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THE CONTROLLER DESIGN FOR SINGULAR FRACTIONAL-ORDER SYSTEMS WITH FRACTIONAL ORDER $0 < \alpha < 1$

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Abstract

We study the problem of pseudostate and static output feedback stabilization for singular fractional-order linear systems with fractional order α when $0 < \alpha < 1$. All the results are given by linear matrix inequalities. First, a new sufficient and necessary condition for the admissibility of singular fractional-order systems is presented. Then based on the admissible result, not only are sufficient conditions for designing pseudostate and static output feedback controllers obtained, but also sufficient and necessary conditions are presented by using different methods that guarantee the admissibility of the closed-loop systems. Finally, the effectiveness of the proposed approach is demonstrated by numerical simulations and a real-world example.

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1. Introduction

Although the study of fractional-order calculus (FOC) dates back to 300 years ago, it has attracted very little attention in the fields of engineering and technology for a long time. With the fast development of modern science and technology, especially of computers, it is now recognized that the theory of FOC provides a new theoretical basis and mathematical tool for the development of many subjects [2, 6, 13]. The nature of fractional-order operators is heredity; they are particularly suitable for describing physical processes with memory characteristics and some historical dependencies. In practical systems, many research objects such as viscoelastic systems, electrode systems, Isabel hurricane images and so on [6, 13], have these characteristics. The most typical example is that of heat conduction through a wall or a sphere which has been shown analytically to be of a fractional order of 0.5 [2]. Recently,

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applications of FOC in the field of control have attracted the increasing interest of many researchers [15, 17]. There have been many important results involving fractional-order systems (FOSs), such as system modelling [16], controllability and observability [4], stability analysis [3, 9] and controller synthesis [5, 10, 19, 25].

Singular systems [22], also known as differential-algebraic systems, semistate systems, descriptor systems, or generalized state-space systems, have been widely studied in the past few years. Practically, many physical systems can be better described by singular systems than by regular systems, for example power grid systems, the Leontief economic model and the Hopfield neural network model. Various stability and robust stabilization problems for integer-order singular systems [21, 27] have been investigated. In fact, if each circuit contains at least one mesh consisting of a branch with only an ideal supercapacitor and voltage source or at least one node with a supercoil, then it is a fractional singular system [8]. Therefore, it is of significant interest to study singular fractional-order systems (SFOSs). Recently, many authors have presented a series of results for stability and stabilization of SFOSs with order between zero and two by linear matrix inequality (LMI) methods [14, 23]. As is well known, admissibility analysis is an important property in singular control theory when stability analysis is referred to, but only a few papers give the relevant results for SFOSs. First the admissibility conditions [20] of SFOSs with an order $0 < \alpha < 2$ were proposed, and then Yu et al. [24] and Zhang et al. [26] discussed some admissibility and stabilization problems of SFOSs. However, these admissibility criteria are complicated, and they are not convenient for controller design.

In addition, the stability domain of FOSs is different when the order belongs to the intervals $(1, 2)$ and $(0, 1)$. Since the former is convex, the latter is non-convex and thus is not an LMI region, which increases the difficulty of studying relative control problems in the interval $(0, 1)$. Recently, Ji and Qiu [7] and Marir et al. [11, 12] have investigated admissibility conditions and stabilization problems for SFOSs with an order between zero and two, but the obtained stabilization conditions were not strict LMIs, and they involved some limited matrix variables, which may lead to an increase in system conservatism. This motivates us to study much better stability and stabilization conditions for SFOSs in $(0, 1)$, which can make the resulting controllers less conservative and more suitable to practical applications. Related controller design problems are also discussed in this paper. The organization of this paper is as follows. In Section 2, a description of the model and some preliminaries are introduced. In Section 3, a new sufficient and necessary condition of admissibility for SFOSs is given. In Section 4, different sufficient conditions for designing feedback controllers are obtained in terms of LMIs, including some sufficient and necessary criteria. Finally, two examples are given to validate the effectiveness of the proposed results. The paper concludes with a brief discussion in Section 5.

2. Preliminaries

In this section, we introduce some notation, definitions and lemmas, which will be used later in the paper.

Let $X \geq 0$ ($X > 0$) denote the semipositive definite (positive definite) matrix, and let 0 and I stand for the zero matrix and identity matrix, respectively, with appropriate dimensions. The sets of integers, real numbers, positive real numbers and complex numbers are denoted by $\mathbf{Z}$, $\mathbf{R}, \mathbf{R}^{+}$ and $\mathbf{C}$, respectively. The symbols $\mathbf{R}^{n}$, $\mathbf{C}^{n}$, $\mathbf{R}^{m\times n}$ and $\mathbf{C}^{n\times n}$ denote, respectively, the n-dimensional Euclidean space, the n-dimensional complex number space, the set of all $m \times n$ real matrices and the set of all $n \times n$ complex matrices. For any complex matrix X, $X^{\star}$ denotes its conjugate transpose, $\bar{X}$ represents its conjugate, $\Re(X)$ denotes its real part, $\Im(X)$ denotes the imaginary part and $\mathrm{sym}(X)$ represents $X + X^{T}$. The symbol $\divideontimes$ denotes the transposed elements in the symmetric positions of a matrix, and $\mathrm{diag}\{\ldots\}$ stands for a block diagonal matrix. In addition, $|\arg(\lambda)|$ represents the absolute value of principal value of any complex number λ, satisfying $0 \leq |\arg(\lambda)| \leq \pi$ and $\arg(\lambda) \neq -\pi$. The symbol i denotes the imaginary unit satisfying $i^{2} = -1$.

Oldham and Spanier [15] and Podlubny [17] defined the Caputo fractional integral

$$I^{\alpha} f(s) = \frac{1}{\Gamma(\alpha)} \int_{0}^{s} (s-\tau)^{\alpha-1} f(\tau)\, d\tau \quad s > 0,$$

where $f(s)$ is any continuously differentiable function, $\alpha \in \mathbf{R}^{+}$ is the order of fractional integration and

$$\Gamma(\alpha) = \int_{0}^{\infty} e^{-z} z^{\alpha-1}\, dz.$$

The Caputo fractional derivative with the order $\alpha \in \mathbf{R}^{+}$ is defined by

$$D^{\alpha} f(s) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{s} (s-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau,$$

where $m - 1 < \alpha \leq m$, $m \in \mathbf{Z}$.

Consider the singular fractional-order pseudostate system described by the form

$$\begin{cases} ED^{\alpha} x(t) = Ax(t) + Bu(t) \\ \qquad\ \ y(t) = Cx(t), \end{cases} \tag{2.1}$$

where $0 < \alpha < 1$ is the noninteger order, $x(t) \in \mathbf{R}^{n}$ is the pseudostate vector, $u(t) \in \mathbf{R}^{m}$ is the control input and $y(t) \in \mathbf{R}^{p}$ is the system measured output. The matrix $E \in \mathbf{R}^{n\times n}$ is a singular matrix with $\mathrm{rank}(E) = n_e < n$, and $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{p\times n}$ are constant matrices. D^{α} denotes the Caputo derivative operator.

The nominal unforced SFOS of (2.1) can be written as

$$ED^{\alpha} x(t) = Ax(t). \tag{2.2}$$

The following definitions are similar to integer-order singular systems.

Definition 2.1 [24]. The finite roots of $|\lambda E - A| = 0$ in SFOS (2.2) are called finite dynamic modes of the pair (E, A).

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Definition 2.2 [7, 24]. For SFOS (2.2):

(i) the pair (E, A) is called regular if there exists a constant scalar $s \in \mathbf{C}$ such that $|s^{\alpha}E - A| \neq 0$;
(ii) the pair (E, A) is called impulse free if $\deg(\det(\lambda E - A)) = \operatorname{rank}(E)$, where $\lambda \in \mathbf{C}$;
(iii) the pair (E, A) is called asymptotically stable if all the finite dynamic modes satisfy $|\arg(\lambda)| > \alpha\pi/2$; and
(iv) the pair (E, A) is called admissible if it is regular, impulse free and asymptotically stable.

Without loss of generality, we choose nonsingular matrices M and N satisfying

$$MEN = \begin{bmatrix} I_{n_e} & 0 \\ 0 & 0 \end{bmatrix}, \quad MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad x(t) = N \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{2.3}$$

Then SFOS (2.2) can be transformed into

$$\begin{cases} D^{\alpha} x_1(t) = A_{11}x_1(t) + A_{12}x_2(t) \\ \quad\; 0 = A_{21}x_1(t) + A_{22}x_2(t), \end{cases} \tag{2.4}$$

where $x_1(t) \in \mathbf{R}^{n_e}$, $x_2(t) \in \mathbf{R}^{n-n_e}$.

Remark 2.3. In (2.3), let $\tilde{E} = MEN$ and $\tilde{A} = MAN$. Then

$$\det(\lambda\tilde{E} - \tilde{A}) = \det(M)\det(N)\det(\lambda E - A)$$

for any $\lambda \in \mathbf{C}$. According to Definitions 2.1 and 2.2, system (2.2) and system (2.4) have the same regularity, nonimpulsiveness and stability. From the second equation of (2.4), the unique solution of state $x_2(t)$ can be obtained when matrix A_{22} is nonsingular. Similar to the integer-order singular systems in [5], and from Definition 2.2, system (2.2) is impulse free if and only if matrix A_{22} is nonsingular. From Definition 2.2, it can also be deduced that system (2.2) being impulse free implies that system is regular.

Remark 2.4. When $t = 0$, $x(0)$ is the initial value of system (2.2), from (2.3), $N^{-1}x(0) = [x_1^T(0) \;\; x_2^T(0)]^T$. However, $x_2(0)$ should satisfy (2.4). In order to guarantee the compatibility of the initial value, the initial condition is given as $Ex(0) = x_0$. In other words, if (E, A) is impulse free, then $x_2(0)$ can be obtained by $x_2(0) = -A_{22}{}^{-1}A_{21}x_1(0)$ when $x_1(0)$ is given.

Lemma 2.5 [7]. *SFOS (2.2) is regular if and only if there exists two nonsingular matrices Q and P such that*

$$QEP = \operatorname{diag}(I_{n_1}, \mathcal{N}), \quad QAP = \operatorname{diag}(A_1, I_{n_2}),$$

where $n_1 + n_2 = n$, $A \in \mathbf{R}^{n_1 \times n_1}$, $\mathcal{N} \in \mathbf{R}^{n_2 \times n_2}$ is nilpotent.

Remark 2.6. According to Lemma 2.5, system (2.2) is impulse free if and only if $\mathcal{N} = 0$.

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LEMMA 2.7 [1]. *For matrices W, P, R and A with appropriate dimensions and a scalar β, the following propositions are equivalent.*

$$\text{(i)} \qquad \begin{bmatrix} W & * \\ \beta P^T + RA & -\beta R - \beta R^T \end{bmatrix} < 0.$$

$$\text{(ii)} \qquad W < 0, W + PA + A^T P^T < 0.$$

LEMMA 2.8 [3]. *Let $A \in \mathbf{R}^{n\times n}$, $0 < \alpha < 1$. FOS $D^\alpha x(t) = Ax(t)$ is asymptotically stable if and only if there exist $X = X^\star \in \mathbf{C}^{n\times n}$, $X > 0$ such that*

$$(rX + \bar{r}\bar{X})^T A^T + A(rX + \bar{r}\bar{X}) < 0,$$

where $r = e^{i(1-\alpha)\pi/2}$.

In this paper, a new admissibility condition for SFOSs with the order belonging to $(0, 1)$ is obtained. Based on this, the corresponding feedback controllers are designed in terms of LMIs to guarantee the admissibility of the resulting closed-loop systems.

3. Main results

3.1. Admissibility condition

THEOREM 3.1. SFOS (2.2) *is admissible, if and only if there exist matrices $X = X^\star \in \mathbf{C}^{n\times n}$, $X > 0$ and $Q \in \mathbf{R}^{(n-n_e)\times n}$ such that*

$$(PE^T + SQ)^T A^T + A(PE^T + SQ) < 0, \tag{3.1}$$

where $r = e^{i(1-\alpha)\pi/2}$, $P = rX + \bar{r}\bar{X} \in \mathbf{R}^{n\times n}$ and $S \in \mathbf{R}^{n\times(n-n_e)}$ is an arbitrary full column rank matrix that satisfies $ES = 0$.

PROOF. (Sufficient) Assume that there exist matrices $X = X^\star > 0$ and Q such that inequality (3.1) holds. First, we show that the pair (E, A) is regular and impulse free. From (2.3) and (2.4), it is equivalent to proving that A_{22} is nonsingular. Set

$$N^{-1}PN^{-T} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}, \quad N^{-1}S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \quad QM^T = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}. \tag{3.2}$$

Since $ES = 0$, we obtain $S_1 = 0$ and that $S_2 \in \mathbf{R}^{(n-n_e)\times(n-n_e)}$ is of full rank. Then, pre- and post-multiplying (3.1) by M and M^T yields

$$\mathrm{sym}\{MANN^{-1}PN^{-T}N^T E^T M^T + MANN^{-1}SQM^T\} < 0. \tag{3.3}$$

Substituting (2.3) and (3.2) in (3.3) gives

$$\begin{bmatrix} \circledast & \circledast \\ \circledast & A_{22}S_2Q_2 + Q_2{}^T S_2{}^T A_{22}{}^T \end{bmatrix} < 0, \tag{3.4}$$

where $\circledast$ represents the unrelated matrix block in the discussion. From (3.4),

$$A_{22}S_2Q_2 + Q_2{}^T S_2{}^T A_{22}{}^T < 0.$$

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Obviously, A_{22} is nonsingular. Otherwise, if A_{22} is singular, then there exists a vector $\xi \in \mathbf{R}^{n-n_e}$ ($\xi \neq 0$) such that $A_{22}^T\xi = 0$ and $\xi^T(A_{22}S_2Q_2 + Q_2{}^TS_2{}^TA_{22}{}^T)\xi = 0$, which conflicts with $A_{22}S_2Q_2 + Q_2{}^TS_2{}^TA_{22}{}^T < 0$. Thus the pair (E, A) is regular and impulse free.

Next, we show that the pair (E, A) is stable. Let λ be any finite eigenvalue of the pair (E^T, A^T) and v be the corresponding eigenvector, that is, $\lambda E^T v = A^T v$, $\bar{\lambda}v^\star E = v^\star A$. From (3.1) and $ES = 0$,

$$v^\star(PE^T + SQ)^TA^Tv + v^\star A(PE^T + SQ)v < 0,$$

that is

$$v^\star E(\lambda P^T + \bar{\lambda}P)E^Tv < 0.$$

Since

$$\begin{aligned}\lambda P^T + \bar{\lambda}P &= \lambda(rX + \bar{r}\bar{X})^T + \bar{\lambda}(rX + \bar{r}\bar{X})\\ &= 2\Re(\lambda r)\bar{X} + 2\Re(\lambda\bar{r})X\\ &= 2[\Re(\lambda r) + \Re(\lambda\bar{r})]\Re(X) - 2[\Re(\lambda r) - \Re(\lambda\bar{r})]\Im(X)i,\end{aligned}$$

and from $X = X^\star > 0$, we have $\Re(X) = \Re(X)^T > 0$ and $\Im(X) = -\Im(X)^T$. Furthermore, $v^\star\Im(X)v = 0$ for any vector $v \in \mathbf{C}^n$. Therefore,

$$v^\star E(\lambda P^T + \bar{\lambda}P)E^Tv = 2[\Re(\lambda r) + \Re(\lambda\bar{r})]v^\star E\Re(X)E^Tv < 0,$$

which implies that $\Re(\lambda r) + \Re(\lambda\bar{r}) < 0$. Then it follows that $\Re(\lambda r) < 0$ or $\Re(\lambda\bar{r}) < 0$, that is, $\lambda \in \{\lambda \in \mathbf{C} : \Re(\lambda e^{i(1-\alpha)\pi/2}) < 0\} \cup \{\lambda \in \mathbf{C} \mid \Re(\lambda e^{-i(1-\alpha)\pi/2}) < 0\}$. According to Kaczorek [8] and Xu [20], all the finite eigenvalues of the pair (E^T, A^T) lie in $\{\lambda \in \mathbf{C} \mid |\arg(\lambda)| > \alpha\pi/2\}$. Then the SFOS (2.2) is stable.

(Necessary) The proof is similar to a result of Yin et al. [12, Theorem 2], so we omit it here. □

Remark 3.2. In Theorem 3.1, if $E = I$, then SFOS (2.2) reduces to a regular fractional-order system; obviously if $S = 0$, then Theorem 3.1 becomes the sufficient and necessary stability condition as in [3, 5]. If the order $\alpha = 1$, then system (2.2) reduces to an integer-order singular system. From Theorem 3.1, system $E\dot{x} = Ax(t)$ is admissible if and only if there exist $P > 0$ and Q such that $(PE^T + SQ)^TA^T + A(PE^T + SQ) < 0$, which is the same as the result Xu and Lam [22]. Therefore, SFOSs are extensions of integer-order singular systems, that is, the admissibility condition given in [22] is a special case of the Theorem 3.1.

Remark 3.3. Marir et al. [11] investigated the admissibility conditions for SFOSs with fractional order between one and two, while the fractional order that we study in this paper belongs to $(0, 1)$. The two issues are entirely different in nature and independent of each other. Therefore, Theorem 2 in [11] cannot solve the problems of this paper. In addition, when the fractional order belongs to $(0, 1)$ or $(1, 2)$, the corresponding stability domain is different. The former is nonconvex while the latter is convex. So it is found that a stability domain between zero and one makes it more difficult to study stability and stabilization problems.

Remark 3.4. Compared with the admissible condition given in [24], Theorem 3.1 is relatively simple in form, and the number of variables are reduced. Moreover, the inequality dimension of Theorem 3.1 is $n \times n$, which is less than that in [24]. So Theorem 3.1 significantly reduces the computational burden in the data analysis phase, and improves the practicality. In addition, Wei et al. [18] gave another admissibility condition for SFOSs; when the order $\alpha = 1$, their result [18] is not available, while it is a special case of the result given in Theorem 3.1.

From Definition 2.2, it can be verified that the admissibility of the pair (E, A) is equivalent to the admissibility of the pair (E^T, A^T). Then we obtain the following corollary.

Corollary 3.5. SFOS (2.2) *is admissible if and only if there exist matrices* $X = X^\star \in \mathbf{C}^{n\times n}$, $X > 0$ *and* $Q \in \mathbf{R}^{(n-n_e)\times n}$ *such that*

$$(PE + SQ)^T A + A^T (PE + SQ) < 0,$$

where $r = e^{i(1-\alpha)\pi/2}$, $P = rX + \bar{r}\bar{X} \in \mathbf{R}^{n\times n}$ *and* $S \in \mathbf{R}^{n\times(n-n_e)}$ *is an arbitrary full column rank matrix that satisfies* $E^T S = 0$.

3.2. Stabilization problems

Consider SFOS (2.1) with the following pseudostate feedback controller

$$u(t) = Kx(t), \quad K \in \mathbf{R}^{m\times n},$$

and output feedback controller

$$u(t) = Ly(t), \quad L \in \mathbf{R}^{m\times p}.$$

Applying the above two controllers, respectively, to system (2.1), the closed-loop system can be obtained as

$$ED^\alpha x(t) = (A + BK)x(t) \tag{3.5}$$

and

$$ED^\alpha x(t) = (A + BLC)x(t). \tag{3.6}$$

In the following section, stabilization conditions for SFOS (2.1) are investigated by LMIs.

3.2.1 *Pseudostate feedback control design.*

Theorem 3.6. *The singular fractional-order closed-loop control system* (3.5) *with the fractional-order* $0 < \alpha < 1$ *is admissible if, for a given scalar* $\beta > 0$, *there exist matrices* $X = X^\star \in \mathbf{C}^{n\times n}$, $X > 0$, $Q \in \mathbf{R}^{(n-n_e)\times n}$ *and* $\tilde{K} \in \mathbf{R}^{m\times n}$ *such that*

$$\begin{bmatrix} \mathrm{sym}\{\Pi^T A^T + E\tilde{K}^T B^T\} & B\tilde{K} + \beta Q^T S^T \\ \divideontimes & -\beta P - \beta P^T \end{bmatrix} < 0, \tag{3.7}$$

where $\Pi = PE^T + SQ$, $P = rX + \bar{r}\bar{X} \in \mathbf{R}^{n\times n}$, $r = e^{i(1-\alpha)\pi/2}$, $S \in \mathbf{R}^{n\times(n-n_e)}$ *is an arbitrary full column rank matrix and satisfies* $ES = 0$. *Moreover, the pseudostate feedback gain matrix is given by*

$$K = \tilde{K}(rX + \bar{r}\bar{X})^{-1}.$$

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Proof. From Theorem 3.1, the closed-loop system (3.5) is admissible if there exist matrices $X = X^\star > 0$ and $Q \in \mathbf{R}^{(n-n_e)\times n}$ such that

$$(PE^T + SQ)^T(A + BK)^T + (A + BK)(PE^T + SQ) < 0,$$

which is equivalent to

$$\mathrm{sym}\,\{(PE^T + SQ)^T A^T + BKPE^T + BKSQ\} < 0, \tag{3.8}$$

where $P = rX + \bar{r}\bar{X}$. According to Kaczorek [8, Theorem 17], the real matrix P is nonsingular; based on the result of Lemma 2.7, if the matrix inequality

$$\begin{bmatrix} \mathrm{sym}\,\{\Pi^T A^T + BKPE^T\} & \beta Q^T S^T + BKP \\ \divideontimes & -\beta P - \beta P^T \end{bmatrix} < 0 \tag{3.9}$$

is established, then (3.8) holds. By setting $\tilde{K} = KP$, (3.9) can be rewritten as (3.7). From (3.7), it follows that P is nonsingular, so $K = \tilde{K}P^{-1}$. This completes the proof. □

Remark 3.7. In the work of Ji and Qiu [7], state feedback analysis based on the admissible condition of Yu et al. [24] is proposed by giving a limit on the input matrix B and assuming that two real symmetric positive definite matrices are equal and two skew-symmetric matrices are null, which greatly increases the conservatism compared with our result in Theorem 3.6.

Theorem 3.8. *A singular fractional-order closed-loop control system (3.5) with the order $0 < \alpha < 1$ is admissible, if and only if there exist matrices $X = X^\star \in \mathbf{C}^{n\times n}$, $X > 0$, $Q \in \mathbf{R}^{(n-n_e)\times n}$ and $\tilde{K} \in \mathbf{R}^{m\times n}$ such that $(PE^T + SQ)$ is nonsingular and*

$$\mathrm{sym}\{(PE^T + SQ)^T A^T + B\tilde{K}\} < 0, \tag{3.10}$$

where $P = rX + \bar{r}\bar{X} \in \mathbf{R}^{n\times n}$, $r = e^{i(1-\alpha)\pi/2}$, $S \in \mathbf{R}^{n\times(n-n_e)}$ is an arbitrary full column rank matrix and satisfies $ES = 0$. Moreover, the pseudostate feedback gain matrix is

$$K = \tilde{K}(PE^T + SQ)^{-1}.$$

Proof. This can be directly obtained from Theorem 3.6. □

Remark 3.9. In Theorem 3.8, if $PE^T + SQ$ is singular, by using the singular value decomposition of the matrix E,

$$E = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T,$$

where Σ is a nonsingular matrix and U, V are orthogonal matrices. Let $X = X_1 + X_2 i$, $r = r_1 + r_2 i$. We have $X_1 = {X_1}^T \in \mathbf{R}^{n\times n}$, $X_1 > 0$, $X_2 = -{X_2}^T$, $r_1, r_2 > 0$. Let

$$V^T X_1 V = \begin{bmatrix} X_{11} & X_{12} \\ {X_{12}}^T & X_{13} \end{bmatrix}, \quad V^T X_2 V = \begin{bmatrix} X_{21} & X_{22} \\ -{X_{22}}^T & X_{23} \end{bmatrix},$$

where $X_{11} > 0$, $X_{13} > 0$, X_{21}, X_{23} are skew-symmetric matrices. Let

$$QU = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}, \quad V^T S = \begin{bmatrix} S_1^T & S_2^T \end{bmatrix}^T,$$

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and from $ES = 0$ it follows that

$$V^T S = \begin{bmatrix} 0 \\ S_2 \end{bmatrix}, \quad S_2 \in \mathbf{R}^{(n-n_e)\times(n-n_e)}, \quad \mathrm{rank}(S_2) = n - n_e.$$

Then

$$PE^T + SQ = V \begin{bmatrix} P_1\Sigma & 0 \\ P_2\Sigma + S_2Q_1 & S_2Q_2 \end{bmatrix} U^T, \tag{3.11}$$

where $P_1 = 2r_1X_{11} - 2r_2X_{21}$ and $P_2 = 2r_1{X_{12}}^T + 2r_2{X_{22}}^T$. Moreover, P_1 is nonsingular. Otherwise, if P_1 is singular, then there exists a vector $\xi \in \mathbf{R}^{n\times n}(\xi \neq 0)$ such that $P_1\xi = 0$ and $\xi^T P_1\xi = 0$. However, for any $x \in \mathbf{R}^{n\times n}(x \neq 0)$,

$$\begin{aligned} x^T P_1 x &= x^T(2r_1X_{11} - 2r_2X_{21})x \\ &= 2r_1x^T X_{11}x - 2r_2x^T X_{21}x \\ &= 2r_1x^T X_{11}x > 0, \end{aligned}$$

which conflicts with the above hypothesis. Thus, from (3.11), if Q_2 is nonsingular, then $PE^T + SQ$ is nonsingular. However, if Q_2 is singular, let

$$\bar{Q} = \begin{bmatrix} 0 & \bar{Q}_2 \end{bmatrix} U^T,$$

where $\bar{Q}_2$ is nonsingular. Choose a sufficiently small constant ϵ that is not the eigenvalue of the matrix $-Q_2\bar{Q}_2^{-1}$ and let

$$\hat{Q} = Q + \epsilon\bar{Q} = \begin{bmatrix} Q_1 & Q_2 + \epsilon\bar{Q}_2 \end{bmatrix} U^T.$$

Then

$$PE^T + S\hat{Q} = V \begin{bmatrix} P_1\Sigma & 0 \\ P_3\Sigma + S_2Q_1 & S_2(Q_2 + \epsilon\bar{Q}_2) \end{bmatrix} U^T,$$

where $Q_2 + \epsilon\bar{Q}_2$ is nonsingular. Thus, it follows that $PE^T + S\hat{Q}$ is nonsingular. From (3.10),

$$\mathrm{sym}\{(PE^T + S\hat{Q})^T A^T + B\tilde{K}\} = \mathrm{sym}\{(PE^T + SQ)^T A^T + B\tilde{K}\} + \epsilon\,\mathrm{sym}\{(S\bar{Q})^T A^T\} < 0,$$

and the gain matrix of the controller is given by $K = \tilde{K}(PE^T + S\hat{Q})^{-1}$. In conclusion, if $PE^T + SQ$ is singular, we can choose a sufficiently small constant ϵ that is not the eigenvalue of matrix $-Q_2\bar{Q}_2^{-1}$ such that $PE^T + S(Q + \epsilon\bar{Q})$ is nonsingular and satisfies (3.10).

REMARK 3.10. In this case, although there is a limitation that the matrix $PE^T + SQ$ is nonsingular, the sufficient and necessary condition provided by the LMI in Theorem 3.8 does not increase the conservatism since, if $PE^T + SQ$ is singular, it can be modified as nonsingular based on Remark 3.9. Further, Examples 4.2 and 4.4 also verify that the conservatism does not increase. In addition, when the order $\alpha = 1$, Theorem 3.8 becomes the stabilization result of an integer-order singular system [22].

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THEOREM 3.11. *A singular fractional-order closed-loop control system* (3.5) *with the order* $0 < \alpha < 1$ *is admissible if and only if there exist matrices* $X = X^\star \in \mathbf{C}^{n\times n}$, $X > 0$, $Q \in \mathbf{R}^{(n-n_e)\times n}$ *and* $\gamma_1 > 0$ *such that*

$$\mathrm{sym}\{A^T(PE + SQ)\} - \frac{1}{\gamma_1}F^T F < 0,$$

where

$$F = B^T(PE + SQ),$$

$P = rX + \bar{r}\bar{X} \in \mathbf{R}^{n\times n}$, $r = e^{i(1-\alpha)\pi/2}$ *and* $S \in \mathbf{R}^{n\times(n-n_e)}$ *is an arbitrary full column rank matrix that satisfies* $E^T S = 0$. *Moreover, the pseudostate feedback gain matrix is*

$$K = -\frac{1}{\gamma_1}B^T(PE + SQ).$$

PROOF. The proof is similar to that of [12, Theorem 3]. □

3.2.2 *Output feedback control design.*

THEOREM 3.12. *The singular fractional-order closed-loop control system* (3.6) *with the order* $0 < \alpha < 1$ *is admissible if, for given a scalar* β, *there exist matrices* $X = X^\star \in \mathbf{C}^{n\times n}$, $X > 0$, $Q \in \mathbf{R}^{(n-n_e)\times n}$, $R \in \mathbf{R}^{p\times p}$ *and* $H \in \mathbf{R}^{m\times p}$ *such that*

$$\begin{bmatrix} \mathrm{sym}\{\Pi^T A^T + BHC\} & \beta\Pi^T C^T - \beta C^T R^T + BH \\ \divideontimes & -\beta R - \beta R^T \end{bmatrix} < 0, \tag{3.12}$$

where $\Pi = PE^T + SQ$, $P = rX + \bar{r}\bar{X} \in \mathbf{R}^{n\times n}$, $r = e^{i(1-\alpha)\pi/2}$ *and* $S \in \mathbf{R}^{n\times(n-n_e)}$ *is an arbitrary full column rank matrix that satisfies* $ES = 0$. *Moreover, the stabilizing output feedback controller is given by*

$$L = HR^{-1}. \tag{3.13}$$

PROOF. From (3.12), it follows that $-\beta R - \beta R^T < 0$, which yields that R is nonsingular, and along with (3.13), we have $H = LR$. Substituting this into inequality (3.12) we obtain

$$\begin{bmatrix} \mathrm{sym}\{\Pi^T A^T + BLRC\} & \beta\Pi^T C^T - \beta C^T R^T + BLR \\ \divideontimes & -\beta R - \beta R^T \end{bmatrix} < 0,$$

which, by Lemma 2.7, implies that

$$\mathrm{sym}\{(PE^T + SQ)^T A^T + BLRC\} + \mathrm{sym}\{BLC(PE^T + SQ) - BLRC\} < 0,$$

that is,

$$\mathrm{sym}\{(A + BLC)(PE^T + SQ)\} < 0.$$

Thus, from Theorem 3.1, the closed-loop system (3.6) is admissible. □

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REMARK 3.13. In Theorem 3.12, there is no limitation that the input matrix B^T is of full row rank, which is different from the condition given by Ji and Qiu [7]. So the condition in Theorem 3.12 is less conservative.

THEOREM 3.14. *A singular fractional-order closed-loop control system* (3.6) *with the order* $0 < \alpha < 1$ *is admissible if and only if there exist matrices* $X = X^\star \in \mathbf{C}^{n\times n}$, $X > 0$, $Q \in \mathbf{R}^{(n-n_e)\times n}$ *and* $\gamma_2 > 0$ *such that*

$$\mathrm{sym}\{A^T(PE + SQ)\} - \frac{1}{\gamma_2}F^T F < 0,$$

where $F = B^T(PE + SQ)$, $P = rX + \bar{r}\bar{X} \in \mathbf{R}^{n\times n}$, $r = e^{i(1-\alpha)\pi/2}$ *and* $S \in \mathbf{R}^{n\times(n-n_e)}$ *is an arbitrary full column rank matrix that satisfies* $E^T S = 0$. *Moreover, the pseudostate feedback gain matrix is given by*

$$K = -\frac{1}{\gamma_2}B^T(PE + SQ)V_1\begin{bmatrix}\Sigma^{-1} & 0\\ 0 & 0\end{bmatrix}U_1{}^{-1},$$

where U_1 *and* V_1 *are orthogonal matrices and* Σ *is nonsingular matrix; these are obtained by using the singular value decomposition of the matrix* C,

$$C = U_1\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}V_1{}^T.$$

PROOF. The proof process is similar to that of [12, Theorem 2]. □

REMARK 3.15. Although Marir et al. [11, 12] gave sufficient and necessary conditions to investigate output feedback and observer-based control, the results were represented by the quadratic matrix inequalities, not LMIs, which made them more difficult to solve. Moreover, if we use a method similar to that of Mrir et al. [12], the sufficient and necessary conditions in the form of the quadratic matrix inequalities can also be obtained from Theorems 3.11 and 3.14. Compared with Theorems 3.6 and 3.12, the conditions (3.7) and (3.12) are only sufficient, but they are all strict LMIs, and can be solved directly by Matlab. Obviously, Theorems 3.6 and 3.12 are easier to solve than Theorems 3.11 and 3.14. In addition, in Theorem 3.12, a sufficient and necessary condition in the form of an LMI is also presented, which is convenient for being utilised in practice with the help of Matlab.

4. Examples

In this section, four numerical examples are presented to show the validity of the results obtained. Example 4.1 serves to illustrate the effectiveness of our proposed admissibility condition. In Example 4.2, we cite an example of Ji and Qiu [7, Example 17] to make a comparison with our results for demonstrating the superiority in the aspect of designing feedback controllers. When the system is not regular or impulse free and unstable, Example 4.3 shows that it is also solvable by applying our results. Finally, we give an electrical circuit which is a singular fractional-order linear system; we note that studying the topics in this paper is of significant use in practical systems.

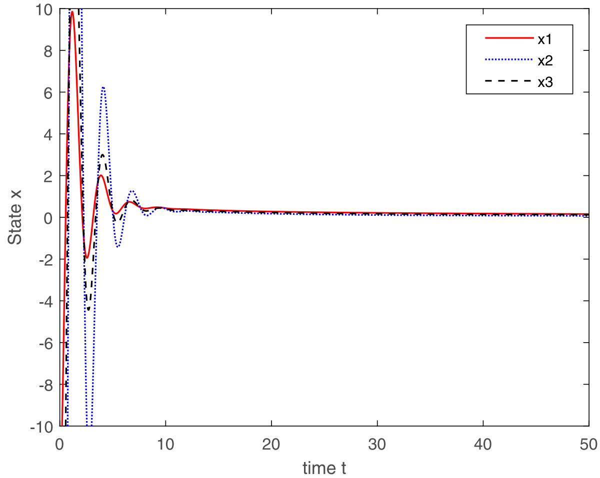

FIGURE 1. Time responses of the system.

EXAMPLE 4.1. Consider an SFOS (2.2) with $\alpha = 0.6$ and

$$E = \begin{bmatrix} 1 & -0.5 & 1 \\ 1 & -0.5 & 1 \\ -1 & -0.5 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} -4 & 1 & -3 \\ 0 & -2 & 4 \\ 5 & 2 & 0 \end{bmatrix}.$$

We choose $S = [1, 0, -1]^T$ to satisfy $ES = 0$. By solving the LMI in Theorem 3.1, a feasible solution is obtained as

$$X = \begin{bmatrix} 0.5967 & 0.1821 + 0.0436i & -0.4602 - 0.0106i \\ 0.1821 - 0.0436i & 0.8943 & 0.1821 - 0.0436i \\ -0.4602 + 0.0106i & 0.1821 + 0.0436i & 0.5967 \end{bmatrix},$$

$$Q = \begin{bmatrix} -0.1292 & 0.2531 & 0.7994 \end{bmatrix}.$$

Therefore, from Theorem 3.1, this system is admissible. The time responses of the system with the initial condition $Ex(0) = [-5.6 \ -5.6 \ 14]^T$ are shown in Figure 1.

EXAMPLE 4.2. Consider an SFOS (2.1) with coefficients $\alpha = 0.5$ and

$$E = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

This example was presented by Marir et al. [12]. The finite dynamic mode of the pair (E, A) is 0.3333, which concludes that the corresponding system is not asymptotically stable. The time responses of the system with $u(t) = 0$ are shown in Figure 2, when the initial condition $Ex(0) = [0.6 \ 1.9]^T$ is satisfied. In this example, a pseudostate feedback controller is designed to satisfy the admissibility of the closed-loop system (3.5).

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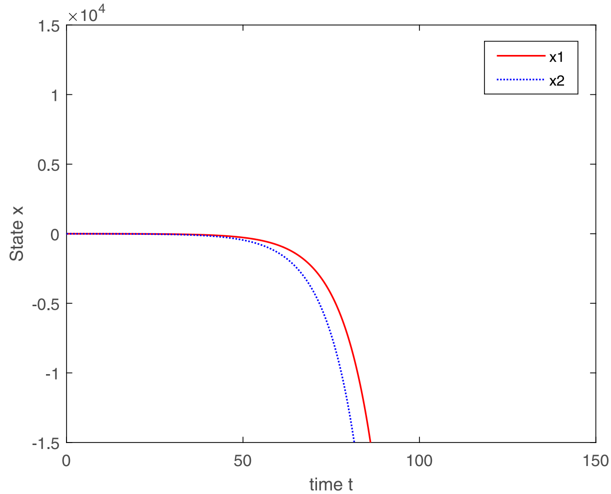

FIGURE 2. State trajectories of the open-loop system with $u(t) = 0$.

Let $S = [3 \quad -1]^T$ and $\beta = 2.5$. Solving the LMI (3.7) in Theorem 3.6 yields

$$X = \begin{bmatrix} 0.2412 & -0.0459 - 0.0603i \\ -0.0459 + 0.0603i & 0.0296 \end{bmatrix},$$

$$Q = \begin{bmatrix} -0.0834 & -0.0112 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 0.1171 & -0.1772 \end{bmatrix}.$$

The pseudostate feedback controller is given by

$$K_1 = \begin{bmatrix} -0.4832 & -1.3695 \end{bmatrix}.$$

Moreover, we know that Theorem 3.8 is a sufficient and necessary condition to guarantee that the closed-loop system is admissible. Thus, solving LMI (3.10), the corresponding feedback gain matrix is presented as

$$K_2 = \begin{bmatrix} -1.2520 & -3.6237 \end{bmatrix}.$$

Comparing the above two gain matrices with that given by Marir et al. [12], we find that the nonsingular limitation of matrix $PE^T + SQ$ in Theorem 3.8 does not have much influence on the conservatism. Therefore, Theorems 3.6 and 3.8 are valid.

Consider the output matrix of system (2.1),

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix},$$

which was given by Ji and Qiu [7, Example 17]. Next, to design an output feedback law for this system, suppose that $\beta = 0.1$. Solving LMI (3.12) yields

$$X = \begin{bmatrix} 1.1033 & -0.2883 - 0.0010i \\ -0.2883 + 0.0010i & 0.3346 \end{bmatrix},$$

$$R = \begin{bmatrix} 2.8265 & -0.5163 \\ -0.5163 & 1.4496 \end{bmatrix}, \quad \text{and} \quad H = \begin{bmatrix} -0.1783 & -0.5348 \end{bmatrix}.$$

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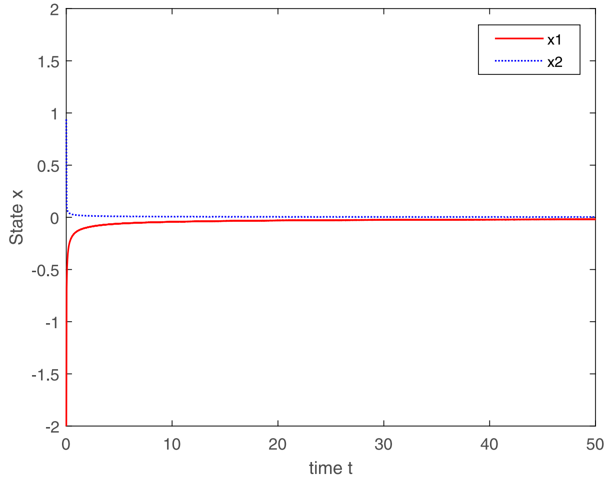

FIGURE 3. State trajectories of the closed-loop system with $u(t) = K_1 x(t)$.

From (3.13), the static output feedback gain matrix is

$$L = \begin{bmatrix} -0.1395 & -0.4186 \end{bmatrix}.$$

Obviously, this result is more optimized than that given by Yu et al. [24] since the magnitude of the obtained output feedback gain matrix is reduced too much. Thus, for SFOSs, condition (3.12) in Theorem 3.12 is more effective than the result of Ji and Qiu [7]. The simulation results of state trajectories of the closed-loop system are shown in Figures 3 and 4, respectively.

EXAMPLE 4.3. One SFOS is considered; the coefficients of the system (2.1) are given by $\alpha = 0.3$ and

$$E = \begin{bmatrix} -10.5 & -7 & 7 \\ -18 & 7.5 & 5.5 \\ 7.5 & -4 & -2 \end{bmatrix}, \quad A = \begin{bmatrix} 31.5 & 0 & -14 \\ 15 & 13 & -11 \\ -4.5 & -6 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2.1 & -0.3 \\ 2.3 & 0.2 \\ 1.5 & 1.3 \end{bmatrix},$$

and $C = [5.3 \quad -2 \quad -2.8]$. Using Definition 2.2, we verify that this system is not regular or impulse free and unstable. Taking $S = [0.8 \quad 0.6 \quad 1.8]^T$ and $\beta = 1.5$, we find that the LMI in (3.7) is feasible, and a set of solution is obtained as

$$X = 10^4 \times \begin{bmatrix} 1.0128 & 0.7570 - 0.0004i & 2.0557 + 0.0001i \\ 0.7570 + 0.0004i & 0.5661 & 1.5435 - 0.0002i \\ 2.0557 - 0.0001i & 1.5435 + 0.0002i & 4.7776 \end{bmatrix},$$

$$Q = 10^7 \times \begin{bmatrix} -1.2125 & -1.4563 & -1.2337 \end{bmatrix},$$

$$\tilde{K} = 10^7 \times \begin{bmatrix} 0.7338 & 0.5504 & 1.6530 \\ 0.2925 & 0.2192 & 0.6547 \end{bmatrix}.$$

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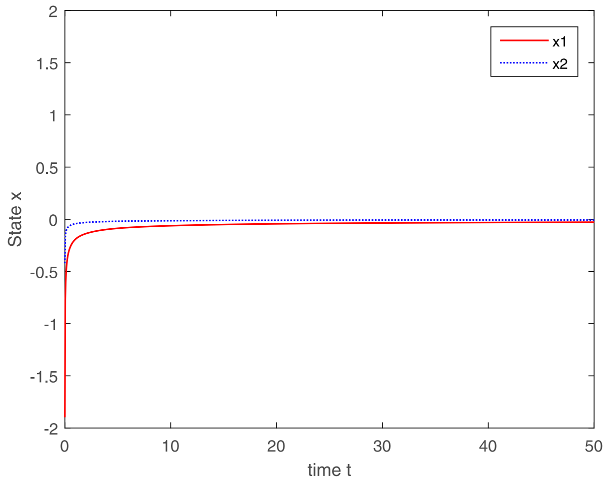

FIGURE 4. State trajectories of the closed-loop system with $u(t) = Ly(t)$.

By Theorem 3.6, we obtain a pseudostate feedback gain matrix

$$K = \begin{bmatrix} -28.1285 & 308.5294 & 293.4556 \\ 320.4891 & -318.7349 & 116.0456 \end{bmatrix}.$$

Similarly, taking $\beta = 0.05$ and using Theorem 3.12 to solve LMI (3.12), the static output feedback control law is obtained as

$$L = \begin{bmatrix} -0.0022 \\ 0.0039 \end{bmatrix}.$$

EXAMPLE 4.4. As in Kaczorek's work [8], consider the following electrical circuit in Figure 5, where R_1, R_2, R_3 are resistances, L_1, L_2, L_3 are inductances and e_1, e_2 represent source voltages. By Kirchhoff's laws, we have the following equations.

$$\begin{aligned} e_1 &= R_1 I_1 + L_1 \frac{d^\alpha I_1}{dt^\alpha} + R_3 I_3 + L_3 \frac{d^\alpha I_3}{dt^\alpha}, \\ e_2 &= R_2 I_2 + L_2 \frac{d^\alpha I_2}{dt^\alpha} + R_3 I_3 + L_3 \frac{d^\alpha I_3}{dt^\alpha}, \\ &I_1 + I_2 - I_3 = 0. \end{aligned} \tag{4.1}$$

Equations (4.1) can be written in the form

$$\begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

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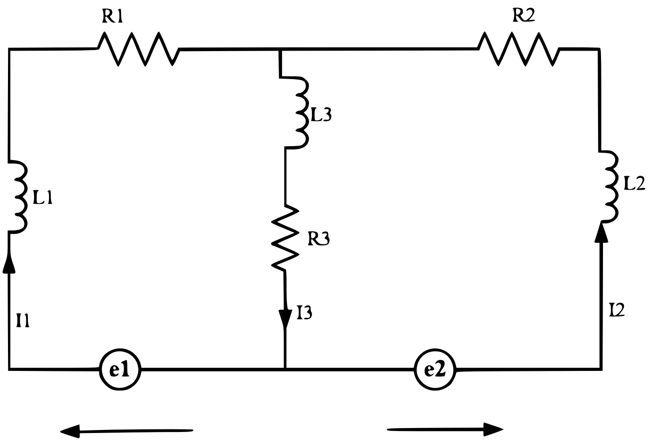

FIGURE 5. Electrical circuit of Example 4.4.

Then a singular fractional-order linear system is given by

$$ED^{\alpha}x(t) = Ax(t) + Bu(t),$$

where

$$E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}, \quad u = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Let $\alpha = 0.3, L_1 = 5, L_2 = 10, L_3 = 2, R_1 = 1, R_2 = 1, R_3 = 2$; we find that this system is unstable. Let $S = [2\ 1\ -5]^T$ and $\beta = 1.5$. Solving LMI (3.7) and LMI (3.10) in Theorems 3.6 and 3.8, respectively, we obtain the pseudostate feedback controllers

$$K_1 = \begin{bmatrix} -6.8748 & -7.1414 & 0.5150 \\ -4.7797 & -3.5045 & 2.6598 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1.7384 & 2.0133 & 6.8024 \\ 1.0051 & 3.9202 & 9.7610 \end{bmatrix}.$$

If we suppose that output matrix $C = [6.3\ -2\ -2]$, then, using Theorem 3.12 and solving LMI (3.12), we obtain an output feedback controller

$$L = \begin{bmatrix} -2.0171 \\ -1.3117 \end{bmatrix}.$$

Setting the initial condition $Ex(0) = [-1.57 - 2.930]^T$, Figure 6 gives the state trajectory of the open-loop system, while Figures 7 and 8 give simulations for the state trajectories of the closed-loop system.

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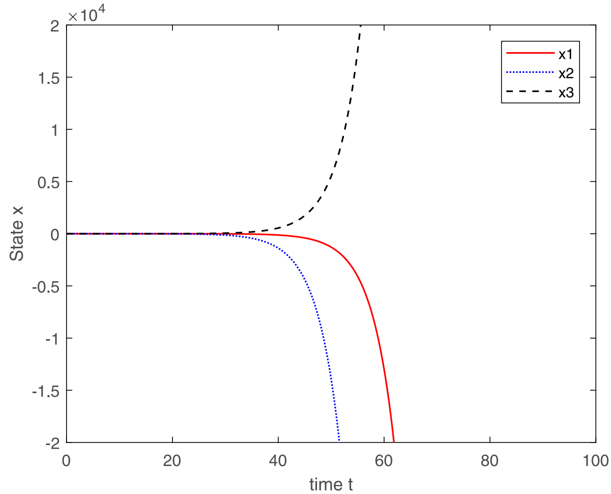

FIGURE 6. State trajectories of the open-loop system with $u(t) = 0$.

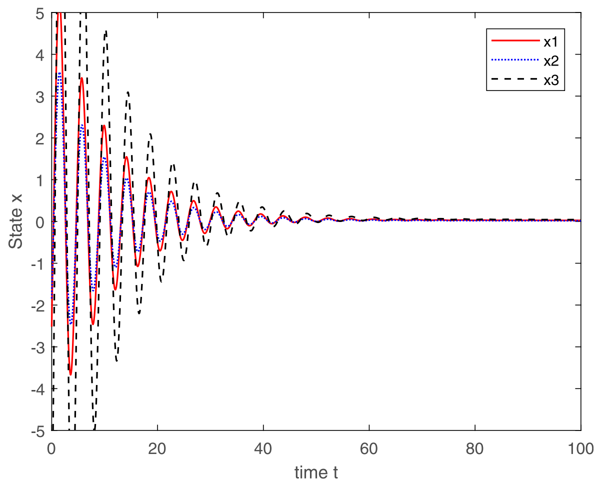

FIGURE 7. State trajectories of the closed-loop system with $u(t) = K_1 x(t)$.

5. Conclusion

In this paper, stabilization problems have been studied, based on a new admissible condition for singular fractional-order control systems with the order $0 < \alpha < 1$. We present sufficient and necessary conditions that guarantee that the closed-loop systems are admissible. Finally, the validity of our approach is illustrated by numerical simulations to make comparisons with some existing results. In the future,

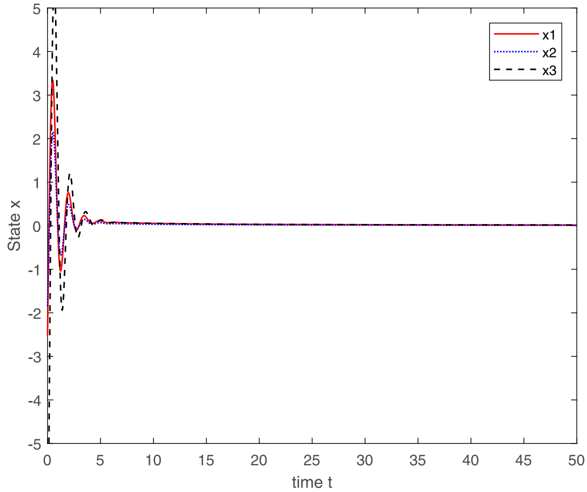

FIGURE 8. State trajectories of the closed-loop system with $u(t) = Ly(t)$.

observer-based control designs for singular fractional-order nonlinear uncertain systems can be investigated.

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