GROUPS WITH FEW NON-NILPOTENT SUBGROUPS

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1. Introduction. Let G be a non-nilpotent group in which all proper subgroups are nilpotent. If G is finite then G is soluble [18], and a classification of such groups is given in [14]. The paper [12] of Newman and Wiegold discusses infinite groups with this property. Clearly such a group is either finitely generated or locally nilpotent. Many interesting results concerning the finitely generated case are established in [12]. Since the publication of that paper there have appeared the examples due to Ol'shanskii and Rips (see [13]) of finitely generated infinite simple p-groups all of whose proper nontrivial subgroups have order p, a prime. Following [12], let us say that a group G is an AN-group if it is locally nilpotent and non-nilpotent with all proper subgroups nilpotent. A complete description is given in Section 4 of [12] of AN-groups having maximal subgroups. Every soluble AN-group has locally cyclic derived factor group and is a p-group for some prime p ([12; Lemma 4.2]). The only further information provided in [12] on AN-groups without maximal subgroups is that they are countable. Four years or so after the publication of [12], there appeared the examples of Heineken and Mohamed [5]: for every prime p there exists a metabelian, non-nilpotent p-group G having all proper subgroups nilpotent and subnormal; further, G has no maximal subgroups and so G/G' is a Prüfer p-group in each case.

One purpose of the present article is to indicate some further properties of AN-groups without maximal subgroups; another is to show that every AN-group is a p-group for some prime p. Also discussed are groups which have certain finiteness conditions with regard to non-nilpotent subgroups, in particular, conditions related to the existence of infinite chains of non-nilpotent subgroups and to conjugacy classes of such subgroups. By collecting all of these results together, it is hoped that this paper constitutes, on the one hand an updating of the survey provided in [12], and on the other hand an indication that, in some cases, a condition which is (apparently) considerably weaker than that of the nilpotency of all proper subgroups is sufficient to ensure nilpotency of the whole group. This is an appropriate juncture at which to thank James Wiegold for pointing out to me many of the properties of AN-groups without maximals that are given in Theorem 3.1 below.

2. Chains of non-nilpotent subgroups. We begin this section by recording the following result.

THEOREM 2.1. *Let G be a torsionfree locally nilpotent group with all proper subgroups nilpotent. Then G is nilpotent.*

This result will be superseded by Theorem 2.2, and for now we remark only that Theorem 2.1 is an easy consequence of Lemma 2 of [10], using a basic result on isolators (referred to below).

Given a property χ pertaining to subgroups, a group G is said to satisfy max-∞ for χ-subgroups if G has no infinite ascending chain $H_1 < H_2 < \ldots$ of χ-subgroups in which all indices $|H_{i+1}:H_i|$ are infinite. The property min-∞ (for χ-subgroups) is defined similarly. The properties max-∞ and min-∞ were first considered by Zaicev [21], and many papers concerned with max-∞ and min-∞ for various properties χ have appeared in recent years (see, for example, [8]). Here we are concerned with the property of being non-nilpotent. Theorem 2.1 presents a special case of the following.

THEOREM 2.2. *Let G be a torsionfree locally nilpotent group which satisfies either max-∞ or min-∞ for non-nilpotent subgroups. Then G is nilpotent.*

Certainly there is no corresponding result for periodic locally nilpotent groups, even for soluble ones, as the Heineken-Mohamed example shows. As will become evident in Section 3, every AN-group has its finite residual of finite index (this is almost, but not quite, shown in [12]). Accordingly, our next result may be of some interest.

THEOREM 2.3. *Let p be a prime and let G be a locally nilpotent p-group, R the finite residual of G. Suppose that G/R is infinite. If G satisfies either max-∞ or min-∞ for non-nilpotent subgroups then G is nilpotent.*

Before turning to the proof of Theorem 2.2, we record some well-known properties of locally nilpotent groups. Suitable references for these are [3], [6; 16.2.8] and [16; 5.4.16].

Let G be a torsionfree locally nilpotent group and H a subgroup of G. For each set π of primes, the π-isolator of H in G, which is the set $\{g \in G : g^n \in H$ for some π-number $n\}$, is a subgroup of G. In the case where π is the set of all primes we refer simply to the isolator of H in G, denoted $I_G(H)$, and H is said to be isolated in G if $I_G(H) = H$. If H is countable then so is $I_G(H)$; this is an easy consequence of the fact that, for $x,y \in G$ and $n \in \mathbb{N}$, $x^n = y^n$ implies $x = y$. If H is nilpotent of class c then so is $I_G(H)$ (and this fact, together with Lemma 2 of [10], suffices to establish Theorem 2.1). Finally, if G is finitely generated and $\{N_i : i = 1, 2, \ldots\}$ is the set of all normal subgroups of finite index in G then

$$H = \bigcap_{i=1}^{\infty} HN_i.$$ These facts will be used in the proof of Lemma 2.4, and this in turn will be used for the proof of Theorem 2.1.

LEMMA 2.4. *Suppose that B is a finitely generated torsionfree nilpotent group, A is a subgroup whose nilpotency class is less than that of B and C is a subgroup of finite index in A. Let p be an arbitrary prime. Then there exists a normal subgroup N of finite index in B such that $NC \cap A = C$ and $|B:NA|$ is divisible by p.*

Proof. Let $\{a_1, \ldots, a_n\}$ be a transversal for C in A such that $a_1 = 1$ and, for $i = 2, \ldots, n$, let D_i be a normal subgroup of finite index in B such that $a_i \notin D_iC$. Set $D = \bigcap_{i=2}^{n} D_i$. Then B/D is finite and $DC \cap A = C$. There exists a subnormal series $A = A_k \lhd \ldots \lhd A_1 \lhd A_0 = B$ from A to B whose factors are abelian and either finite or torsionfree. Since $|B:A|$ is infinite, there is a least integer $j \geq 0$ such that A_j/A_{j+1} is infinite. Define E to be the normal core in B of the subgroup $A_{j+1}A_j^p$. Then B/E is finite

and $|B:EA|$ is divisible by p. It is easily verified that the subgroup $N = D \cap E$ satisfies the conditions stated in the lemma. □

Proof of Theorem 2.2. Assume, for a contradiction, that G satisfies the given hypotheses and that G is not nilpotent. Then G has a countable non-nilpotent subgroup and hence an isolated such subgroup, which we denote by K. Write $K = \bigcup_{i=1}^{\infty} K_i$, where $1 = K_0 < K_1 < K_2 < \ldots$ is a chain of finitely generated subgroups of increasing nilpotency class. Let $\{p_1, p_2, \ldots\}$ be an infinite set of primes. Choose a normal subgroup H_1 of K_1 such that the index $|K_1 : H_1|$ is finite and divisible by p_1. Now let N_2 be a normal subgroup of finite index in K_2 such that $|K_2 : N_2 K_1|$ is divisible by p_2 and $N_2 H_1 \cap K_1 = H_1$. Write $H_2 = N_2 H_1$. Inductively, having defined N_i and H_i for some $i \geq 2$, let N_{i+1} be a normal subgroup of finite index in K_{i+1} such that $|K_{i+1} : N_{i+1} K_i|$ is divisible by p_{i+1} and $N_{i+1} H_i \cap K_i = H_i$ and write $H_{i+1} = N_{i+1} H_i$. We obtain an infinite chain $H_1 \leq H_2 \leq \ldots$ such that, in particular, $|K_i : H_i|$ is finite for each i. Thus, setting $H = \bigcup_{i=1}^{\infty} H_i$, we have $I_K(H) = K$ and hence H non-nilpotent. Now define $L_0 = H$ and, for each $i \geq 1$, $L_i = \langle H, K_i \rangle$. Thus $K = \bigcup_{i=1}^{\infty} L_i$. We shall establish the following facts.

(1) For each $i \geq 1$, $H \cap K_i = H_i$.

(2) $|L_1 : L_0| = |K_1 : H_1|$ and, for $i \geq 1$, $|L_{i+1} : L_i| = |K_{i+1} : N_{i+1} K_i|$.

To prove (1) we need to show that, for arbitrary i, $H_j \cap K_i \leq H_i$ for all j. Certainly this is true for all $k \leq i$; assume as inductive hypothesis that $H_t \cap K_i \leq H_i$ for some $t \geq i$. Then $H_{t+1} \cap K_i = N_{t+1} H_t \cap K_i = N_{t+1} H_t \cap K_t \cap K_i = H_t \cap K_i$, and the result follows. Next, we have $L_1 = \langle H, K_1 \rangle = \langle H_1, N_2, N_3, \ldots; K_1 \rangle = M K_1$, where $M = \langle N_2, N_3, \ldots \rangle$ (which is normalised by K_1). Similarly, $L_0 = M H_1$, and so $|L_1 : L_0| = |M K_1 : M H_1| = |K_1 : K_1 \cap M H_1| = |K_1 : H_1 (K_1 \cap M)|$. But $K_1 \cap M \leq K_1 \cap H = H_1$, giving $|L_1 : L_0| = |K_1 : H_1|$. For $i \geq 1$ we have $L_i = \langle H_i, N_{i+1}, N_{i+2}, \ldots; K_i \rangle = N K_i$; where $N = \langle N_{i+1}, N_{i+2}, \ldots \rangle$. Similarly, $L_{i+1} = N K_{i+1}$, and so $|L_{i+1} : L_i| = |K_{i+1} : K_{i+1} \cap N K_i| = |K_{i+1} : K_i (K_{i+1} \cap N)|$. But $K_{i+1} \cap N \leq K_{i+1} \cap H = H_{i+1}$, by (1), and so $K_i (K_{i+1} \cap N) \leq K_i N_{i+1}$. Since the reverse inclusion certainly holds, (2) now follows.

From the choice of the subgroups N_i, we see that $|L_{i+1} : L_i|$ is divisible by p_{i+1} for all $i \geq 0$. In particular, the chain $L_0 \leq L_1 \leq \ldots$ is not a finite one. We now obtain a similar chain where the indices are all infinite. Let π denote the set of all primes p which divide at least one of the indices $|L_{i+1} : L_i|$. Then π is infinite and we may write it as a disjoint union of infinitely many infinite subsets $\{T_i\}_{i=1}^{\infty}$. Let I_1 denote the π_1-isolator of H in K and, for $n \geq 1$, let I_{n+1} denote the π_{n+1}-isolator of I_n in K. Then each of the indices $|I_{n+1} : I_n|$ is infinite (as is the index of H in I_1). Thus G does not satisfy max-∞ for non-nilpotent subgroups. On the other hand, if J_1 denotes the π_1'-isolator of H in K and, for $n \geq 1$, J_{n+1} denotes the π_{n+1}'-isolator of H in J_n, then each of the indices $|J_n : J_{n+1}|$ is infinite and G does not satisfy min-∞ for non-nilpotent subgroups. We thus have a contradiction, and the theorem is proved. □

Proof of Theorem 2.3. Firstly, let X be an arbitrary locally finite group which is infinite and residually finite, and let F_1 be an arbitrary nontrivial finite subgroup of X. There is a normal subgroup N_1 of finite index in X such that $F_1 \cap N_1 = 1$. Write $U_1 = F_1$, choose an arbitrary nontrivial finite subgroup F_2 of N_1, and set $U_2 = \langle F_1, F_2 \rangle$. Now choose

an X-invariant subgroup N_2 of finite index in N_1 such that $U_2 \cap N_2 = 1$. Continuing in the obvious manner, we obtain an infinite descending chain $X = N_0 > N_1 > N_2 > \ldots$ of normal subgroups of finite index in X and an infinite ascending chain $U_1 < U_2 < U_3 < \ldots$ of finite subgroups of X such that $U_i \cap N_i = 1$ for all i. Further, for each i we have $U_{i+1} = \langle U_i, F_{i+1} \rangle$, where F_{i+1} is a (nontrivial, finite) subgroup of N_i. Set $H = \bigcup_{i=1}^{\infty} U_i$. Next, let M_1, M_2 be disjoint subsets of $\mathbb{N}$, not both empty, and, for $i = 1, 2$, let $B_i = \langle F_j : j \in M_i \rangle$. Suppose first that M_1 and M_2 are both finite and, without loss of generality, that the maximum element k, say, of M_2 is greater than that of M_1 (if M_1 is non-empty). Write $B_2 = \langle C, F_k \rangle$, where $C = \langle F_j : j \in M_2, j < k \rangle$ (=1 if $M_2 = \{k\}$), and suppose that $B_1 \cap B_2 \neq 1$. Since $F_k \leq N_{k-1}$ and $\langle B_1, C \rangle \cap N_{k-1} = 1$, we have $B_1 \cap B_2 = B_1 \cap C$. An easy induction on $|M_1| + |M_2|$ now gives a contradiction which establishes that $B_1 \cap B_2 = 1$. It follows easily that $B_1 \cap B_2$ is also trivial in the case where at least one of M_1, M_2 is infinite. Now let $\emptyset = S_0 \subset S_1 \subset S_2 \ldots$, $\mathbb{N} = T_0 \supset T_1 \supset T_2 \supset \ldots$ be (infinite) chains of subsets of $\mathbb{N}$ such that $S_i \setminus S_{i-1}$ and $T_{i-1} \setminus T_i$ are infinite for each i and $\bigcup_{i=1}^{\infty} S_i = \mathbb{N}$. It is now clear how to construct an infinite ascending chain of subgroups $K_0 < K_1 < \ldots$, with union H, such that K_0 is infinite and each of the indices $|K_i : K_{i-1}|$ is infinite, and a descending chain $H = H_0 > H_1 > \ldots$ such that $|H_{i-1} : H_i|$ is infinite.

Let G be a group satisfying the hypotheses of the theorem and suppose, for a contradiction, that G is not nilpotent. If G has a normal nilpotent subgroup N of finite index then there is a finite subgroup F of G such that $G = FN$. Thus FL is non-nilpotent for every normal subgroup L of finite index in G (else F acts nilpotently on $N \cap L$ and hence on N, a contradiction). Certainly there exists a finite subgroup F with this latter property if G is not nilpotent-by-finite. In either case, with $X = G/R$, the subgroups F_i in the above construction may be chosen such that $F_1 = FR/R$ and such that the pre-image in G of $\langle F_1, F_{i+1} \rangle$ is either non-nilpotent or of nilpotency class which increases with i. By arranging for all of the subgroups H_i, K_i in our infinite chains to contain F_1, we obtain the required contradiction. $\square$

3. Groups with all proper subgroups nilpotent.

THEOREM 3.1. *Let G be a soluble non-nilpotent group with all proper subgroups nilpotent, and suppose that G has no maximal subgroups. Then:*
  (i) *G is a countable p-group for some prime p and $G/G' \cong C_{p^\infty}$.*
  (ii) *Every subgroup of G is subnormal.*
  (iii) *$(G')^p \neq G'$, and every hypercentral image of G is abelian. In particular, $G' = \gamma_n(G)$ for all $n \geq 2$.*
  (iv) *Every radicable subgroup of G is central.*
  (v) *The centraliser of G' is abelian, and G' is omissible (that is, $HG' = G$ implies $H = G$). In particular, G has no proper subgroups of finite index.*
  (vi) *G' is not the normal closure in G of a finite subgroup.*
  (vii) *The hypercentre of G coincides with the centre.*

There is just one auxiliary result that we require, namely the following, which is well-known and is easily proved by induction on the subnormal defect of H and application of Lemma 3.13 of [15].

LEMMA 3.2. *Let G be a nilpotent p-group and let H be a radicable abelian subgroup of G. Then H is central in G.*

Proof of Theorem 3.1. (i) Since G has no maximal subgroups it is certainly not finitely generated, and we may apply Lemmas 4.1 and 4.2 of [12].

(ii) Suppose that G has a non-subnormal subgroup H. Since $HG' \lhd G$ we have HG' non-nilpotent and hence equal to G. Let $K = H \cap G'$, so that K is normal in H. If $G = HK^G$ then $K^G = K^{K^G}$. Since $K^G \leq G'$ we have K^G nilpotent and hence K sn K^G, and it follows that $K = K^G$. This gives the contradiction $G = H$, and so HK^G is properly contained in G and is therefore nilpotent. But now HK^G is also non-subnormal, and we may assume that $K^G \leq H$. Factoring by K^G, we may further assume that $H \cap G' = 1$ and hence that $H \cong C_p^\infty$. Let $g \in G$. Then either $\langle g, H \rangle$ is nilpotent or $\langle g, H \rangle = G$. Since $\langle g, H \rangle$ nilpotent implies $[H, g] = 1$, by Lemma 3.2, we see that $\langle g, H \rangle = G$ for all $g \notin C_G(H)$. Since G has no maximal subgroups we deduce that $G = G_G(H)$. This gives the contradiction $H \lhd G$.

(iii) From (ii) and Lemma 1 and Corollary 2 of [5], $(G')^p \neq G'$ and every hypercentral image is nilpotent. The result follows by Corollary 2.11 of [12].

(iv) Let H be a radicable subgroup of G, and suppose first that $H \neq G$. Then H^G is nilpotent and $H^G/(H^G)'$ is abelian and generated by radicable subgroups and is therefore radicable. Thus H^G is abelian [15; 9.23] and therefore central, by Lemma 3.2. It remains only to show that G is not radicable. Assuming otherwise, $G/(G')^p$ is radical and non-abelian, by (iii), and we may assume that $(G')^p = 1$. Write $A = G'$; then every element of A is the pth power of some element of G and hence of some element of $A\langle y \rangle = H$, say, where y has order p mod G'. Now A/H' is contained in $(H/H')^p$, which is finite, so H/H' is finite. Since H is nilpotent, it too is finite and so G' is finite and hence (since G is locally nilpotent) G is hypercentral, contradicting (iii). Thus G is not radicable.

(v) If $x, y \in C_G(G')$ then, since G/G' is locally cyclic, we may assume without loss of generality that $y = x^n g$ for some $g \in G'$ and $n \in \mathbb{N}$. Thus $[x, y] = 1$ and $C_G(G')$ is abelian. We may apply Lemma 2(b) of [5] to the (non-abelian) group $G/(G')^p$ to deduce that G' is omissible.

(vi) If $G' = H^G$ for some finite subgroup H then $G'/[H, G]$ is finite and so $G/[H, G]$ is hypercentral and hence abelian, by (iii). Thus $G' = [H, G]$, and there is a finite subgroup K of G such that $H \leq [H, K]$. But then $H < [H, {}_r K] = 1$, for some integer r (since $\langle H, K \rangle$ is nilpotent). This is a contradiction.

(vii) Let x be an element of $Z_2(G)$ and suppose that x has order p^n mod $Z(G)$. By considering the map $g \to [g, x]$ for all $g \in G$, we see that $[x, G]$ is an image of C_{p^∞} and has finite exponent. Thus x is central and the result follows. $\square$

With regard to (vii) above, we note that Bruno and Phillips [1] have shown that it is possible for the centre of G to be nontrivial. We also remark that there is no bound for the derived length of a group G satisfying the hypotheses of Theorem 3.1, for Menegazzo [9] has constructed such "Heineken–Mohamed groups" of arbitrary (finite) derived length at least two. In the same paper, he has constructed metabelian Heineken–Mohamed groups whose derived groups have infinite exponent.

Not much appears to be known about insoluble AN-groups, and this state of affairs is not about to be remedied by the present article. The following result indicates just a few of the properties that such a group must have (if, indeed, such a group exists).

THEOREM 3.3. *Suppose that G is an insoluble (and hence perfect) locally nilpotent group with all proper subgroups nilpotent. Then*:
  (i) *G is a p-group for some prime p.*
  (ii) *G is a Fitting group and satisfies the normaliser condition.*
  (iii) *There is a nilpotent subgroup H of G such that $H^G = G$.*
  (iv) *Every proper radicable subgroup of G is central, and $Z(G)$ is the hypercentre of G.*

Proof. (i) Let T denote the torsion subgroup of G. By Theorem 2.1, G/T is nilpotent and hence trivial. If G is not a p-group then every primary component of G is nilpotent and we have the contradiction that G is not perfect.

(ii) Let F denote the Fitting radical of G. If $F \neq G$ then F is nilpotent and G/F has no nontrivial normal subgroups. But then G/F has order p, a contradiction. Now let H be an arbitrary proper subgroup of G. Since G has no maximal subgroups, H is properly contained in a nilpotent subgroup K, and of course $H < N_K(H)$, so (ii) is proved.

(iii) Since G is insoluble it has a non-subnormal subgroup H, by [11]. Then H^G is not nilpotent and hence equals G.

(iv) Let A be a proper radicable subgroup of G and write $C = C_G(A)$. Since A is abelian, by Theorem 9.23 of [15], we have $A \leqslant C$. If $C \neq G$ then C is nilpotent and, for each $g \in G \setminus C$, $\langle g, A \rangle$ is non-abelian and hence non-nilpotent, by Lemma 3.2. Thus $g \notin C$ implies $\langle g, A \rangle = G$, giving the contradiction that C is a maximal subgroup of G. Finally, in every perfect group the centre is the hypercentre, and this concludes the proof of the theorem. $\qquad \square$

Note that the (Chernikov) AN-groups of Section 4 of [12] are not Fitting groups; indeed, they are not even Baer groups. Note also that every AN-group has now been shown to be a p-group for some prime p.

4. Conjugacy classes of non-nilpotent subgroups.
The main aim of this section is to establish the following.

THEOREM 4.1. *Let G be an infinite, locally graded group and suppose that the set of non-nilpotent subgroups of G is a union of finitely many conjugacy classes. Then G is locally nilpotent and has only finitely many non-nilpotent subgroups.*

We recall that a group G is locally graded if every finitely generated nontrivial subgroup of G has a finite nontrivial image. If we replace the property of being non-nilpotent by that of being non-(nilpotent of class at most c), where c is a fixed positive integer, then G is in fact nilpotent of class at most c; this is proved in [19]. Certainly we cannot conclude from the hypotheses of Theorem 4.1 that G has all proper subgroups nilpotent, as may be seen by considering the direct product of a Heineken–Mohamed p-group and a finite nilpotent p'-group. Proposition 4.8 below says a little more about the structure of locally nilpotent groups having finitely many non-nilpotent subgroups.

For torsionfree groups we may assert the following.

THEOREM 4.2. *Let G be a torsionfree locally nilpotent group which is not nilpotent. Then G contains $2^{\aleph_0}$ pairwise non-conjugate non-nilpotent subgroups.*

Proof. With the notation as in the proof of Theorem 2.2, let $\{\sigma_\lambda : \lambda \in \Lambda\}$ be the set of

all subsets of π and, for each λ, let σ'_λ denote the complement of σ_λ in π, J_λ the σ_λ-isolator of H in K. If $J^g_\lambda = J_\mu$ for some λ, $\mu \in \Lambda$ and $g \in G$ then, since K is the isolator in G of each of J_λ and J_μ, we have $K^g = K$. But then K is both the σ'_λ-isolator and the σ'_μ-isolator of J_μ in G, so that $\sigma_\lambda = \sigma_\mu$ and $\lambda = \mu$. Since $|\Lambda| = 2^{\aleph_0}$, the result follows. $\square$

There is a similar result for residually finite p-groups.

THEOREM 4.3. *Let G be a locally nilpotent p-group for some prime p, and let R denote the finite residual of G. If G/R is infinite and G is not nilpotent then G contains $2^{\aleph_0}$ pairwise non-conjugate non-nilpotent subgroups.*

Proof. Let X and the finite subgroups F_i be as in the proof of Theorem 2.3. From the details of that proof, we see that it suffices here to establish the following claim.

Claim. If S and T are distinct subsets of $\mathbb{N}$, $Y = \langle F_j : j \in S \rangle$ and $Z = \langle F_j : j \in T \rangle$, then Y and Z are not conjugate in X.

To prove this, we may assume without loss of generality that there is an integer k in S but not in T. If $Y^x = Z$ for some $x \in X$ then (again with the notation as before) we have $(YN_k)^x = ZN_k$. But then we have $F^x_k \leqslant (YN_k \cap N_{k-1})^x = ZN_k \cap N_{k-1} \leqslant U_{k-1}N_k \cap N_{k-1} = N_k(U_{k-1} \cap N_{k-1}) = N_k$, which gives the contradiction $F_k \leqslant N_k$. This establishes the claim and hence concludes the proof of the theorem. $\square$

Theorem 4.3 has the following easy consequence.

COROLLARY 4.4. *Let G be a locally nilpotent group and R the finite residual of G. Suppose that G is periodic and G/R is infinite. If G is not nilpotent then it has $2^{\aleph_0}$ pairwise non-conjugate non-nilpotent subgroups.*

Proof. If G has nontrivial p-components for infinitely many primes p then there is no bound for the nilpotency class of these p-components, and so there is a non-nilpotent normal subgroup N such that G/N involves infinitely many primes and therefore has $2^{\aleph_0}$ pairwise non-conjugate subgroups. If the number of primary components is finite then at least one such, P, say, is infinite modulo its finite residual. Write $G = P \times Q$. If P satisfies the condition on conjugacy classes then so does G. Otherwise, by Theorem 4.3, P is nilpotent. But then Q is non-nilpotent and, as in the proof of Theorem 4.3, P (and hence G/Q) has $2^{\aleph_0}$ pairwise non-conjugate subgroups. The result follows. $\square$

Note that we cannot remove from the statement of this corollary the condition that G is periodic, as is shown, for example, by the group $G = A]\langle x \rangle$, where $A \cong C_{2^\infty}$, $\langle x \rangle$ is infinite cyclic and $a^x = a^{-1}$ for all $a \in A$.

For the proof of Theorem 4.1, let us denote by N^* the given conjugacy class property on non-nilpotent subgroups. The first step is to establish that G is locally (soluble-by-finite), and this we now proceed to do. Much of the proof here is similar to that of Proposition 1 of [19], although there are one or two amendments required.

LEMMA 4.5. *Let G be an infinite locally graded group with the property N^*. Then G is locally (soluble-by-finite).*

Proof. Suppose the result false and let F be a finitely generated subgroup of G which is not soluble-by-finite. Let R be the finite residual of F; then F/R is infinite. Let K/R be an arbitrary normal subgroup of finite index in F/R. Every subgroup S of F which

contains K is finitely generated and non-nilpotent, and so S is at most r-generated for some integer r depending only on G. Thus F/K has rank at most r. By Theorem 6.10 of [2], F/R has finite rank and is soluble-by-finite. Let H/R be the soluble radical of F/R. Then H/R is finitely generated and infinite and therefore non-periodic. Assume first that F is not even soluble-by-finite modulo its Hirsch–Plotkin radical and let H/N be an arbitrary soluble image of H, L/N a finitely generated subgroup of H/N. We may write $L = MN$, where M is finitely generated. Since N is not locally nilpotent, M may be chosen non-nilpotent. Hence L/N is at most r-generated for all such L and thus H/N has rank at most r. By Lemma 8 of [17], we may choose N so that the Hirsch length of H/N is maximal (subject to H/N being soluble). Then N is insoluble and N/N' is periodic. We may now proceed exactly as in the proof of Proposition 1 of [19] to construct, for each prime p, a subgroup Y_p of G containing N such that Y_p/N is isomorphic to the additive group of p-adic rationals, and to use the conjugacy class property to obtain a contradiction. Thus, if K denotes the Hirsch–Plotkin radical of F then F/K is soluble-by-finite. Let J/K be the soluble radical of F/K, so that F/J is finite. Write U/K for the Fitting radical of J/K. Every finitely generated subgroup of U/K is subnormal and hence, for each $u \in U \setminus K$, $\langle u \rangle K$ is not locally nilpotent (otherwise $\langle x \rangle K$ is contained in the Hirsch–Plotkin radical of U, which is precisely K). Thus K is the Hirsch–Plotkin radical of every subgroup of U which contains K. Since the set of all such subgroups is a union of finitely many conjugacy classes, there are only finitely many isomorphism types of subgroups of U/K. If U/K is periodic it is therefore finite. Then the centraliser of U/K in J/K has finite index and, since this centraliser is contained in U/K (see Lemma 2.17 of [15]), we deduce that J/K is finite. But this gives F/K finite and hence K finitely generated nilpotent, a contradiction. Hence may choose an element x of U which has infinite order modulo K. Let p be a prime and write $X_i = \langle x^{p^i} \rangle K$, for $i = 1, 2, \ldots$. For each i, X_i is non-nilpotent, and so there are integers m, n with $m < n$ such that, for some $g \in G$, $x_n^g = X_m$. Also, K is the Hirsch–Plotkin radical of each X_i and so $K^g = K$. Arguing as in the proof of Proposition 1 of [19], we obtain a subgroup Y_p of G such that $K \leq Y_p$, $x^{p^k} \in Y_p$ for some $k \geq 0$, and Y_p/K is isomorphic to the additive group of p-adic rationals. There exist distinct primes p and q and an element h of G such that $Y_p^h = Y_q$. But $K^h = K$, and we have the contradiction $Y_p/K \cong Y_q/K$. This concludes the proof of the lemma. □

LEMMA 4.6. *Let G be an infinite group with the property N^*. If G is locally soluble-by-finite then G is locally nilpotent.*

Proof. Suppose that G is a counterexample to the statement of the lemma. Then G has a local system of finitely generated non-nilpotent subgroups, each of which is therefore (soluble of bounded derived length)-by-(finite of bounded order), by the N^*-property. Thus G is soluble-by-finite, by Proposition 1.K.2 of [7]. By induction on the derived length of the soluble radical, we may assume that G is either abelian-by-finite or abelian-by-locally nilpotent. In the case where G is abelian-by-finite, let r be a positive integer such that every finitely generated non-nilpotent subgroup is r-generated, and let A be a normal abelian subgroup of finite index t, say, in G. If F is an arbitrary finitely generated subgroup of G then F embeds in an r-generated subgroup F_1, and $F_1 \cap A$ is at most s-generated for some $s = s(r, t)$. Since A is abelian, $F \cap A$ is also s-generated, F is $(s + t)$-generated, and G has finite rank. There is an s-generated, G-invariant subgroup B of A such that G/B is periodic. If G/B is finite then, arguing as in the proof of Theorem

A of [19], we may assume that B is torsion free and deduce that every finite image of G is nilpotent. But then, for each prime p, $B/B^p \le Z_s(G/B^p)$, that is, $[B, {}_sG] \le B^p$. Since the intersection of all the subgroups B^p is trivial, we have the contradiction that G is nilpotent. So G/B is infinite and locally finite. This implies that G/B is locally nilpotent (otherwise, by the N^*-property, there would be a finite subgroup U of G/B such that all finite subgroups containing U were of bounded order). So we may assume from now on that G is abelian-by-locally nilpotent. Let H be the Hirsch–Plotkin radical of G. Every normal subgroup of G properly containing H is non-nilpotent and has Hirsch–Plotkin radical H, and so there are only finitely many such subgroups. Since G/H is also locally nilpotent, every chief factor is finite and it follows tht G/H is finite. Let F be a finitely generated non-nilpotent subgroup of G, and suppose that there exists an infinite ascending chain $1 < H_1 < H_2 < \ldots$ of G-invariant subgroups of H. By the N^*-property, there is an integer k such that infinitely many of the subgroups FH_i are conjugate to FH_k.

Since $H_i \lhd G$, we deduce that $H_i \le FH_k$ for all such i. Writing $K = \bigcup_{j=1}^{\infty} H_j$, we thus have $K \le FH_k$ and hence $K = H_k(F \cap K)$. Since $F \cap H$ is finitely generated nilpotent, $F \cap K$ is finitely generated, and so $K = H_l$ for some l, a contradiction which shows that H satisfies max-G. Thus G satisfies max-n and so H satisfies max-n [20] and is therefore finitely generated nilpotent [15; 5.37]. Since H is infinite, so is H/H'. But G/H' is abelian-by-finite and therefore, by our previous argument, locally nilpotent and hence nilpotent. Theorem 7 of [4] now gives the contradiction that G is nilpotent, and the lemma is proved. □

Our final requirement is the following result, which is surely well-known.

LEMMA 4.7. *Let h be a subgroup and x an element of the (arbitrary) locally nilpotent group G. If $H^x \le H$ then $H^x = H$.*

Proof. Let $a \in H$ and write $F = \langle a, x \rangle$. Then F is nilpotent and satisfies max; in particular, $L = \langle a \rangle^{\langle x \rangle}$ is finitely generated, and there are integers i, j with $i < j$ such that $L = \langle a^{x^k} : i \le k \le j \rangle$. We have $L = L^{x^{-i}} = \langle a^{x^l} : 0 \le l \le j - 1 \rangle \le \langle H^{x^l} : l \ge 0 \rangle \le H$, giving $H^{\langle x \rangle} \le H$ and hence $H^{\langle x \rangle} = H$, as required. □

Proof of Theorem 4.1. By Lemmas 4.5 and 4.6, a group G satisfying the hypotheses of the theorem is locally nilpotent. From the N^*-property and Lemma 4.7, we see that G has no infinite (proper) chains of non-nilpotent subgroups. Hence, if G has a proper non-nilpotent subgroup H, then H is contained in a maximal subgroup H_1, say, of G. If $H \ne H_1$ then H is contained in a maximal subgroup H_2 of H_1. Continuing in this way, we have $H = H_i$ for some integer i. Since a maximal subgroup of a locally nilpotent group has prime index, we see that $|G:H|$ is finite and hence that H has only finitely many conjugates in G. The result follows. □

We conclude with a few words about groups which are locally nilpotent and have finitely many non-nilpotent subgroups. The argument of the previous proof shows that a non-nilpotent group G with these properties has a minimal non-nilpotent subgroup H of finite index. As remarked in Section 3 above, H is a p-group for some prime p, and so G is the direct product of a p-group and a finite p'-group. Let us consider, then, the case where G is itself a p-group. There are three possibilities (see Section 3):

(i) H is a Chernikov group;

(ii) H is a "Heineken–Mohamed group";

(iii) H is insoluble, and a Fitting group.

In cases (ii) and (iii) H is normal in G since H has no proper subgroups of finite index, and then $G = HF$ for some finite subgroup F. In case (ii), every proper subgroup U of H which contains H' is normalised by F and FU is nilpotent. Beyond that, we have nothing further to say about cases (ii) and (iii); (for case (iii), we do not even have much idea as to what H looks like). Case (i) is far clearer. Here H has a G-invariant radicable abelian subgroup A of finite index and G/A is of course finite also, and $G = AF$ for some finite subgroup F. Let C be the centraliser of A in F; then C is normal in G. If $C \le D \le F$ and $[A, D] \le C$ then $[A, {}_rD] = 1$ for some integer r, and so AD is nilpotent and hence abelian, by Lemma 3.2, and we get $D = C$. This shows that $C_{F/C}(AC/C) = 1$. Since our description of the groups G in which we are interested will be in terms of the action of F on A, we shall assume henceforth that $C = 1$. In particular, G is now a semi-direct product $A]F$. Let g be an arbitrary nontrivial element of F, and suppose that g has order p^n. Write $h = g^{p^{n-1}}$, $K = A\langle h \rangle$. If B is an $\langle h \rangle$-invariant subgroup of A and $B\langle h \rangle$ is non-nilpotent then, by our hypothesis, A/B has only finitely many $\langle h \rangle$-invariant subgroups and is therefore trivial. If U is a non-nilpotent subgroup of K then U contains an element ah for some $a \in A$, and we see that $U \cap A$ is normal in K and hence equals A. Thus K is minimal non-nilpotent and, by [12; 4.6], every proper $\langle h \rangle$-invariant subgroup of A is finite. Write $L = A\langle g \rangle$. We now show that every non-nilpotent subgroup of L contains A. By induction on n we may assume that every non-nilpotent subgroup of N contains A, where $N = A\langle g^p \rangle$. If $S < L$ and $S \nleq N$ then $L = SN$. But clearly N is the Frattini subgroup of L and so $L = SA$. Thus we have $S \cap A \lhd L$ and hence $S \cap A$ finite, which gives S finite and hence nilpotent. Thus every proper non-nilpotent subgroup of L is contained in N and hence contains A, as required. It now follows easily that every non-nilpotent subgroup of G contains A. Observing that this final part of the argument uses only the fact that every proper $\langle g \rangle$-invariant subgroup of A is finite for all $g \in F \setminus \{1\}$, we are now able to state the following result which, when viewed in conjunction with the remarks of the preceding paragraphs, provides us to some extent with a classification of the Chernikov groups which have (only) finitely many non-nilpotent subgroups.

PROPOSITION 4.8. *Let p be a prime and suppose that $G = A]F$ is a p-group, for some nontrivial, radicable, abelian normal subgroup A and finite subgroup F satisfying $C_F(A) = 1$. Then G has finitely many non-nilpotent subgroups if and only if, for every nontrivial element g of F, every proper $\langle g \rangle$-invariant subgroup of A is finite. The non-nilpotent subgroups of G are then precisely those subgroups which properly contain A.*

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