


Figure 2. Two triangulations of the hexagon P_6.

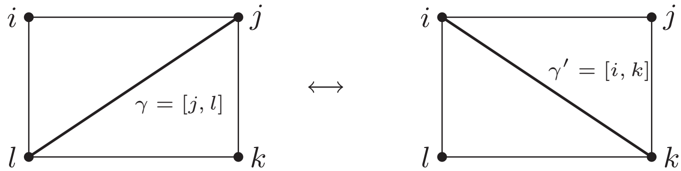


Figure 3. Flip of the arc γ.

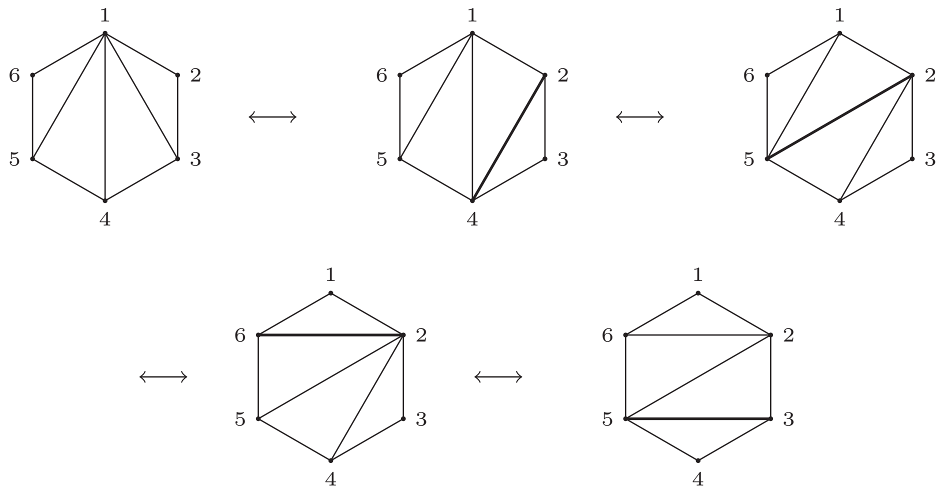


Figure 4. Sequence of flips.

DEFINITION 2.4. $C_{p,q}$ denotes the annulus with $p > 0$ points marked on the outer boundary component ∂, and $q > 0$ marked points on the inner boundary component ∂'. Without loss of generality, we assume that $p \geq q$.

DEFINITION 2.5. An arc in $C_{p,q}$ is called *peripheral* if its two endpoints lie on the same boundary component. It is called *bridging* otherwise.

DEFINITION 2.6. Let T be a triangulation of $C_{p,q}$. A peripheral arc $\gamma \in T$ is called *bounding* (with respect to T) if the flip μ_γ of arc γ is a bridging arc.

Figure 5 shows an example of a triangulated annulus.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

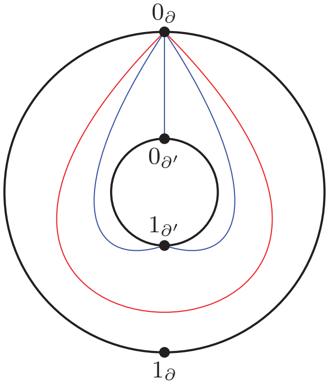


Figure 5. (Colour online) A triangulation of $C_{2,2}$. The bridging arcs are marked in blue and the peripheral arc is marked in red.

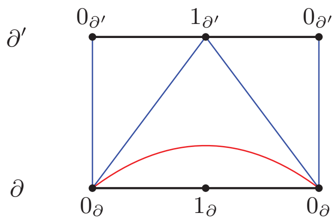


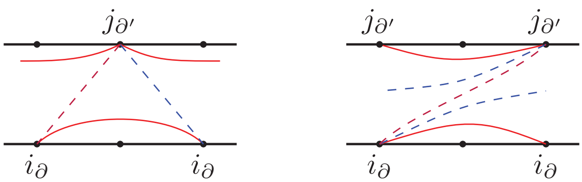
Figure 6. (Colour online) A triangulation of $C_{2,2}$ represented as a cylinder.

It is usually more convenient to work with an opened-up picture of the annulus. We can identify the annulus with a cylinder of height 1, where the bottom boundary of the cylinder corresponds to the outer boundary of the annulus, and the upper boundary of the cylinder corresponds to the inner boundary of the annulus. See Figure 6 for an example of a triangulation of $C_{2,2}$ drawn as a cylinder $Cyl_{2,2}$.

The following result appears in [**2**, Lemma 1.7]. For convenience, we include a proof below.

LEMMA 2.7. *A triangulation T of the annulus contains at least two bridging arcs.*

Proof. Let T be a triangulation of $C_{p,q}$. Then there is at least one point on each boundary component which does not have a peripheral arc lying above it. The triangulation T requires at least two bridging arcs connecting these two points. Two examples are drawn below: The leftmost figure depicts a triangulation of the annulus that has two bridging arcs $[i_\partial, j_{\partial'}]$, and the rightmost figure shows an example where the blue arc $[i_\partial, j_{\partial'}]$ spirals once around the centre of the annulus, and is drawn as leaving the frame on the right and entering back in on the left. Each frame contains two bridging arcs.



□

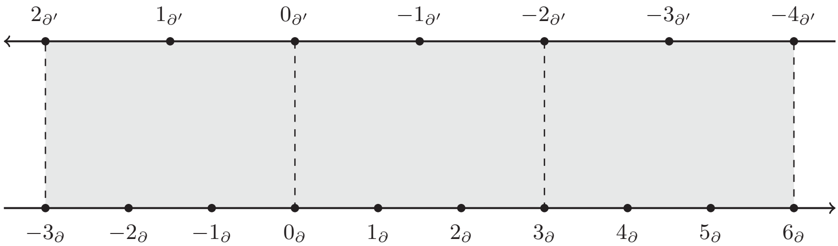


Figure 7. Universal cover of the annulus $C_{3,2}$.

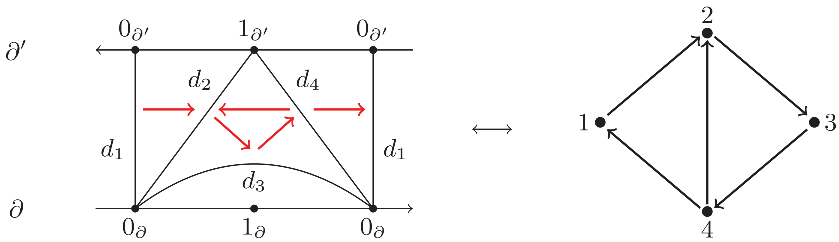


Figure 8. (Colour online) Triangulation T and associated quiver Q_T.

2.2. Universal cover of the annulus. It is convenient to work with the universal cover of the annulus. For an explicit construction, see [**3**].

We tile an infinite strip with rectangles (which we call *frames*), and in each rectangle we mark q points at the top border, and p points at the bottom border, where the points are placed equidistant from each other. We consider each frame to be a copy of $C_{p,q}$ drawn as a cylinder. We keep the orientation of the annulus, so the marked points are labelled left to right on the lower boundary, and right to left on the upper boundary. We choose an initial frame, label the bottom and top left-most points by 0_∂ and $0_{\partial'}$, respectively. We label the rest of the points (increasing or decreasing by increments of 1) following the orientation of the boundaries, as illustrated in Figure 7.

2.3. Quivers. A *quiver* $Q = (Q_0, Q_1)$ is an oriented graph. Q_0 denotes the set of vertices of Q, and Q_1 denotes the set of arrows between vertices. The right-hand side of Figure 8 gives an example of a quiver. Given a triangulation T of a surface S, we can associate a quiver to T.

DEFINITION 2.8. The quiver Q_T associated to a triangulation T is obtained as follows:

(1) The vertices of Q_T correspond to the arcs in T, with vertex i corresponding to the arc d_i.
(2) There is an arrow from i to j in Q_T if d_i and d_j in T bound a common triangle, and d_j is a clockwise rotation of d_i.

For two arcs to bound a common triangle, they must have a common endpoint. Let $d_i = [a, b]$ and $d_j = [b, c]$ be two arcs in a triangulation T of S. Then d_j is a clockwise rotation of d_i if the endpoint of d_i at the marked point a can be rotated clockwise to the marked point c of S such that the new arc $d_i' = [b, c]$ is isotopic to d_j. An example of a quiver associated to a triangulation is illustrated in Figure 8.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

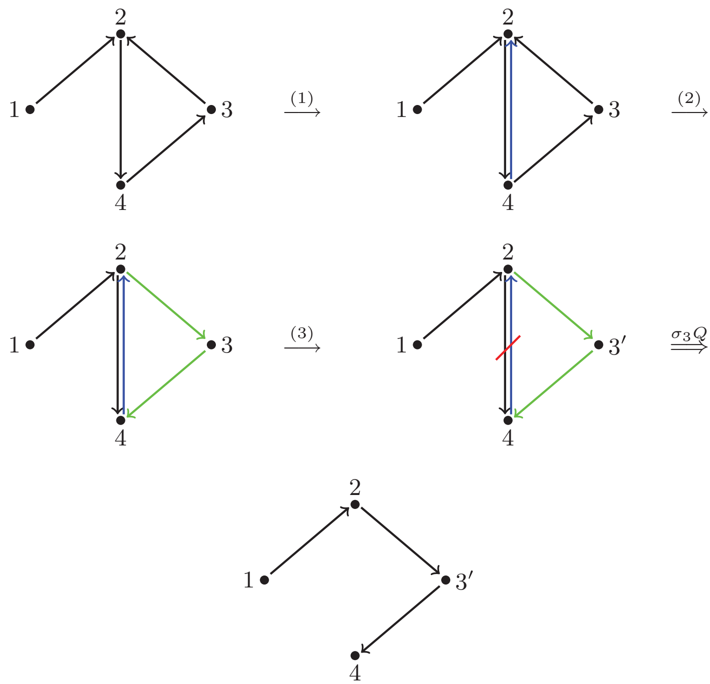
Recall that we can perform flips of arcs in a triangulation. On the level of quivers, there exists a procedure called mutation of vertices.

DEFINITION 2.9. Let Q be a quiver without loops or two-cycles. The mutation of a vertex $k \in Q_0$ is defined as follows:

(1) For all paths of the form $i \xrightarrow{a} k \xrightarrow{b} j$, where a, b denote the multiplicity of the arrows, add arrow $i \xrightarrow{ab} j$ to Q.
(2) Reverse all arrows incident with k.
(3) Cancel a maximal number of two cycles created in (1).

Mutation of a vertex k will be denoted by σ_k.

EXAMPLE 2.10. Let Q be the following quiver, and consider mutation at vertex 3. The blue arrow $4 \to 2$ is the arrow added in step (1) of Definition 2.9, since there is a path $4 \to 3 \to 2$. We then reverse the arrows incident to the vertex 3 (Definition 2.9, step (2)). We cancel out the newly arising two-cycle (Definition 2.9, step (3)):



The flips of arcs in a triangulation T and mutations of the associated quiver Q_T correspond to each other.

3. Asymptotic triangulations. In this section, we recall *asymptotic triangulations*, which are defined by the presence of *strictly asymptotic arcs*, and were first defined by Baur and Dupont in [**2**], where they view asymptotic triangulations as limits of triangulations under the action of the mapping class group. We will first define strictly asymptotic arcs, asymptotic triangulations, and flips of asymptotic arcs. To any asymptotic triangulation, we can associate a quiver as in Definition 2.8. Such a quiver may have loops and two-cycles, and hence classical quiver mutation cannot

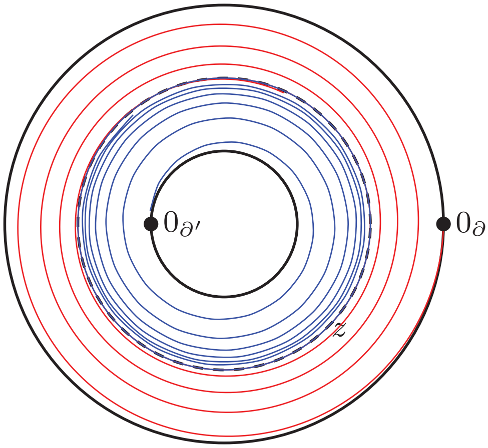


Figure 9. (Colour online) Adic arc π_{0_∂} in red, Prüfer arc $\pi_{0_{\partial'}}$ in blue.

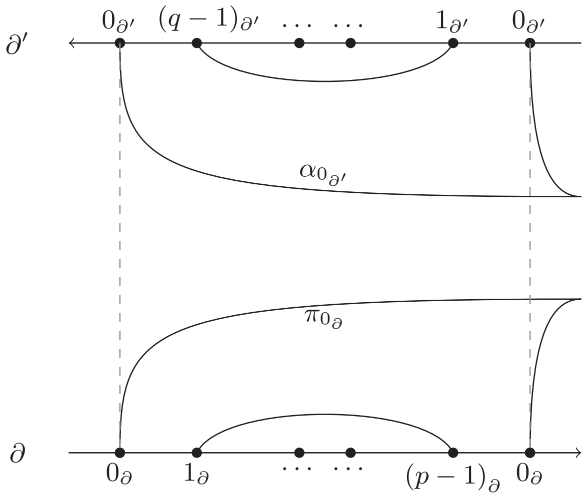


Figure 10. Asymptotic arcs in $C_{p,q}$.

be applied. In Section 3.1, we introduce a modified version of quivers of asymptotic triangulations in order to deal with this issue.

We denote by z a non-contractible closed curve in the annulus.

DEFINITION 3.1. Let m be a marked point of $C_{p,q}$. Let π_m be the isotopy class of the arc starting at m and spiraling positively around the annulus. We call π_m the *Prüfer arc* at m. Similarly, let α_m be the isotopy class of the arc starting at m and spiraling negatively around the annulus. We call α_m the *adic arc* at m (cf. Figures 9 and 10).

We call π_m and α_m *strictly asymptotic* arcs. We define the set of asymptotic arcs to be the union of the finite arcs and the strictly asymptotic arcs in a triangulation. Two arcs of $C_{p,q}$ are compatible if they do not intersect.

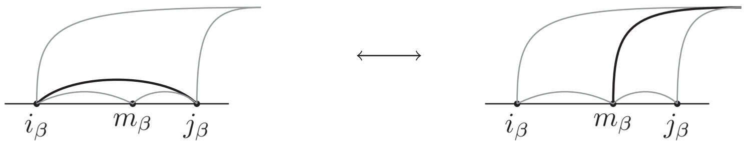


Figure 11. Flips of asymptotic arcs.

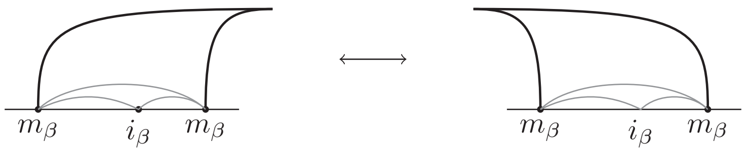


Figure 12. Flips of asymptotic arcs with one strictly asymptotic in the partial triangulation.

Figure 13. Bounding arcs $\gamma, \gamma_1, \gamma_2$.

DEFINITION 3.2. An *asymptotic triangulation of the annulus* is a maximal collection of pairwise distinct and compatible asymptotic arcs, and contains strictly asymptotic arcs.

Figure 9 shows two asymptotic arcs in the annulus, spiraling around z, and Figure 10 shows examples of asymptotic arcs drawn in the cylinder $Cyl_{p,q}$.

DEFINITION 3.3. Let $\beta \in \{\partial, \partial'\}$ be a boundary component. We say that an asymptotic arc is *based at* β if it is either a peripheral arc with both endpoints on β, or it is a strictly asymptotic arc with its unique endpoint on β. A *partial asymptotic triangulation* T_β is the collection of arcs of an asymptotic triangulation based on the boundary component β.

The following result is from **[2]** (see paper for proof).

LEMMA 3.4. *Let T be an asymptotic triangulation of $C_{p,q}$. Then T contains at least two strictly asymptotic arcs, and there are two partial asymptotic triangulations T_∂, $T_{\partial'}$ based at ∂, ∂', respectively, such that $T = T_\partial \sqcup T_{\partial'}$.*

Flips of asymptotic arcs are defined in the same way as flips of finite arcs, except we may consider quadrilaterals formed with strictly asymptotic arcs (cf. Figure 11). When there is only one strictly asymptotic arc in the partial asymptotic triangulation of a boundary component, based at a marked point m, then $\mu_{\pi_m} = \alpha_m$, and $\mu_{\alpha_m} = \pi_m$ (Figure 12). This is because the strictly asymptotic arc α_m (π_m) is the only arc compatible with $T \setminus \{\pi_m\}$ ($T \setminus \{\alpha_m\}$).

We can extend Definition 2.6 to the context of asymptotic arcs.

DEFINITION 3.5. A *bounding arc* γ is a finite arc in an asymptotic triangulation such that the flipped arc μ_γ is a strictly asymptotic arc.

Figure 13 shows three examples of bounding arcs. Bounding arcs are the finite arcs "closest" to the asymptotic arcs in a triangulation. We call them bounding arcs

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

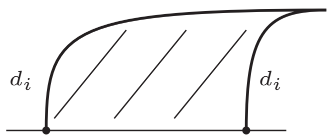
because they separate all other non-bounding finite arcs from the strictly asymptotic arcs in the asymptotic triangulation.

Now just as in the finite case, given an asymptotic triangulation T, we can associate a quiver to T (cf. Definition 2.8).

3.1. Quivers of asymptotic triangulations. It is easy to see that the quiver associated to an asymptotic triangulation always has two connected components. It contains a quiver Q_∂ corresponding to the triangulation based on the outer boundary T_∂, and the quiver $Q_{\partial'}$ corresponding to the triangulation based on the inner boundary $T_{\partial'}$, and $Q_T = Q_\partial \sqcup Q_{\partial'}$.

Quiver mutation as in Definition 2.9 only works for loop-free quivers without two-cycles. If we associate quivers to asymptotic triangulations as in Definition 2.8 then loops and two-cycles may appear. In order to define quiver mutation in this set-up, we need to modify the definition of a quiver associated to an asymptotic triangulation.

Recall that a frame of T_β is one lift of T_β in the universal cover. We start by choosing a frame of T_β for $\beta \in \{\partial, \partial'\}$ with two copies of a strictly asymptotic arc as the end arcs of the frame. If we need to specify, we refer to this as a d_i frame, with d_i the *framing arc*, and denote it by $T_\beta(d_i)$.



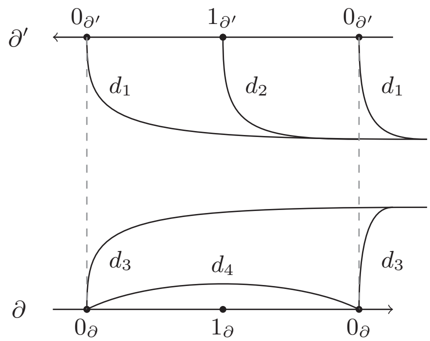
In the frame of T_β, each arc gives rise to a vertex in Q_β, and in particular, each copy of d_i gives rise to separate vertex in Q_β. We denote the quiver corresponding to $T_\beta(d_i)$ by $Q_\beta(i)$.

We call these two i vertices *framing vertices*. These framing vertices do not get mutated during the quiver mutation. However, we don't consider them to be frozen because we allow arrows between framing vertices. If we want to mutate these vertices, we need to switch from our quiver $Q_\beta(i)$ to a new quiver $Q_\beta(j)$, for d_j another strictly asymptotic arc in T_β. The corresponding operation in our triangulation is switching frames in the universal cover. We can go between a frame $T_\beta(d_i)$ and another frame $T_\beta(d_j)$ by shifting in one direction in our universal cover until we reach another strictly asymptotic arc d_j, which we now choose to be our framing arc. If there is no other strictly asymptotic arc in T_β, then we cannot switch frames, and therefore we cannot mutate the vertex $i \in Q_\beta(i)$. Recall from Figure 12 that when we only have one strictly asymptotic arc γ, then a flip will only reverse the orientation of γ without affecting the quiver. Since $\sigma_\gamma Q_\beta = Q_\beta$ for γ the only strictly asymptotic arc in T_β, we can use this definition of quiver mutation for a framing quiver.

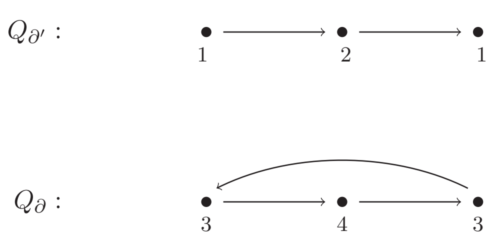
If there is another strictly asymptotic arc in the associated triangulation $T_\beta(d_i)$, then we can mutate our frozen vertices by modifying our quiver $Q_\beta(i)$. Let d_j be another strictly asymptotic arc in T_β. Then we can move between $Q_\beta(i)$ and $Q_\beta(j)$ by identifying the i vertices in $Q_\beta(i)$, and then break the quiver at vertex j so that our quiver now has two j vertices. All arrows remain the same.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

EXAMPLE 3.6. Let T be the following asymptotic triangulation of $C_{2,2}$:



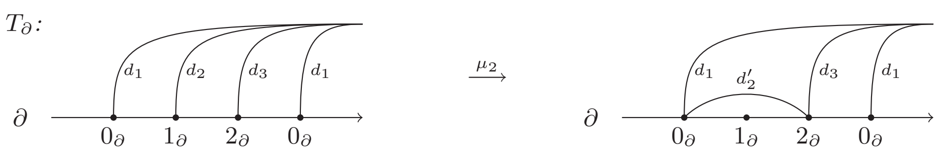
Then the quivers $Q_{\partial'}$ and Q_{∂} are



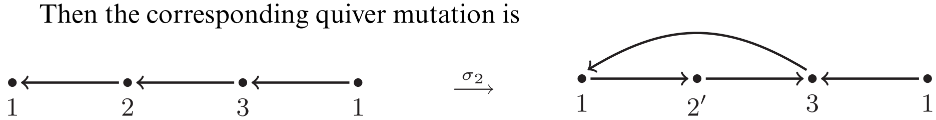
Now we can perform the classical quiver mutation as per Definition 2.9.

EXAMPLE 3.7.

Let T_{∂} be the following asymptotic triangulation based on boundary component ∂, and let Q_{∂} be the quiver associated to T_{∂}. Consider what happens when we flip the arc d_2 in T.



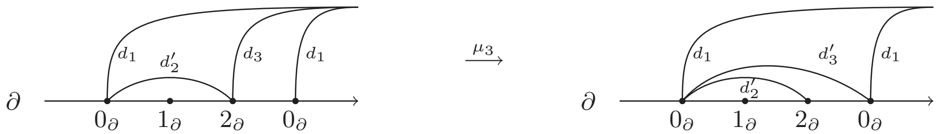
Then the corresponding quiver mutation is

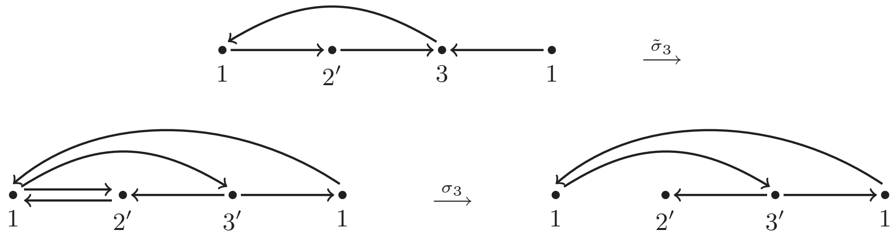


And we have the quiver $\sigma_2 Q$. Note that if we were to identify the framing vertices, we would have a two-cycle between $1 \leftrightarrows 3$ that would be cancelled using the classical definition of quiver mutation. However, by drawing the quiver with two framing vertices, we keep these arrows and our quiver is the quiver associated to $\mu_2 T$.

Now consider the flip μ_3 and the corresponding quiver mutation σ_3. We denote by $\tilde{\sigma}_i$ the *premutation at vertex i*, that is, the process of applying the first two steps of quiver mutation at vertex i (before cancelling two-cycles):

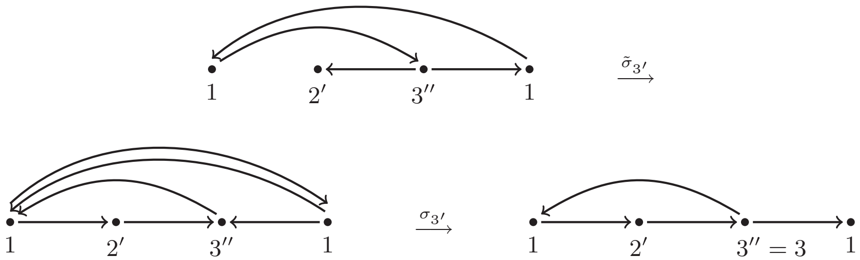
https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press





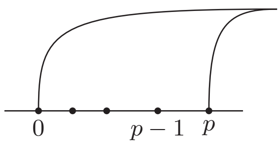
Our resulting quiver is $\sigma_3\sigma_2 Q$.

If we flip d_3' in T, we get the previous triangulation back. The quiver mutation rules should also give us the previous quiver $\sigma_2 Q$ back.

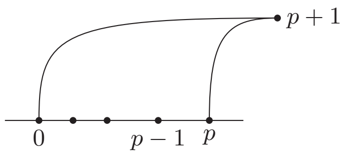


PROPOSITION 3.8. *Flips of arcs in a frame* $T_\beta(d_i)$ *correspond to mutations of vertices in the associated framing quiver* $Q_\beta(i)$.

Proof. Let $T_\beta(d_i)$ be a d_i-frame of T_β, and let $Q_\beta(i)$ be the quiver associated to $T_\beta(d_i)$. Without loss of generality, we relabel the marked points of the frame from $0, \ldots, p-1, p$,



and give a name to the "point" where the two strictly asymptotic arcs meet.



Then this is equivalent to a triangulated polygon on $p+2$ vertices with $p-1$ arcs.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

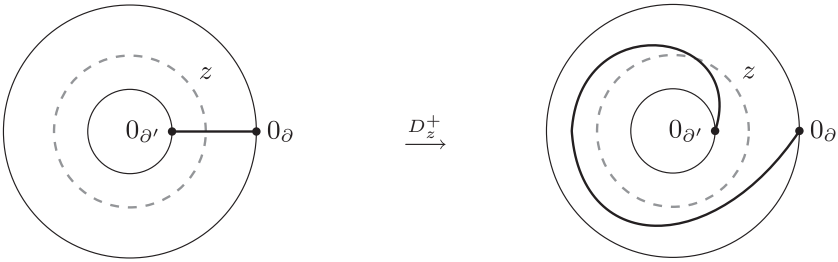
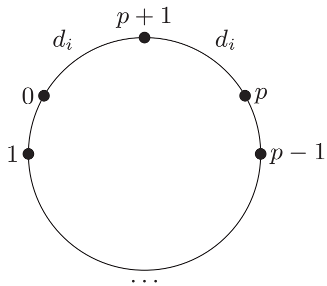


Figure 14. Dehn twist around closed curve z.



Now it is known that flips of arcs in a triangulation of an unpunctured polygon correspond to mutation of vertices of the associated quiver. Therefore, any sequence of flips of arcs in $T_\beta(d_i)$ does the same thing as the corresponding sequence of mutations of vertices in the associated quiver $Q_\beta(i)$. □

An alternative way to mutate quivers associated to asymptotic triangulations is by using quivers with potentials (cf. Appendix A).

4. Dehn twist. In the previous section, we defined asymptotic triangulations. In this section, we describe the process of going from a finite triangulation to an asymptotic triangulation. This process constitutes applying the Dehn twist infinitely many times to a triangulation. Applying the Dehn twist infinitely many times causes some arcs of the triangulation to become identified, while breaking other arcs into two parts so that we are left with two triangulations – one based at each boundary component of our annulus.

Let z be a non-contractible closed curve in $C_{p,q}$. Consider the homeomorphism of $C_{p,q}$ obtained by cutting $C_{p,q}$ along z and gluing it back after rotating the inner boundary by 2π. This homeomorphism is called a *Dehn twist*. We have chosen a counter-clockwise orientation of our surface, so when applying a positive Dehn twist, we rotate the inner boundary ∂' of the annulus clockwise by 2π (Figure 14). A negative Dehn twist would be a rotation of ∂' by 2π in the counter-clockwise direction.

The Dehn twist results in a lengthening or shortening of bridging arcs.

NOTATION 4.1. *D_z^+ denotes the positive Dehn twist with respect to z, and D_z^- denotes the negative Dehn twist with respect to z.*

D_z^n is the nth Dehn twist (the Dehn twist applied n times). We define

$$D_z^{+\infty} = \lim_{n\to\infty} D_z^n,$$

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

and

$$D_z^{-\infty} = \lim_{n \to -\infty} D_z^n.$$

We use the construction of [2] to get to the limit of the Dehn twist. In particular, we have the following cases. Recall that π_i denotes the Prüfer arc based at marked point i, and α_i denotes the adic arc based at marked point i. Let $\gamma = [i, j]$ be an arc in a triangulation T of $C_{p,q}$. If i and j lie on different boundary components, then

$$D_z^{+\infty} \cdot \gamma = \{\pi_i, \pi_j\}, \tag{1}$$

and

$$D_z^{-\infty} \cdot \gamma = \{\alpha_i, \alpha_j\}. \tag{2}$$

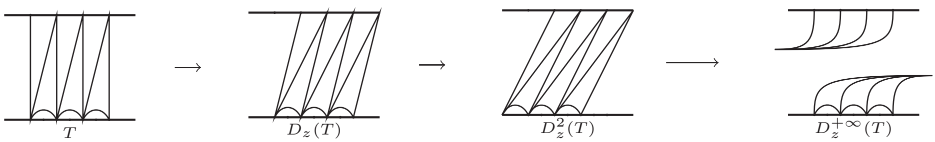
If i and j lie on the same boundary component, then $D_z^{\pm\infty} \cdot \gamma = \gamma$.

Given a triangulation T of the annulus $C_{p,q}$, we define

$$D_z^{+\infty}(T) = \bigcup_{\gamma \in T} D_z^{+\infty} \cdot \gamma \quad \text{and} \quad D_z^{-\infty}(T) = \bigcup_{\gamma \in T} D_z^{-\infty} \cdot \gamma.$$

Note that since T contains at least two bridging arcs (Proposition 2.7), $D_z^{\pm\infty}(T)$ is always asymptotic.

EXAMPLE 4.2. Consider the following triangulation T of $C_{p,q}$. Each application of the Dehn twist lengthens the bridging arcs of T. After infinitely many Dehn twists, the bridging arcs have infinite length and "break" into Prüfer arcs stemming from both boundary components.



From now on, we will only consider the positive Dehn twist D_z^+, but everything can be defined analogously for D_z^-.

Notice that in equations (1) and (2), our arc γ gives rise to two new arcs in the limit. However, we do not end up with twice as many arcs in the asymptotic triangulation. This is because all bridging arcs originating at the same boundary vertex become identified in the limit: Consider two bridging arcs $d_{[i,j]}, d_{[i,k]}$ in T where i lies on ∂ and j, k lie on ∂' (possibly, $j = k$). Then $D_z^+ \cdot d_{[i,j]} = \{\pi_i, \pi_j\}$ and $D_z^+ \cdot d_{[i,j]} = \{\pi_i, \pi_k\}$. So two of the vertices arising from $D_z^+ \cdot d_{[i,j]}$ and $D_z^+ \cdot d_{[i,k]}$ in Q_T are one vertex π_i in Q_∂. This is illustrated in Figures 15 and 16. A finite triangulation of an annulus $C_{p,q}$ has $p + q$ arcs. The number of asymptotic arcs of an asymptotic triangulation is also $p + q$, which we can see by the decomposition $T = T_\partial \sqcup T_{\partial'}$ where $|T_\partial| = p$, $|T_{\partial'}| = q$.

5. Coxeter transformations. The Dehn twist provides us with a topological way of obtaining asymptotic triangulations. In this section, we describe a combinatorial method of obtaining asymptotic triangulations. This *Coxeter transformation* is done by applying a sequence of flips to the arcs of the triangulation. On the level of

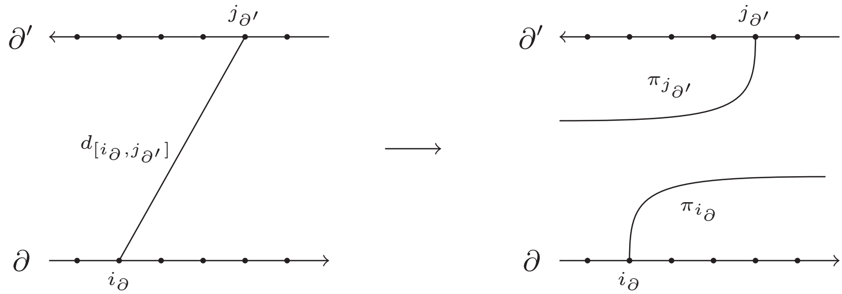


Figure 15. Bridging arc becoming two asymptotic arcs.

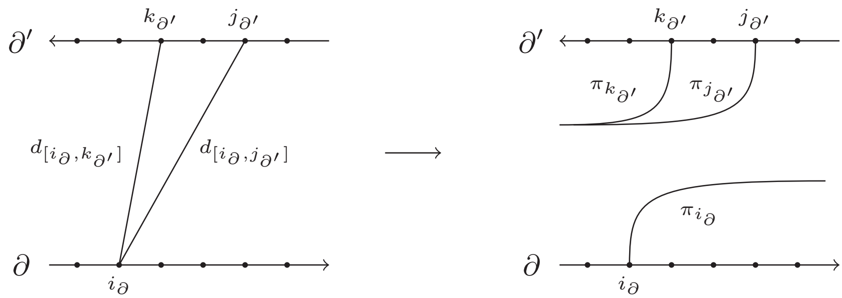


Figure 16. Two bridging arcs becoming one asymptotic arc.

quivers, we perform the sequence of corresponding mutations. The benefit of having a combinatorial method to describe this process is that we can now study other variables and systems associated to the surface. For example, we can look at root systems and (cluster) variables associated to arcs of the triangulation, and we expect this to provide a way to define cluster structures on asymptotic triangulations.

5.1. Quivers. Given a source i of a quiver Q, the quiver $\sigma_i Q$ is obtained by reversing all arrows in Q which start or end at i.

Recall from Definition 1.1, that an ordering $i_1, \ldots, i_n$ of the vertices of Q is (source-) admissible if for each p the vertex i_p is a source in the quiver $\sigma_{i_{p-1}} \ldots \sigma_{i_1} Q$.

The following lemma is a well-known result from graph theory. For a proof, see [**7**, Lemma 3.1.1].

LEMMA 5.1. *There exists an admissible ordering of the vertices of Q if and only if there are no oriented cycles in Q.*

DEFINITION 5.2. If $\underline{i} = i_1, \ldots, i_n$ is an admissible ordering on a quiver Q, then the Coxeter transformation of Q is

$$\mathrm{cox}_{\underline{i}}(Q) = \sigma_{i_n} \ldots \sigma_{i_1} Q.$$

Recall from Section 1 that $\mathrm{cox}_{\underline{i}}(Q) \cong Q$, and note that $\mathrm{cox}_{\underline{i}}(Q)$ is independent of the chosen admissible ordering $\underline{i}$. Thus, we will drop the index and just write $\mathrm{cox}(Q)$.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

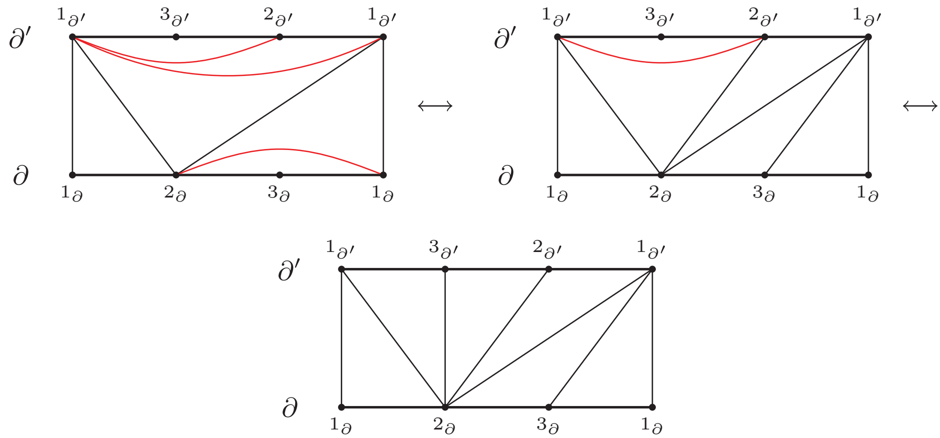


Figure 17. (Colour online) Flipping the peripheral arcs of a triangulation T.

As described in Definition 2.8, we have quivers associated to triangulations. We consider what happens to an associated triangulation when we mutate the arcs of the triangulation and the vertices of the associated quiver concurrently.

Let T be a triangulation of $C_{p,q}$. Cycles in Q_T occur when there are peripheral arcs in T. If there are any peripheral arcs in T, we can flip them until we obtain a triangulation $\tilde{T}$ consisting only of bridging arcs. For an example, see Figure 17.

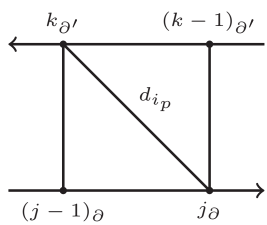
We call such a triangulation $\tilde{T}$ a *bridging triangulation*.

DEFINITION 5.3. Let $\tilde{T}$ be a bridging triangulation. Let $Q_{\tilde{T}}$ be the associated quiver, and $\underline{i} = i_1 \dots i_n$ an admissible ordering of the vertices of $Q_{\tilde{T}}$. Then the *Coxeter transformation* of $\tilde{T}$ is

$$\mathrm{cox}_{\underline{i}}(\tilde{T}) = \mu_{d_{i_n}} \dots \mu_{d_{i_1}} \tilde{T}.$$

LEMMA 5.4. *Let $\tilde{T}$ be a bridging triangulation and $\underline{i} = i_1 \dots i_n$ an admissible ordering of $Q_{\tilde{T}}$. Then* $\mathrm{cox}_{\underline{i}}(\tilde{T})$ *moves endpoints of all arcs by -1 on both ∂ and ∂'.*

Proof. Let $\tilde{T}$ be a bridging triangulation, and $i_1 \dots i_n$ an admissible ordering of the vertices of the associated quiver $Q_{\tilde{T}}$. Then for every $p = 2, \dots, n$, vertex i_p is a source in the quiver $\sigma_{i_{p-1}} \dots \sigma_{i_1} Q$, and the corresponding arc $d_{i_p} = [j_\partial, k_{\partial'}]$ lies in such a quadrilateral in $\tilde{T}$:



The flip corresponding to the mutation σ_{i_p} is $\mu_{d_{i_p}}$. It replaces d_{i_p} in $\mu_{i_{p-1}} \cdots \mu_{i_1}(T)$ with the other diagonal $d'_{i_p} = [(j-1)_\partial, (k-1)_{\partial'}]$ in the quadrilateral. This holds for all $1 \le p \le n$. Thus, we have that the map $\mathrm{cox}_{\underline{i}}(\tilde{T})$ sends $[j_\partial, k_{\partial'}]$ to $[(j-1)_\partial, (k-1)_{\partial'}]$.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

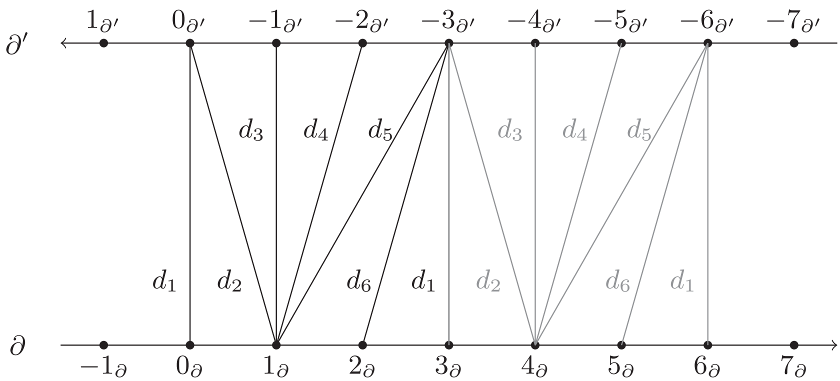
So the Coxeter transformation moves the endpoints of every arc in $\tilde{T}$ by -1 on each boundary component. □

The following corollary is a direct consequence of Lemma 5.4.

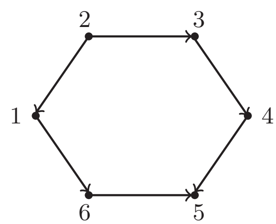
COROLLARY 5.5. *Let $\tilde{T}$ be a bridging triangulation with admissible ordering* ı. *Then* $\mathrm{cox}_{\underline{i}}(\tilde{T})$ *is independent of the choice of* $\underline{i}$.

We will thus simply write $\mathrm{cox}(\tilde{T})$ for the Coxeter transformation of a bridging triangulation $\tilde{T}$.

EXAMPLE 5.6. Let T be the following triangulation of $C_{3,3}$ drawn in black, with labelled arcs. For convenience and clarity, we draw in a second copy of T in gray.



The associated quiver Q_T:



We use this quiver to obtain an admissible ordering of the vertices (and therefore also of arcs). Going from sources to sinks, we have an ordering $\underline{i} = 2, 3, 1, 6, 4, 5$. So we will perform flips in the order $\mu_5\mu_4\mu_6\mu_1\mu_3\mu_2 T$:

As shown in Lemma 5.4, this Coxeter transformation shifts the endpoints of each arc in the triangulation by -1 on each boundary component. The same effect can be achieved by rotating the outer boundary component clockwise by $\frac{2\pi}{p}$, and the inner boundary component counter-clockwise by $\frac{2\pi}{q}$. In the example, $d_1 = [0_\partial, 0_{\partial'}] \mapsto [-1_\partial, -1_{\partial'}]$. Applying the Coxeter transformation p times would result in moving the endpoints on ∂ a full turn in the clockwise direction. Similarly, applying the Coxeter transformation q times would result in moving the endpoints on ∂' a full turn around in the counter-clockwise direction.

In the example above, $p = q = 3$, so applying the Coxeter transformation two more times $(\mathrm{cox}^2(\mathrm{cox}(T)) = \mathrm{cox}^3(T))$ to Figure 18 would result in Figure 19.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

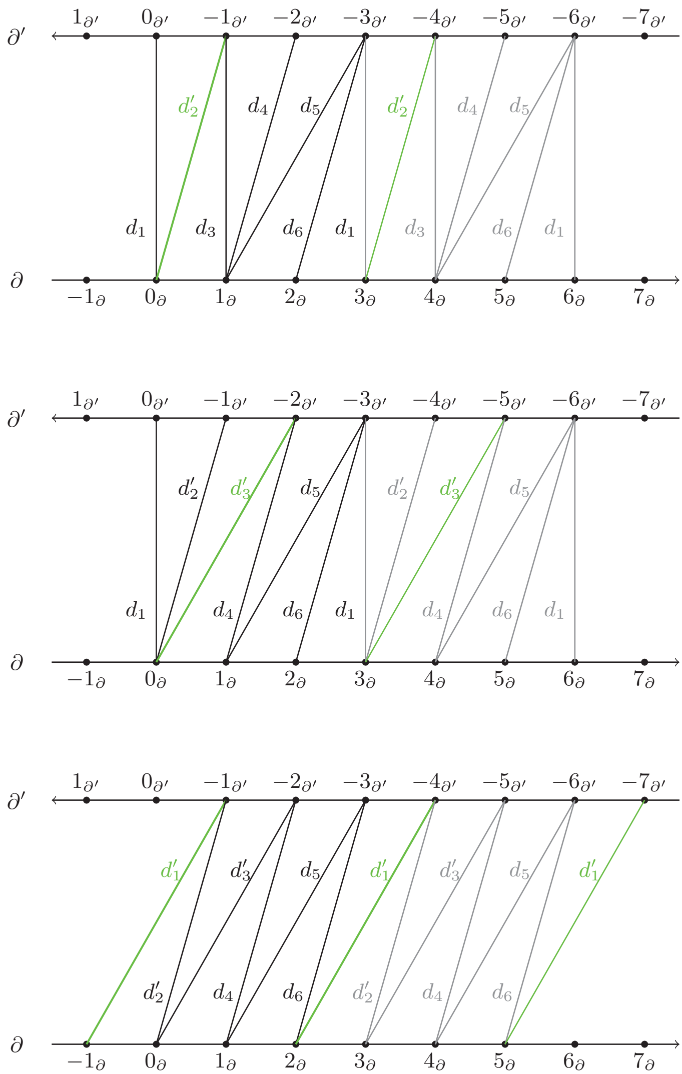


Figure 18. (Colour online) $\text{cox}(T) = \mu_5\mu_4\mu_6\mu_1\mu_3\mu_2 T$.

We can extend the definition of the Coxeter transformation to arbitrary triangulations of $C_{p,q}$ by first flipping all peripheral arcs to reach a bridging triangulation. Let T be a triangulation with $k \geq 1$ peripheral arcs. Then there exists a finite sequence of flips $\mu_\alpha := \mu_{\alpha_1}, \ldots, \mu_{\alpha_k}$, where α_i is a peripheral arc, for every i, and α_p is a bounding arc in the triangulation $\mu_{\alpha_{p-1}} \cdots \mu_{\alpha_1} T$. Note that this sequence is not necessarily unique, since we may have more than one bounding arc in our triangulation at any given time. Other sequences of flips (flipping non-bounding arcs, for example) may also result in a bridging triangulation, but flipping only bounding arcs will give us a minimal sequence of flips. However, our resulting triangulation is independent of the order in which we choose to mutate the bounding arcs.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

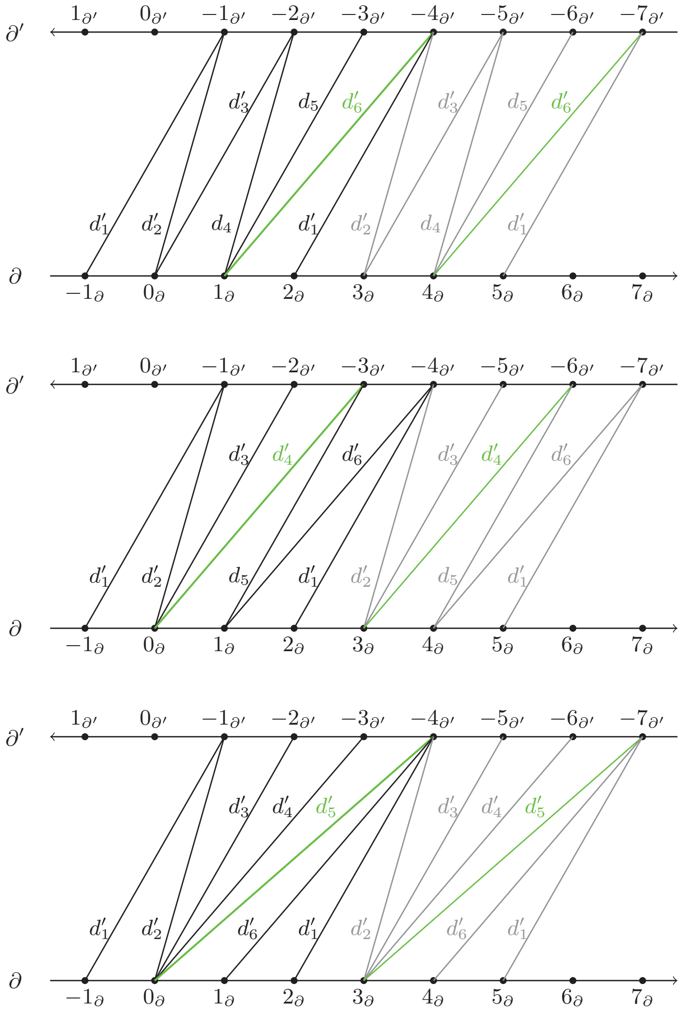


Figure 18. (Colour online) Continued.

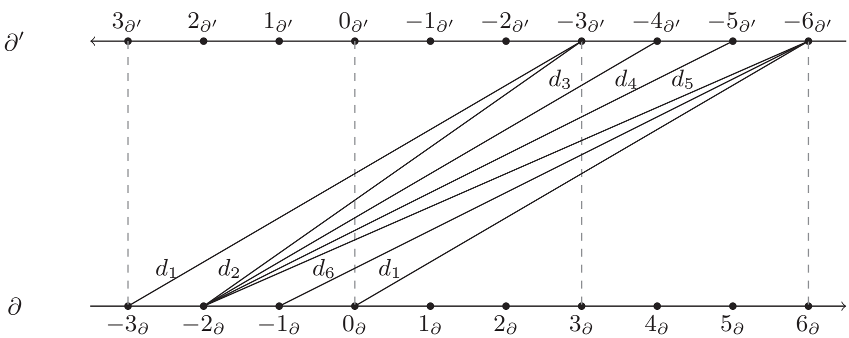
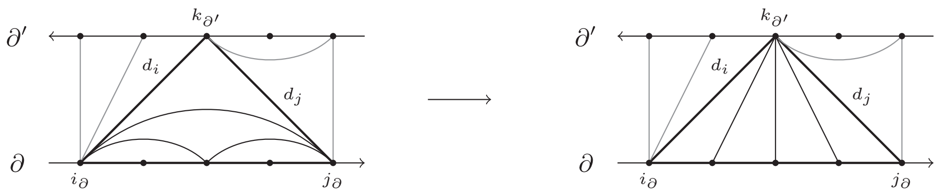


Figure 19. $\mathrm{cox}^3(T)$.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

PROPOSITION 5.7. *Using the same notation as above, the bridging triangulation* $\tilde{T} = \mu_\alpha T$ *is uniquely determined (unique up to labelling of arcs).*

Proof. Let T be a triangulation of $C_{p,q}$. We know by Lemma 2.7 that T contains at least two bridging arcs. We claim that $\tilde{T}$ is determined by the bridging arcs in T. Every bridging arc is an edge of two triangles in T where in both triangles one of the other two edges is also a bridging arc. Without loss of generality, let $d_i = [i_\partial, k_{\partial'}]$ and $d_j = [j_\partial, k_{\partial'}]$ be two bridging arcs in T such that d_i and d_j are two sides of a triangle in T, and $i < j$. We consider the triangulation restricted to the polygon P_k where the boundary of P_k is made up of the arcs of $[i_\partial, k_{\partial'}]$ and $[j_\partial, k_{\partial'}]$, and the boundary segment $[i_\partial, j_\partial]$ of $C_{p,q}$. Then we can find a finite sequence of flips μ_{α_k} such that after performing this sequence, all arcs in P_k have an endpoint at $k_{\partial'}$ (note that if there are no internal arcs in P_k, then we have the empty sequence). Such a sequence exists because any two triangulations of a surface S are related through a sequence of flips. Then we have a fan of bridging arcs $[(i+1)_\partial, k_{\partial'}] \cdots [(j-1)_\partial, k_{\partial'}]$ in our triangulation $\mu_{\alpha_k} T$ stemming from $k_{\partial'}$. We do this for every such triangle where two sides are bridging arcs of T. Our bridging triangulation $\tilde{T} = \mu_\alpha T$ is then comprised of fans of bridging arc originating at the endpoints where two bridging arcs of T meet.



□

DEFINITION 5.8. Let T be a triangulation with peripheral arcs, and $\{\mu_{\alpha_i}\}_{i \in I}$ a finite sequence of flips, α_i peripheral, so that $\tilde{T} = \mu_\alpha T$ consists only of bridging arcs. Then the Coxeter transformation of T is

$$\text{cox}(T) = (\mu_\alpha)^{-1} \, \text{cox}(\mu_\alpha T).$$

THEOREM 5.9. *Let T be a triangulation of $C_{p,q}$.*

(1) We have $\mu D_z^+(T) = D_z^+(\mu T)$ for every arc flip μ. Let $\tilde{T} = \mu_\alpha T$ be a bridging triangulation of $C_{p,q}$, and let $m = \text{lcm}(p, q)$. Then there exist $r, s \in \mathbb{N}$ such that $pr = m$ and $qs = m$. We have the following commutativity relations:

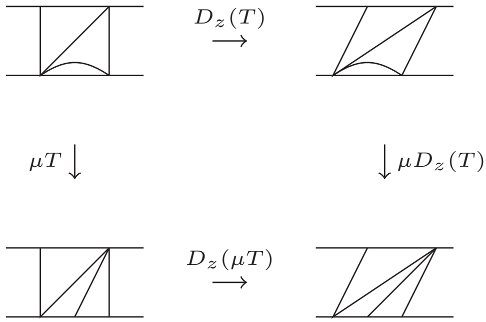
(2) $\text{cox}^m(\tilde{T}) = D_z^{r+s}(\tilde{T})$,

(3) $D_z^{r+s}(T) = \text{cox}^m(T)$.

Proof.

(1) Dehn twists do not change relative positions of arcs in a triangulation, so the arcs involved in a quadrilateral still form a quadrilateral after applying the Dehn twist. Thus, the following diagram commutes for every arc flip μ:

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press



(2) One iteration of $\operatorname{cox}(\tilde{T})$ moves an arc $[i_\partial, j_{\partial'}] \mapsto [(i-1)_\partial, (j-1)_{\partial'}]$. Since $pr = m = qs$, $\operatorname{cox}^m(\tilde{T}) : [i_\partial, j_{\partial'}] \mapsto [(i-m)_\partial, (j-m)_{\partial'}] = [(i-pr)_\partial, (j-qs)_{\partial'}]$. So $\tilde{T}$ has shifted endpoints of arcs r frames in the negative direction on boundary ∂, and s frames in the negative direction on ∂'. In total, the triangulation now stretches $r+s$ frames, and thus $\operatorname{cox}^m(\tilde{T}) = D_z^{r+s}(\tilde{T})$.

(3) We will use parts (1) and (2) to prove (3).

$$D_z^{r+s}(T) = \mu_\alpha^{-1}\mu_\alpha D_z^{r+s}(T) \overset{(1)}{=} \mu_\alpha^{-1} D_z^{r+s}(\mu_\alpha T) \overset{(2)}{=} \mu_\alpha^{-1} \operatorname{cox}^m(\mu_\alpha T) = \operatorname{cox}^m(T).$$

□

We define $\operatorname{Cox} := \operatorname{cox}^m$, where $m = \operatorname{lcm}(p, q)$. The endpoints of the arcs of a triangulation T of $C_{p,q}$ are invariant under $\operatorname{Cox}(T)$.

These commutativity relations provide us with a dictionary to go between the topological and algebraic framework. This becomes useful when considering what happens to the quivers (or root systems) under the Dehn twist, and to see what happens to a triangulation when applying a Coxeter transformation. The Coxeter transformation for triangulations of the annulus can be defined for planar surfaces with several boundary components. This comes down to choose appropriate "boundaries" ∂_1, ∂_2.

5.2. Coxeter transformations of surfaces with several boundary components. Let (S, M) be a marked planar surface with several boundary components, such that each boundary component has at least one marked point, and let $\tilde{T} = T$ be a bridging triangulation of S, i.e. a triangulation where the endpoints of each arc lie on different boundary components. We choose a simple, non-contractible loop z in S. The *interior* of the loop z is to the right of z when moving along z in the clockwise direction. The exterior is then to the left of z. We use the following notation:

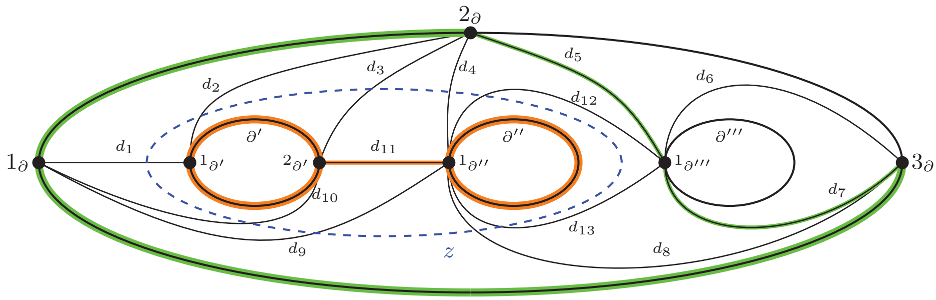
- Let $D(z) = \{d_{i_1}, \ldots, d_{i_d}\}$ denote the set of arcs of T that intersect z.
- Let $V_1(z)$ be the set of marked points in the interior of z such that every marked point in V_1 is the endpoint of at least one $d_i \in D(z)$.
- Let $V_2(z)$ be the set of marked points in the exterior of z such that every marked point in V_2 is the endpoint of at least one $d_i \in D(z)$.

Then there exists a minimal cycle c_1 (not necessarily unique), formed by arcs in T and boundary segments of S (arcs may appear more than once in the cycle), connecting all $m \in V_1$. We set $\partial_1 = c_1$. There also exists a minimal cycle c_2 (not necessarily unique), formed by arcs in T and boundary segments of S (arcs may appear more than once in the cycle), connecting all $m \in V_2$. We set $\partial_2 = c_2$.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

We can then consider T restricted to the region between ∂_1 and ∂_2. This is an annulus triangulated by $D(z)$, and we can apply the machinery from Section 5.1 to T.

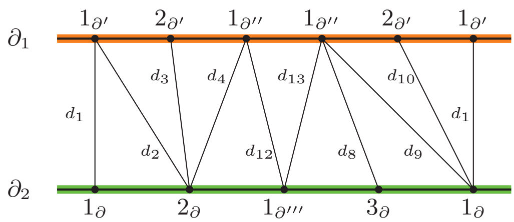
EXAMPLE 5.10. Consider the surface S with four boundary components, drawn below.



We want to perform a Coxeter transformation with respect to (the closed curve) z as chosen. We have

$$
\begin{aligned}
D(z) &= \{d_1, d_2, d_3, d_4, d_8, d_9, d_{10}, d_{12}, d_{13}\},\\
V_1(z) &= \{1_{\partial'}, 2_{\partial'}, 1_{\partial''}\},\\
V_2(z) &= \{1_{\partial}, 2_{\partial}, 1_{\partial'''}, 3_{\partial}\}.
\end{aligned}
$$

We now find minimal cycles c_1 and c_2. The cycle c_1 is marked in orange, and it is $1_{\partial'} \to 2_{\partial'} \to 1_{\partial''} \to 1_{\partial''} \to 2_{\partial'} \to 1_{\partial'}$. Note that $1_{\partial''}$ is repeated twice in a row. This is because the boundary component ∂'' cannot be contracted to a single point. This cycle now becomes our boundary ∂_1. The cycle c_2 is marked in green, and it is $1_\partial \to 2_\partial \to 1_{\partial'''} \to 3_\partial \to 1_\partial$. This cycle now becomes our boundary ∂_2. We can then represent the part of the triangulation between c_1 and c_2 as a cylinder $Cyl_{4,5}$:



From here, we perform a Coxeter transformation in the usual manner. We consider the associated quiver to find an admissible ordering, and then perform a sequence of flips.

We can now reach the asymptotic triangulations using limits of Coxeter transformations. This is discussed in Section 6.

6. Limits. In this section, we aim to show that the Coxeter transformation and the Dehn twist behave the same way in the limit. This allows us to use whichever process of obtaining an asymptotic triangulation that is the most useful in our setting.

Recall from Definition 5.8 that if T is a triangulation with peripheral arcs, and $\{\mu_{\alpha_i}\}_{i\in I}$ is a finite sequence of flips with α_i peripheral, so that $\tilde{T} = \mu_\alpha T$ consists only

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

of bridging arcs, then the Coxeter transformation of T is

$$\mathrm{cox}(T) = (\mu_\alpha)^{-1}\,\mathrm{cox}(\mu_\alpha T).$$

Using the commutativity relations from Theorem 5.9, we have that $\mathrm{Cox}(\tilde{T}) = \mathrm{cox}^m(\tilde{T}) = D_z^{r+s}(\tilde{T})$ for $r, s \in \mathbb{N}$ where $pr = qs = m = \mathrm{lcm}(p, q)$. We have the following proposition.

PROPOSITION 6.1. *Let T be a bridging triangulation of $C_{p,q}$. Then*

$$\lim_{n\to\infty} \mathrm{Cox}^n(T) = D_z^{+\infty}(T).$$

The proof follows from the definitions of Cox and $D_z^{+\infty}$, cf. Section 4.

If T is a bridging triangulation, we define

$$\mathrm{Cox}^{+\infty}(T) = \lim_{n\to\infty} \mathrm{Cox}^n(T).$$

By Proposition 6.1, $\mathrm{Cox}^{+\infty}(T)$ is an asymptotic triangulation of $C_{p,q}$.

6.1. Quivers. As described in Section 1, a Coxeter transformation on a quiver is a sequence of reflections from sources to sinks. We have already defined the limits of the Dehn twist and Coxeter transformation on a bridging triangulation T. Now we want to see what happens to the quiver Q_T under these transformations.

The *mapping class group* of a surface S is the group of orientation-preserving homeomorphisms $S \to S$ whose restriction to ∂S is the identity, up to isotopy among homeomorphisms of the same kind. The mapping class group of the annulus is $MCG(C_{p,q}) \simeq \langle D_z \rangle \simeq \mathbb{Z}$. It is well known that a quiver is fixed under the action of the mapping class group on $C_{p,q}$. However, the quiver behaves differently in the limit. We saw that the quiver $Q_{D_z^{+\infty}(T)}$ becomes disconnected. We now give an algorithm to obtain the quiver $Q_{\mathrm{Cox}^{+\infty}(T)}$ directly from Q_T without passing through the triangulations involved.

Let $Q = Q_T$ be a quiver associated to a bridging triangulation T. Take two copies of Q, draw them as planar graphs (as un-oriented cycles) with the vertices of each quiver labelled in a clockwise manner. Consider a maximal counter-clockwise path P in Q. For there to be such a path $P = i \to \cdots \to j$ in Q, the corresponding arcs $d_i, \ldots, d_j \in T$ must share an endpoint on ∂. In the limit $\mathrm{Cox}^{+\infty}(T)$, the arcs involved collapse to a single Prüfer arc based on ∂ (cf. Figure 20). We also consider maximal clockwise paths in Q_T.

Algorithm 1 Constructing Q_∂, $Q_{\partial'}$ from Q.

1: Replace every maximal counter-clockwise path $P = i \to \cdots \to j$ in Q by a single vertex $w_{i,j}$. Denote the resulting quiver by Q_∂.
2: Replace every maximal clockwise path $P = r \to \cdots \to s$ in Q by a single vertex $u_{r,s}$. Denote the resulting quiver by $Q_{\partial'}$.

By construction, Q_∂ only has arrows forming a clockwise cycle, and $Q_{\partial'}$ only has arrows forming a counter-clockwise cycle. Note that because the quiver does not

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

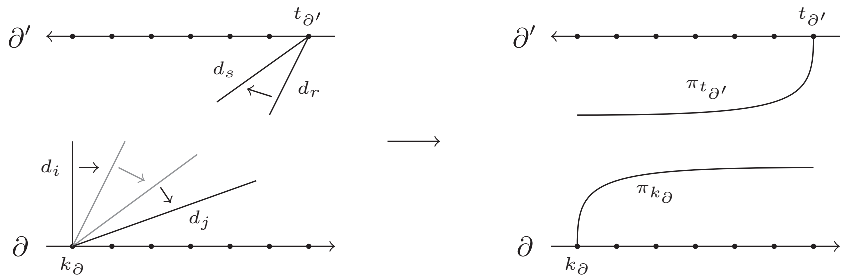


Figure 20. Coxeter transformation on a quiver.

differentiate between Prüfer and adic arcs (see Figure 21 for an illustration), we can get the quiver for all asymptotic triangulations via this algorithm.

PROPOSITION 6.2. *Let T be a bridging triangulation of $C_{p,q}$, Q_T its associated quiver, and Q_∂, $Q_{\partial'}$ as constructed above. Then*

$$Q_{D_z^{+\infty}(T)} \cong Q_\partial \sqcup Q_{\partial'}.$$

Proof. The quivers Q_∂ and $Q_{\partial'}$ are constructed as above from a bridging triangulation of $C_{p,q}$. Thus, the quiver Q_∂ will be a clockwise cycle on p vertices and the quiver $Q_{\partial'}$ will be a counter-clockwise cycle on q vertices. Since T is a bridging triangulation, $D_z^{+\infty}(T)$ will have p Prüfer arcs stemming from the outer boundary ∂, and q Prüfer arcs stemming from the inner boundary ∂'. So $Q_{D_z^{+\infty}(T)}$ has two connected components, one of which is a clockwise cycle on p vertices, and one component is a counter-clockwise cycle on q vertices. This is exactly Q_∂ and $Q_{\partial'}$, and so we have the isomorphism $Q_{D_z^{+\infty}(T)} \cong Q_\partial \sqcup Q_{\partial'}$. □

COROLLARY 6.3. *Let T be a triangulation, and Q_T its associated quiver. Then $Q_{D^{+\infty}(T)} \cong Q_{D^{-\infty}(T)}$.*

Proof. Consider the two triangulations $D_z^{+\infty}(T)$ and $D_z^{-\infty}(T)$ obtained from a triangulation T. All the strictly asymptotic arcs of $D^{+\infty}(T)$ are Prüfer arcs, and all the strictly asymptotic arcs of $D^{-\infty}(T)$ are adic arcs. Now consider the associated quivers. Recall that every arc in a triangulation corresponds to a vertex in the quiver, and there is an arrow between two vertices $i \to j$ in the quiver if the corresponding arc d_i can be rotated clockwise to become an arc isotopic to the corresponding arc d_j. Strictly asymptotic arcs have one endpoint a marked point on the boundary of $C_{p,q}$, and the other endpoint spirals infinitely around z. Thus, for a strictly asymptotic arc d_j to be a clockwise rotation of d_i, the endpoint of d_i that is rotated is the one on the boundary. All arrows will go from left to right between strictly asymptotic arcs on the upper boundary, and all arrows between strictly asymptotic arcs will go from right to left on the lower boundary (cf. Figure 21), independent of whether the arcs spiral positively or negatively around z. Peripheral arcs are unaffected by the Dehn twist, and therefore the internal triangles of T stay fixed, and the cycles in the quivers corresponding to the internal triangles will be the same for both $Q_{D_z^{+\infty}}(T)$ and $Q_{D_z^{-\infty}(T)}$. □

To see an example of how these paths of arcs contract in the triangulation, see Figure 20. The following example shows how we contract the paths in Q to get two quivers Q_∂, $Q_{\partial'}$.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

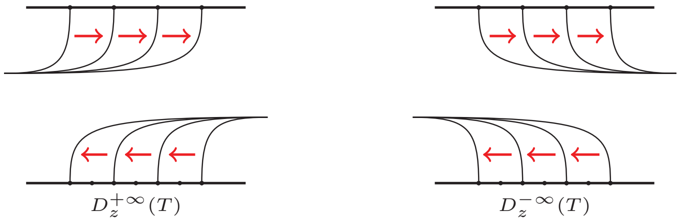
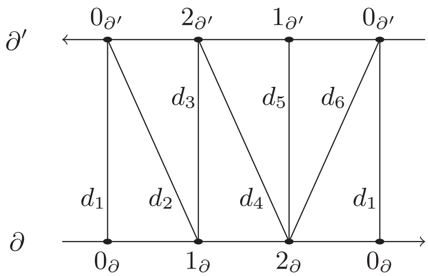
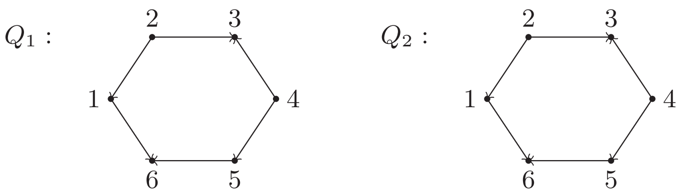


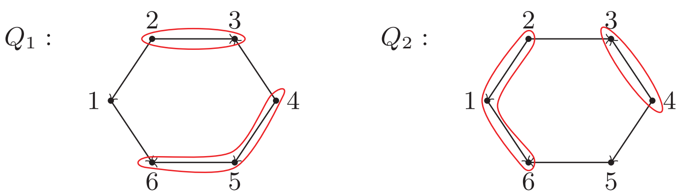
Figure 21. (Colour online) $Q_{D^{+\infty}(T)}$ and $Q_{D^{-\infty}(T)}$.

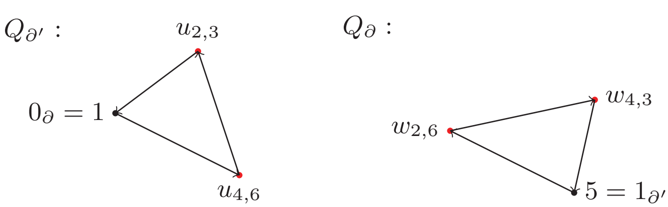
EXAMPLE 6.4. Consider a frame of the following (bridging) triangulation T and two copies of the associated quiver Q_T:



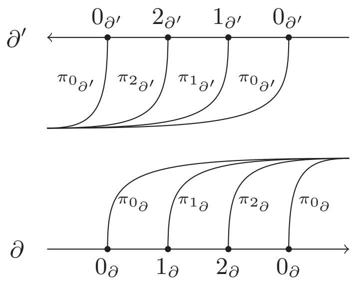


As described, we consider all maximal clockwise paths in Q_1, and all maximal counter-clockwise paths in Q_2 and contract them to a single vertex:





We can check that this is indeed the quiver of the asymptotic triangulation $D_z^{+\infty}(T)$.



THEOREM 6.5. *Let T be a bridging triangulation, and Q_T the associated quiver. Then*

$$Q_{\mathrm{Cox}^{+\infty}(T)} \cong Q_{D_z^{+\infty}(T)} \cong Q_{D_z^{-\infty}(T)}.$$

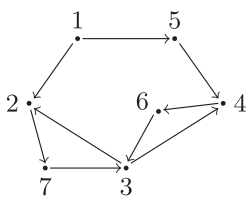
Proof. We have that $\mathrm{Cox}^{+\infty}(T) = D_z^{+\infty}(T)$ as triangulations, and thus $Q_{\mathrm{Cox}^{+\infty}(T)} \cong Q_{D_z^{+\infty}(T)}$. The second isomorphism is the result from Corollary 6.1. □

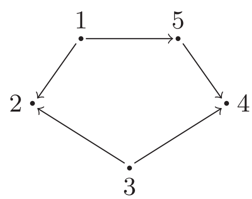
We have described an algorithm for obtaining a quiver from a bridging asymptotic triangulation. A natural question to ask is whether the algorithm can be used on a quiver when we don't know the associated triangulation. To do this, we need to work with the *shape* Q^b, where Q^b is the full subquiver obtained by removing arrows that belong only to internal triangles of Q_T. We have the alternate algorithm as follows:

Algorithm 2 Constructing Q_∂, $Q_{\partial'}$ from Q^b.

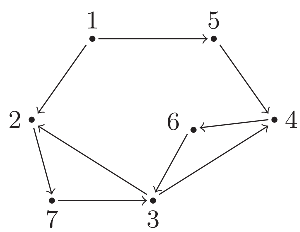
1: Draw two copies Q_1, Q_2 of Q_T.
2: Do 1 & 2 as in Algorithm 1 to paths in $Q^b \cap Q_1$ and $Q^b \cap Q_2$, respectively.
3: Draw result, killing all subgraphs that share an edge with the contracted path P in Step 2 above.

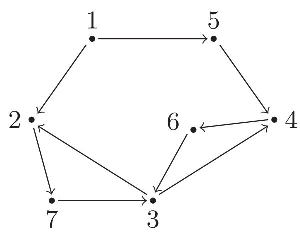
EXAMPLE 6.6. Consider the quiver Q_T and the full subquiver Q^b of Q_T.

Q_T :


Q^b :


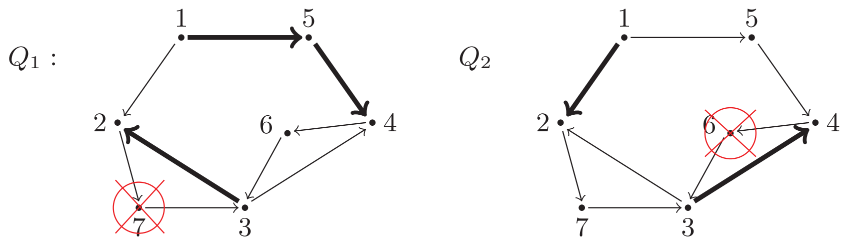
We first draw two copies Q_1 and Q_2 of Q_T:

Q_1 :


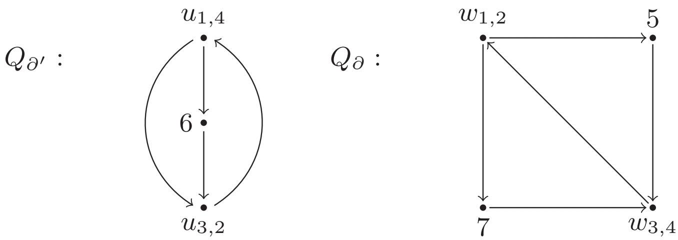
Q_2 :


https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

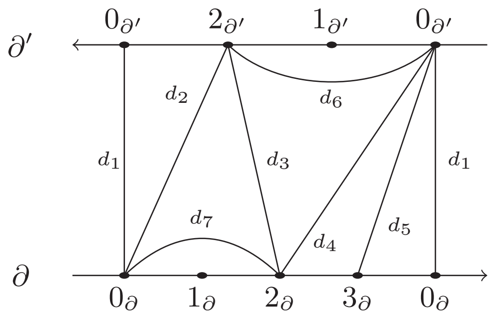
Then we apply Algorithm 1 to $Q_1 \cap Q^b$ and $Q_2 \cap Q^b$ and kill subgraphs:



The resulting quivers are



An example of a triangulation associated to this quiver is



It is possible to distinguish the vertices that correspond to bridging (resp. strictly asymptotic) arcs in the associated triangulation from those that correspond to peripheral arcs. Vertices that correspond to bridging (strictly asymptotic) arcs form an unoriented (clockwise-oriented) cycle C in the quiver. This cycle actually gives us the full subquiver Q^b. Vertices that correspond to peripheral arcs lie in counter-clockwise oriented cycles in the quiver, and these counter-clockwise oriented cycle shares an edge with C, that is, an edge between two vertices corresponding to bridging (strictly asymptotic) arcs. Bastian, in [**1**], denotes these vertices by z_α, and describes the quivers Q_α that branch off from C.

Using the full subquiver allows us to construct an "asymptotic quiver" without knowing the triangulation. There are restrictions on the types of quivers for which these algorithms work. The quivers need to be associated to a triangulation of a surface described in this paper.

A. Quivers with potentials. In this appendix, we describe an alternate way of performing quiver mutation by using quivers with potentials. The authors Derksen, Weymann, and Zelevinsky developed a mutation theory of quivers in [**4**] using potentials, which lifts quiver mutation from the combinatorial level to the algebraic level. This provides a representation-theoretic interpretation of quiver mutation and ultimately leads to the notion of mutation of representations of quivers with potentials. For the convenience of the reader, we will recall the necessary background needed (cf. [**4**]). The definitions and notation in this appendix are taken from [**8**] and [**4**].

Let Q be a quiver. For each vertex $i \in Q_0$, we have the path of length 0, denoted by e_i. A^l denotes the $\mathbb{C}$-vector space with basis the set of paths of length $l \geq 0$. We use the notation $R = A^0$ and $A = A^1$. Note that R is the vector space with basis the set of length-0 paths ($\dim R = |Q_0|$), and A is the vector space with basis the set of arrows of Q. If we define $e_i e_j = \delta_{ij} e_i$, then R becomes a commutative $\mathbb{C}$-algebra. If we define $e_i \alpha = \delta_{i,h(\alpha)}\alpha$ and $\alpha e_i = \delta_{i,t(\alpha)}\alpha$, then $A^{l>0}$ becomes an R-R-bimodule for every $l > 0$.

Definition A.1. The *path algebra of Q* is the $\mathbb{C}$-vector space

$$R\langle Q\rangle = \bigoplus_{l=0}^{\infty} A^l.$$

The path algebra can also be defined as the (graded) tensor algebra, and for each $i, j \in Q_0$, the component $R\langle Q\rangle_{i,j} = e_i R\langle Q\rangle e_j$ is called the *space of paths from j to i*.

Definition A.2. The *complete path algebra of Q* is the $\mathbb{C}$-vector space $R\langle\langle Q\rangle\rangle$ consisting of all possibly infinite linear combinations of paths in Q, that is

$$R\langle\langle Q\rangle\rangle = \prod_{l=0}^{\infty} A^l.$$

$R\langle Q\rangle$ has multiplication induced by concatenation of paths and this multiplication extends naturally to $R\langle\langle Q\rangle\rangle$. $R\langle Q\rangle$ is a dense subalgebra of $R\langle\langle Q\rangle\rangle$ under the $\mathfrak{m}$-adic topology for $\mathfrak{m}$ the two-sided ideal of $R\langle\langle Q\rangle\rangle$ generated by the arrows of Q. The fundamental system of open neighbourhoods of this topology around 0 is given by the powers of the ideal $\mathfrak{m}$.

A.1 Quiver mutation. For a quiver Q, an l-cycle in Q is a path $\alpha_1\alpha_2\ldots\alpha_l$ with $l > 0$ such that $h(\alpha_1) = t(\alpha_l)$. If $\alpha_1\alpha_2\ldots\alpha_l$ is an l-cycle in Q, then so is $\alpha_i\alpha_{i+1}\ldots\alpha_{i-2}\alpha_{i-1}$ for $i = 2, \ldots, l$ (reducing indices mod l). We say that $\alpha_i\alpha_{i+1}\ldots\alpha_{i-1}$ can be obtained from $\alpha_1\alpha_2\ldots\alpha_l$ by *rotation*.

Definition A.3. Let Q be a quiver. An element W of $R\langle\langle Q\rangle\rangle$ is called a *potential* if it is a possibly infinite linear combination of cycles of Q such that no two cycles appearing in W with non-zero coefficients can be obtained from each other by rotation. If W is a potential on Q, we say that the pair (Q, W) is a quiver with potential, or a QP.

Definition A.4. Let Q, Q' be quivers with the same vertex set $Q_0 = Q'_0$.

(1) Two potentials W and W' on Q are *cyclically equivalent* if $W - W'$ lies in the closure of the vector subspace of $R\langle\langle Q\rangle\rangle$ spanned by all the elements of the form $\alpha_1\ldots\alpha_l - \alpha_2\ldots\alpha_l\alpha_1$ with $\alpha_1\ldots\alpha_l$ a cycle of positive length.

(2) We say that two QPs (Q, W) and (Q', W') are *right-equivalent* if there exists a $\mathbb{C}$-algebra isomorphism $\phi : R\langle\langle Q\rangle\rangle \to R\langle\langle Q'\rangle\rangle$ satisfying $\phi(e_i) = e_i\ \forall i \in Q_0 = Q'_0$ and such that $\phi(W)$ is cyclically equivalent to W'.

(3) For each arrow $\alpha \in Q_1$ and each cycle $\alpha_1 \dots \alpha_l$ in Q, we define the *cyclic derivative*

$$\partial_\alpha(\alpha_1 \dots \alpha_l) = \sum_{k=1}^{l} \delta_{\alpha,\alpha_k} \alpha_{k+1} \cdots \alpha_l \alpha_1 \cdots \alpha_{k-1}$$

and extend ∂_α by linearity and continuity so that $\partial_\alpha(W)$ is defined for every potential W.

(4) The *Jacobian ideal* $J(W)$ is the topological closure of the two-sided ideal of $R\langle\langle Q\rangle\rangle$ generated by $\{\partial_\alpha(W) | \alpha \in Q_1\}$, and the *Jacobian algebra* $P(Q, W)$ is the quotient algebra $R\langle\langle Q\rangle\rangle / J(W)$.

(5) A *QP* is *trivial* if $W \in A^2$ and $\{\partial_\alpha(W) | \alpha \in Q_1\}$ spans A as a $\mathbb{C}$-vector space.

(6) A QP is *reduced* if the degree-2 component of W is 0, that is, if the expression of W involves no two-cycles.

(7) The *direct sum* $Q \oplus Q'$ is the quiver whose vertex set is $Q_0 = Q'_0$ and whose arrow set is the disjoint union $Q_1 \sqcup Q'_1$.

(8) The *direct sum of two QPs* (Q, W) and (Q', W') is the QP $(Q, W) \oplus (Q', W') = (Q \oplus Q', W + W')$.

The following proposition then follows.

PROPOSITION A.5. *If $\varphi : R\langle\langle Q\rangle\rangle \to R\langle\langle Q'\rangle\rangle$ is a right-equivalence between (Q, W) and (Q', S'), then φ sends $J(W)$ onto $J(W')$ and therefore induces an algebra isomorphism $P(Q, W) \to P(Q', W')$.*

THEOREM A.6 (Splitting theorem [**4**]). *For every QP (Q, W), there exist a trivial QP (Q_{triv}, W_{triv}) and a reduced QP (Q_{red}, W_{red}) such that (Q, W) is right-equivalent to the direct sum $(Q_{triv}, W_{triv}) \oplus (Q_{red}, W_{red})$. The right-equivalence class of each of the QPs (Q_{triv}, W_{triv}) and (Q_{red}, W_{red}) is determined by the right-equivalence class of (Q, W).*

We will now discuss mutations of quivers with potentials. Let (Q, W) be a QP on the vertex set Q_0, and let $i \in Q_0$. We have no restrictions on Q, so it is possible that Q has a loop or two-cycle incident to i. We can replace W with a cyclically equivalent potential, where none of the cyclic paths of length greater than 1 in the expression of W begin at i. We can now define the potential $[W]$ on Q as the potential obtained from W by replacing every length-2 path $\alpha\beta$ passing through i with the arrow $[\alpha\beta]$. We also define $\Delta_i(Q) = \sum \beta^* \alpha^* [\alpha\beta]$, where the sum runs over all length-2 paths $\alpha\beta$ through i. Now we set $\tilde{\mu}_i(W) = [W] + \Delta_i(Q)$, which is a potential on $\tilde{\mu}_i(Q)$, the quiver obtained by applying the first two steps of quiver mutation as in Definition 2.9.

DEFINITION A.7. The mutation $\mu_i(Q, W)$ of (Q, W) with respect to i is defined as the reduced part of the QP $\tilde{\mu}_i(Q, W) = (\tilde{\mu}_i(Q), \tilde{\mu}_i(W))$.

It's important to note that the underlying quiver of a mutated QP is not necessarily two-acyclic. The potential determines whether or not we keep two-cycles.

A.2 Potential of a triangulation. If we have two triangulations related by a flip, we know that the associated quivers are related by the corresponding quiver mutation.

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

We want to lift this to the level of QPs, and see if the associated QPs are also related by a QP-mutation.

Let T be a triangulation of a marked surface (S, M), possible with punctures. Then the associated quiver has two types of oriented cycles: cycles arising from internal triangles of T, and simple cycles (cycles without repeated arrows) surrounding punctures. As before, we only consider cyclic equivalence classes of cycles.

In this generality (i.e. allowing punctures), Labardini–Fragoso **[8]** provides the following definition of a potential associated to a triangulation.

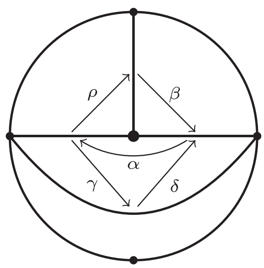
DEFINITION A.8. Let T be a triangulation of a marked surface (S, M). The potential W_T associated to T is the potential on Q_T that results from adding all the three-cycles that arise from internal triangles of T, and all the simple cycles that surround the punctures of (S, M).

In our situation with asymptotic triangulations of the annulus, we have no punctures, so the definition of a potential looks as follows:

$$W_T = \Sigma \text{ internal 3-cycles.}$$

Note that cycles between strictly asymptotic arcs do not appear in the potential.

EXAMPLE A.9. Let T be the following triangulation of the punctured disk D_4:



Then the potential W_T is $W_T = \alpha\gamma\beta + \alpha\beta\rho$.

THEOREM A.10 (**[8]**). *Let T and T' be two triangulations of a marked surface (S, M). If T' is obtained from T by flipping an arc d_i, then the QPs $(Q_{T'}, W_{T'})$ is obtained from the QP (Q_T, W_T) via the QP mutation μ_i.*

A.3 QPs of asymptotic triangulations. Let $T = T_\partial \sqcup T_{\partial'}$ be an asymptotic triangulation. We now look at QPs and QP-mutation of the associated quiver $Q_T = Q_\partial \sqcup Q_{\partial'}$.

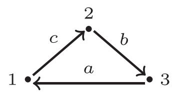
Consider a partial asymptotic triangulation T_β of $C_{p,q}$ and its associated quiver Q_β. Following Definition A.8, the potential $W_\beta = W_{T_\beta}$ associated to T_β results from adding all the three-cycles in the quiver Q_β.

As stated earlier, the two-cyclicity of a quiver relies heavily on the potential. When we work with the triangulation and quiver side-by-side, it's easy to determine the potential. However, if we're given a quiver Q_T, we want to be able to perform QP mutation without seeing what happens in T. We can consider two types of quivers that we associate to T. The first is as described in Section 3.1 with framing vertices. There we can start with any framing quiver associated to an asymptotic triangulation, read off the potential directly from the quiver, and perform the QP mutation.

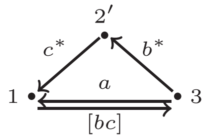
https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

Here we describe how to define mutation on the "classical" quiver (as in Definition 2.8) via QPs. Let $T = T_\partial \sqcup T_{\partial'}$ be an asymptotic triangulation. As mentioned earlier, the quiver $Q_T = Q_\partial \sqcup Q_{\partial'}$ may contain two-cycles or loops. Furthermore, both Q_∂ and $Q_{\partial'}$ contain a negatively oriented cycle around the meridian z. In particular, if all arcs are strictly asymptotic, the quiver Q_∂ and $Q_{\partial'}$ are both simple cycles $\alpha_1 \dots \alpha_p$ and $\alpha_{p+1} \dots \alpha_{p+q}$. This is the type of quiver where framing vertices are identified. In this quiver model, we need to start with a strictly asymptotic triangulation (all arcs of the triangulation T_β are strictly asymptotic), and because we have no internal triangles in T_β our potential $W_\beta = 0$. We make this specification because our quiver may show three-cycles that arise from going around the meridian, and we do not want this to be included in our potential. By starting with a potential $W_\beta = 0$, we can now work with quiver and QP mutation, and we will be able to keep certain loops and two-cycles in Q_β, while eliminating others.

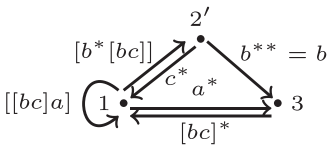
EXAMPLE A.11. Consider the potential $W = 0$ on the quiver



If we perform the *premutation* $\tilde{\mu}_2$ on (Q, W), we get $(\widetilde{Q}, \widetilde{W})$ where $\tilde{Q}$ is the arrow span of the quiver



and $\widetilde{W} = c^*b^*[bc]$. Then $\tilde{\mu}_2(Q, W) = \mu_2(\widetilde{Q}, \widetilde{W}) = (Q', W')$. Now if we want to mutate at vertex 3, we perform the same steps. First, we have the premutation $\tilde{\mu}_3$ on (Q', W'):



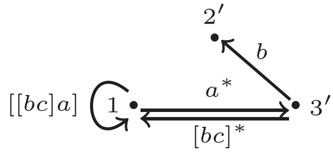
and our potential is

$$\widetilde{W}' = c^*[b^*[bc]] + [bc]^*b[b^*[bc]] + a^*[bc]^*[[bc]a] = (c^* + [bc]^*b)[b^*[bc]] + a^*[bc]^*[[bc]a].$$

Now we can check that $(c^* + [bc]^*b)$ is right-equivalent to an arrow $D : 2' \to 1$, which gives the potential

$$\widetilde{W}' = D[b^*[bc]] + a^*[bc]^*[[bc]a],$$

and our Jacobian algebra is $P(\widetilde{Q}', \widetilde{W}') = R\langle\langle \widetilde{Q}' \rangle\rangle / J(\widetilde{W}')$, where our Jacobian ideal $J(\widetilde{W}')$ gives us the relations $D = 0$ and $[b^*[bc]] = 0$, along with other relations. Thus, our mutated potential $\mu_3(\widetilde{Q}', \widetilde{W}') = (Q'', W'')$ where Q'' is the arrow span of the quiver:



B. Cluster structure on asymptotic triangulations.

The aim of this short addendum is to introduce an alternative cluster structure on asymptotic triangulations and provide a geometric interpretation.

B.1 Double cover and double quiver. As it is shown in Section 3, the most natural way to build a quiver from an asymptotic triangulation (i.e. adjacency quiver) leads to loops and two-cycles. To avoid this, consider a double cover $\widetilde{C}_{p,0}$ of the annulus $C_{p,0}$. An asymptotic triangulation T on $C_{p,0}$ induces an asymptotic triangulation $\widetilde{T}$ on $\widetilde{C}_{p,0}$, and the signed adjacency quiver $Q(\widetilde{T})$ of $\widetilde{T}$ is free of loops and two-cycles, so one can mutate it applying usual rules.

A flip of an arc $d_i \in T$ lifts as a composition of two commuting flips of arcs d_i^1 and d_i^2 in $\widetilde{T}$, so, it has the same effect as a composition of two commuting mutations of $Q(\widetilde{T})$.

B.2 Variables. To the arcs $d_1, \dots d_n$ of T, we assign independent variables $x_1, \dots, x_n$. Lifting this to the double cover results in pairs of identical variables x_i^1, x_i^2. To mutate the variables, we use the usual exchange relations provided by the quiver $Q(\widetilde{T})$. Since the initial quiver $Q(\widetilde{T})$ is symmetric (with symmetrically assigned initial variables) and each mutation is a composition of the symmetric commuting mutations, the symmetric structure on $Q(\widetilde{T})$ is preserved under mutations. We can also consider an exchange graph Γ of $(Q(\widetilde{T}), \{x_1, \dots, x_n\})$ consisting of seeds obtained by composite mutations preserving the initial symmetry.

B.3 Geometric interpretation of variables. Consider the annulus $C_{p,q}$ as a surface with hyperbolic metric. While applying Dehn twists in a closed curve z, we can renormalize the metric on $D_z^\infty(C_{p,q})$ so that the limit of the length of z is equal to 0. Then we can consider $D_z^\infty(C_{p,q})$ as a disjoint union of two hyperbolic punctured discs $\mathcal{C}_p$ and $\mathcal{C}_q$. Hence, we are able to measure lambda lengths of the curves of the asymptotic triangulation (including strictly asymptotic arcs). Combinatorially, z becomes a puncture, Prüfer arcs are tagged plane, adic arcs are tagged notched. See Figure 22 for an example of an exchange graph and corresponding lambda lengths on $\mathcal{C}_2$ obtained in this way.

Now, assign to $x_1, \dots, x_n$ the values equal to the lambda lengths of the lifts of the arcs $d_1, \dots, d_n$ on the double cover of $D_z^\infty(C_{p,q})$. Then the cluster variables in the exchange graph Γ will model the lambda lengths of arcs of asymptotic triangulations of $D_z^\infty(C_{p,q})$. More precisely, assuming (without loss of generality) that the initial

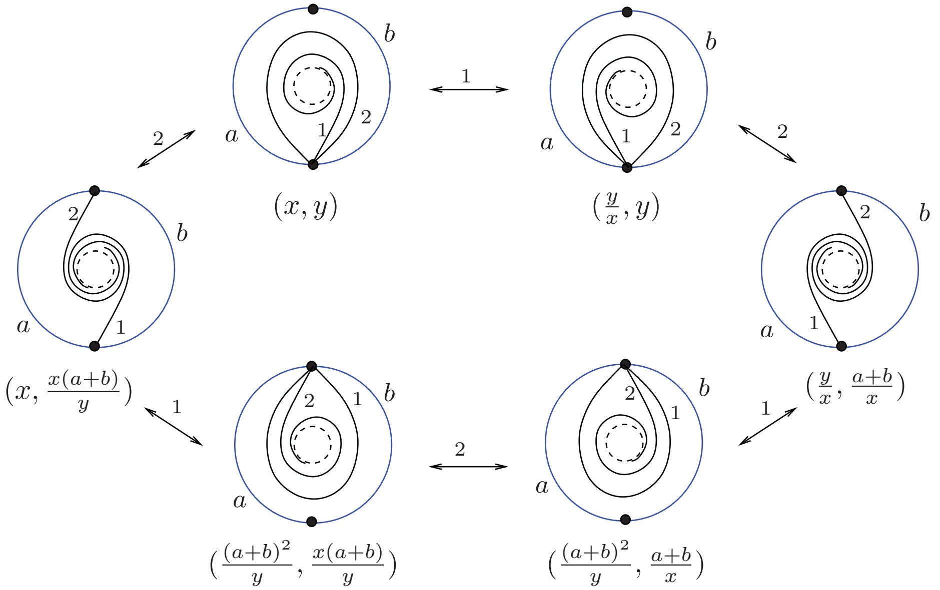


Figure 22. (Colour online) Lambda lengths of the curves on a connected component of $D_z^\infty(C_{2,q})$.

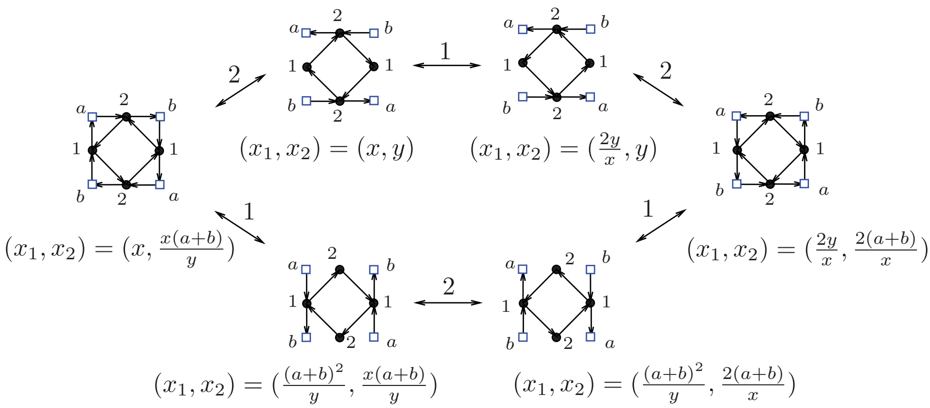


Figure 23. (Colour online) Exchange graph Γ and cluster variables for a connected component of $\widetilde{D}_z^\infty(C_{2,q})$.

asymptotic triangulation T contained no adic arcs, the lambda lengths of finite arcs and Prüfer arcs will be exactly equal to the values of the corresponding cluster variables, and the lambda lengths of the adic arcs will be halves of the corresponding variables (this is caused by the fact that the length of the corresponding horosphere around the limit of z is doubled in the cover $\widetilde{D}_z^\infty(C_{p,q})$ of $D_z^\infty(C_{p,q})$). See Figure 23 for an example.

B.4 From annulus to general hyperbolic surface. The same procedure as described above for an annulus can be done for any triangulated hyperbolic surface S: One can choose any simple closed curve $z \subset S$ and apply a sequence of Dehn twists in z, so that z becomes shorter and shorter in a renormalized metric and turns into a cusp in the

https://doi.org/10.1017/S0017089516000574 Published online by Cambridge University Press

limit. A triangulation T of S turns into an asymptotic triangulation of $D_z^\infty(S)$. If in addition there exists a double cover of S such that the curve z is covered by one (twice longer) curve, then we can build the quiver of the asymptotic triangulation and realize corresponding variables as lambda lengths.

ACKNOWLEDGEMENTS. We are grateful to Karin Baur and Michael Shapiro for inspiring discussions. The author would like to thank her supervisor Karin Baur for the many helpful discussions, as well as Gregg Musiker for his insightful discussions on using quivers with potentials, the anonymous referee for their comments, and Anna Felikson and Pavel Tumarkin for their work that is included in Appendix B. The author was supported by the Austrian Science Fund (FWF): projects No. P25141-N26 and W1230, and acknowledges support from NAWI Graz.

REFERENCES

1. J. Bastian, Mutation classes of $\tilde{A}_n$-quivers and derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$, *Algebra Number Theory* **5**(5) (2011), 567–594.

2. K. Baur and G. Dupont, Compactifying exchange graphs: Annuli and tubes, *Ann. Comb.* **3**(18) (2014), 797–839.

3. K. Baur and R. J. Marsh, A geometric model of tube categories, *J. Algebra* **362** (2012), 178–191.

4. H. Derksen, J. Weyman and A. Zelevinsky, Quivers with potentials and their representations I: Mutations, *Sel. Math*. **2**(14) (2008), 59–119.

5. S. Fomin and A. Zelevinsky, Cluster algebras I. Foundations, *J. Amer. Math. Soc*. **2**(15) (2002), 497–529.

6. A. Hatcher, On triangulations of surfaces, *Topology Appl*. **40**(2) (1991), 189–194.

7. H. Krause, Representations of quivers via reflection functors, arXiv:0804.1428, (2008).

8. D. Labardini-Fragoso, On triangulations, quivers with potentials and mutations, *Mexican Mathematicians Abroad* **657** (2016), 103.