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WEAK CONVERGENCE OF THE EXTREMES OF BRANCHING LÉVY PROCESSES WITH REGULARLY VARYING TAILS

YAN-XIA REN,* *Peking University*
RENMING SONG,** *University of Illinois Urbana-Champaign*
RUI ZHANG,*** *Capital Normal University*

Abstract

We study the weak convergence of the extremes of supercritical branching Lévy processes $\{\mathbb{X}_t, t \geq 0\}$ whose spatial motions are Lévy processes with regularly varying tails. The result is drastically different from the case of branching Brownian motions. We prove that, when properly renormalized, $\mathbb{X}_t$ converges weakly. As a consequence, we obtain a limit theorem for the order statistics of $\mathbb{X}_t$.

Keywords: Branching Lévy process; extremal process; regularly varying; rightmost position

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1. Introduction

We consider a supercritical branching Lévy process. At time 0, we start with a single particle which moves according to a Lévy process $\{(\xi_t)_{t\geq 0}, \mathrm{P}\}$ with Lévy exponent $\psi(\theta) = \log \mathrm{E}\big(\mathrm{e}^{\mathrm{i}\theta\xi_1}\big)$. The lifetime of each particle is exponentially distributed with parameter β, then it splits into k new particles with probability p_k, $k \geq 0$. Once born, each particle will evolve independently, from its parent's place of death, according to the same law as its parent, i.e. move according to the same Lévy process, and branch with the same branching rate and offspring distribution. We use $\mathbb{P}$ to denote the law of the branching Lévy process. Expectations with respect to $\mathbb{P}$ and P will be denoted by $\mathbb{E}$ and E respectively.

In this paper, we use ':=' to denote a definition. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$. We will label each particle using the classical Ulam–Harris system. We write $\mathbb{T}$ for the set of all the particles in the tree, and o for the root of the tree. We also use the following notation:

- For any $u \in \mathbb{T}$, I_u^0 denotes set of all the ancestors of u, $I_u := I_u^0 \cup \{u\}$, and n^u is the number of particles in $I_u \setminus \{o\}$.

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* Postal address: LMAM School of Mathematical Sciences and Center for Statistical Science, Peking University, Beijing, 100871, P. R. China. Email: yxren@math.pku.edu.cn
** Postal address: Department of Mathematics, University of Illinois Urbana-Champaign, Urbana, IL 61801, USA. Email: rsong@illinois.edu
*** Postal address: School of Mathematical Sciences & Academy for Multidisciplinary Studies, Capital Normal University, Beijing, 100048, P.R. China. (Corresponding Author). Email: zhangrui27@cnu.edu.cn

- For any $u \in \mathbb{T}$, τ_u is the life length of u. Then $\{\tau_u, u \in \mathbb{T}\}$ are independent and identically distributed (i.i.d.), and exponentially distributed with parameter β. Let b_u and σ_u be the birth and death times of u respectively. It is clear that $b_u = \sum_{v \in I_u^0} \tau_v$ and $\sigma_u = b_u + \tau_u$. For any $t \geq 0$, let $\mathcal{F}_t^{\mathbb{T}} := \sigma\{b_u \wedge t, \sigma_u \wedge t : u \in \mathbb{T}\}$.
- For any $t \geq 0$, let $\mathcal{L}_t$ be the set of all particles alive at time t.
- Let $\{(X_s^u)_{s \geq 0}, u \in \mathbb{T}\}$ be i.i.d. with the same law as $\{(\xi_s)_{s \geq 0}, \mathrm{P}\}$ and also independent of $\{\tau_u, u \in \mathbb{T}\}$.
- For $u \in \mathcal{L}_t$, let ξ_t^u be the position of u at time t. Then, for $t \in [0, \sigma_o]$, $\xi_t^o = X_t^o$ and, for any other $u \in \mathbb{T}$,

$$\xi_t^u = \xi_{\sigma_{\pi(u)}}^{\pi(u)} + X_{t-b_u}^u = \sum_{v \in I_u^0} X_{\tau_v}^v + X_{t-b_u}^u, \qquad t \in [b_u, \sigma_u], \tag{1.1}$$

  where $\pi(u)$ denotes the parent of u.
- For $t \geq 0$, $v \in \mathcal{L}_t$, and $u \in I_v$, we set $X_{u,t} := \xi_{\sigma_u \wedge t}^u - \xi_{b_u \wedge t}^u$. Note that $X_{v,t} = X_{t-b_v}^v$ and $X_{u,t} = X_{\tau_u}^u$ for all $u \in I_v^0$.

For $t \geq 0$, define $\mathbb{X}_t := \sum_{u \in \mathcal{L}_t} \delta_{\xi_t^u}$. The measure-valued process $(\mathbb{X}_t)_{t \geq 0}$ is called a branching Lévy process. When $\{(\xi_t)_{t \geq 0}, \mathrm{P}\}$ is a Brownian motion, $(\mathbb{X}_t)_{t \geq 0}$ is called a branching Brownian motion.

Denote by Z_t the number of particles alive at time t. It is well known that $(Z_t)_{\geq 0}$ is a continuous-time branching process. In this paper we consider the supercritical case, i.e. $m := \sum_k kp_k > 1$. Then $\mathbb{P}(\mathcal{S}) > 0$, where $\mathcal{S}$ is the event of survival. The extinction probability $\mathbb{P}(\mathcal{S}^c)$ is the smallest root in (0,1) of the equation $\sum_k p_k s^k = s$; see, for instance, [6, Section III.4]. Define

$$\lambda := \beta(m-1). \tag{1.2}$$

The process $\left(\mathrm{e}^{-\lambda t} Z_t\right)_{t \geq 0}$ is a non-negative martingale and hence

$$\lim_{t \to \infty} \mathrm{e}^{-\lambda t} Z_t =: W \quad \text{exists almost surely (a.s.).} \tag{1.3}$$

For any two functions f and g on $[0, \infty)$, $f \sim g$ as $s \to 0_+$ means that $\lim_{s \downarrow 0} (f(s)/g(s)) = 1$. Similarly, $f \sim g$ as $s \to \infty$ means that $\lim_{s \to \infty} (f(s)/g(s)) = 1$. Throughout this paper we assume the following two conditions hold.

The first condition is that the offspring distribution satisfies the Kesten–Stigum condition:

(H1) $\sum_{k \geq 1} (k \log k) p_k < \infty$.

Condition (H1) ensures that W is non-degenerate with $\mathbb{P}(W > 0) = \mathbb{P}(\mathcal{S})$. For more details, see [6, Section III.7].

The second condition is on the spatial motion:

(H2) There exist a complex constant c_* with $\mathrm{Re}(c_*) > 0$, $\alpha \in (0, 2)$, and a function $L(x) : \mathbb{R}_+ \to \mathbb{R}_+$ slowly varying at ∞ such that $\psi(\theta) \sim -c_* \theta^\alpha L(\theta^{-1})$ as $\theta \to 0_+$.

Since $\mathrm{e}^{\psi(\theta)} = \mathrm{E}(\mathrm{e}^{\mathrm{i}\theta\xi_1})$, we have $\mathrm{Re}(\psi) \leq 0$ and $\psi(-\theta) = \overline{\psi(\theta)}$. Thus, $\psi(\theta) \sim -\overline{c_*} |\theta|^\alpha L(|\theta|^{-1})$ as $\theta \to 0_-$. Under condition (H2), we can prove (see Remark 2.1) that $\mathrm{P}(|\xi_s| \geq x) \sim csx^{-\alpha} L(x)$ as $x \to \infty$, i.e. $|\xi_s|$ has regularly varying tails.

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An important example satisfying (H2) is the strictly stable process.

Example 1.1. *(Stable process.)* Let ξ be a strictly α-stable process, $\alpha \in (0, 2)$, on $\mathbb{R}$ with Lévy measure

$$n(\mathrm{d}y) = c_1 x^{-(1+\alpha)} \mathbf{1}_{(0,\infty)}(x)\, \mathrm{d}x + c_2 |x|^{-(1+\alpha)} \mathbf{1}_{(-\infty,0)}(x)\, \mathrm{d}x,$$

where $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$, and if $\alpha = 1$, $c_1 = c_2 = c$. For $\alpha \in (1, 2)$, by [36, Lemma 14.11, (14.19)] and the fact that $\Gamma(-\alpha) = -\alpha\Gamma(1-\alpha)$, we obtain that, for $\theta > 0$,

$$\int_0^\infty \big(\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y\big)\, n(\mathrm{d}y) = -c_1 \alpha \Gamma(1-\alpha) \mathrm{e}^{-\mathrm{i}\pi\alpha/2} \theta^\alpha,$$

and, taking the conjugate on both sides of [36, Lemma 14.11 (14.19)], we get that

$$\int_{-\infty}^0 \big(\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y\big)\, n(\mathrm{d}y) = -c_2 \alpha \Gamma(1-\alpha) \mathrm{e}^{\mathrm{i}\pi\alpha/2} \theta^\alpha.$$

Thus, the Lévy exponent of ξ is given, for $\theta > 0$, by

$$\psi(\theta) = \int \big(\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y\big)\, n(\mathrm{d}y) = -\alpha\Gamma(1-\alpha)\big(c_1 \mathrm{e}^{-\mathrm{i}\pi\alpha/2} + c_2 \mathrm{e}^{\mathrm{i}\pi\alpha/2}\big)\theta^\alpha.$$

Similarly, by [36, Lemma 14.11 (14.18), (114.20)], we have, for $\theta > 0$,

$$\begin{aligned}\psi(\theta) &= \begin{cases} \int \big(\mathrm{e}^{\mathrm{i}\theta y} - 1\big)\, n(\mathrm{d}y), & \alpha \in (0, 1), \\ \int (\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y \mathbf{1}_{|y|\leq 1})\, n(\mathrm{d}y) + \mathrm{i}a\theta, & \alpha = 1 \end{cases} \\ &= \begin{cases} -\alpha\Gamma(1-\alpha)\big(c_1 \mathrm{e}^{-\mathrm{i}\pi\alpha/2} + c_2 \mathrm{e}^{\mathrm{i}\pi\alpha/2}\big)\theta^\alpha, & \alpha \in (0, 1), \\ -c\pi\theta + \mathrm{i}a\theta, & \alpha = 1, \end{cases}\end{aligned} \tag{1.4}$$

where $a \in \mathbb{R}$ is a constant. It is clear that ψ satisfies (H2). For more details on stable processes, we refer the reader to [36, Section 14].

In Section 4 we give more examples satisfying condition (H2). Note that the non-symmetric 1-stable process does not satisfy (H2). However, in Example 4.1 we show that our main result still holds for the non-symmetric 1-stable process.

The maximal position $M_t := \sup_{u \in \mathcal{L}_t} \xi_t^u$ of branching Brownian motions has been studied intensively. Assume that $\beta = 1$, $p_0 = 0$, and $m = 2$. The seminal paper [29] proved that $M_t/t \to \sqrt{2}$ in probability as $t \to \infty$. [15] (see also [16]) proved that, under some moment conditions, $\mathbb{P}(M_t - m(t) \leq x) \to 1 - w(x)$ as $t \to \infty$ for all $x \in \mathbb{R}$, where $m(t) = \sqrt{2}t - 3/2\sqrt{2} \log t$ and $w(x)$ is a traveling wave solution. For more works on M_t, see [19, 20, 30, 35]. For inhomogeneous branching Brownian motions, many papers have discussed the growth rate of the maximal position; see [12–14] for the case with catalytic branching at the origin, and [31, 32, 34, 37] for the case with some general branching mechanisms.

Recently, the full statistics of the extremal configuration of branching Brownian motion have been studied. [3, 4] studied the limit property of the extremal process of branching Brownian motion, proving that the random measure defined by $\mathcal{E}_t := \sum_{u \in \mathcal{L}_t} \delta_{\xi_t^u - m(t)}$ converges weakly, and that the limiting process is a (randomly shifted) Poisson cluster process. Almost at the same time, [2] proved similar results using a totally different method.

For branching random walks, several authors have studied similar problems under an exponential moment assumption on the displacements of the offspring from the parent [1, 18, 27, 33]. When the displacements of the offspring from the parents are i.i.d. and have regularly varying tails, [23] studied the limit property of its maximum displacement M_n. More precisely, [23] proved that $a_n^{-1}M_n$ converges weakly, where $a_n = m^{n/\alpha}L_0(m^n)$ and L_0 is slowly varying at ∞. Recently, the extremal processes of the branching random walks with regularly varying steps were studied in [8, 9], where it was proved that the point random measure $\sum_{|v|=n} \delta_{a_n^{-1}S_v}$, where S_v is the position of v, converges weakly to a Cox cluster process, which is quite different from the case with exponential moments. See also [10, 25] for related works on branching random walks with heavy-tailed displacements.

[38] studied branching symmetric stable processes with branching rate μ being a measure on $\mathbb{R}$ in a Kato class with compact support (i.e. the support of μ is compact) and the offspring distribution $\{p_n(x), n \geq 0\}$ being spatially dependent. Under some conditions on μ and $\{p_n(x), n \geq 0\}$, [38] proved that the growth rate of the maximal displacement is exponential with rate given by the principal eigenvalue of the mean semigroup of the branching symmetric stable process. In this paper, we study the extremes of branching Lévy processes with constant branching rate β (that is, $\mu(\mathrm{d}x) = \beta\,\mathrm{d}x$) and spatial motion having regularly varying tails (see condition (H2)). Since $\beta\,\mathrm{d}x$ is not compactly supported, we cannot get the growth rate of the maximal displacement from [38]. As a corollary of our extreme limit result we get the growth rate of the maximal displacement, see Corollary 1.2.

The key idea of the proof in this paper is the 'one large jump principle' inspired by [8, 9, 23]. Along the discrete times $n\delta$, the branching Lévy process $\{\mathbb{X}_{n\delta}, n \geq 1\}$ is a branching random walk and the displacements from parents has the same law as $\mathbb{X}_\delta$. It is natural to think that we may get the results of this paper from the results for branching random walks by letting the time grid become finer and finer, and appropriately controlling the behavior between the time gaps. However, we cannot apply the results for branching random walks in [8, 9, 33] to $\{\mathbb{X}_{n\delta}, n \geq 1\}$. First, under condition (H2), the exponential moment assumption in [33] is not satisfied. Second, [8] assumes that the displacements are i.i.d., while the atoms of the random measure $\mathbb{X}_\delta$, being particles alive at time δ in our branching Lévy process, are not independent. Last, although the displacements of offspring coming from the same parent are allowed to be dependent in [9], [9, Assumption 2.5], where the displacements from parents are given by a special form [9, (2.9) and (2.10)]), seems to be very difficult to check for $\mathbb{X}_\delta$.

Branching Lévy processes are closely related to the Fisher–Kolmogorov–Petrovsky–Piskunov (Fisher–KPP) equation when the classical Laplacian Δ is replaced by the infinitesimal generator of the corresponding Lévy process. For any $g \in C_{\mathrm{b}}^{+}(\mathbb{R})$, define $u_g(t, x) = \mathbb{E}\big(\exp\big\{-\sum_{v\in\mathcal{L}_t} g\big(\xi_t^v + x\big)\big\}\big)$. By the Markov and branching properties, we have

$$u_g(t, x) = \mathrm{E}\big(\mathrm{e}^{-g(\xi_t+x)}\big) + \mathrm{E}\int_0^t \varphi\big(u_g\big(t-s, \xi_s + x\big)\big)\,\mathrm{d}s, \tag{1.5}$$

where $\varphi(s) = \beta\big(\sum_k s^k p_k - s\big)$. Then $1 - u_g$ is a mild solution to

$$\partial_t u - \mathcal{A}u = -\varphi(1-u), \tag{1.6}$$

with initial data $u(0, x) = 1 - \mathrm{e}^{-g(x)}$, where $\mathcal{A}$ is the infinitesimal generator of ξ. [17] proved that, under the assumption that the density of ξ is comparable to that of a symmetric α-stable process, the frontal position of $1-u$ is exponential in time. Using our main result, we give another proof of [17, Theorem 1.5] and also partially generalize it; see Remark 5.1.

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1.1. Main results

Put $\mathbb{R}_0 = (-\infty, \infty) \setminus \{0\}$, and $\overline{\mathbb{R}}_0 = [-\infty, \infty] \setminus \{0\}$ with the topology generated by the set $\{(a, b), (-b, -a), (a, \infty], [-\infty, -a) : 0 < a < b \le \infty\}$. Note that, for any $a > 0$, $[a, \infty]$ and $[-\infty, -a]$ are compact subsets of $\overline{\mathbb{R}}_0$. Denote by $\mathcal{B}_b^+(\overline{\mathbb{R}}_0)$ the set of all bounded non-negative Borel functions on $\overline{\mathbb{R}}_0$. Let $C_c^+(\overline{\mathbb{R}}_0)$ be the set of all non-negative continuous functions on $\overline{\mathbb{R}}_0$ such that $g = 0$ on $(-\delta, 0) \cup (0, \delta)$ for some $\delta > 0$. Denote by $\mathcal{M}(\overline{\mathbb{R}}_0)$ the set of all Radon measures endowed with the topology of vague convergence (denoted by $\stackrel{v}{\to}$), generated by the maps $\mu \to \int f\, d\mu$ for all $f \in C_c^+(\overline{\mathbb{R}}_0)$. Then $\mathcal{M}(\overline{\mathbb{R}}_0)$ is a metrizable space, see [28, Theorem 4.2, p. 112]. For any $g \in \mathcal{B}_b^+(\overline{\mathbb{R}}_0)$ and $\mu \in \mathcal{M}(\overline{\mathbb{R}}_0)$, we write $\mu(g) := \int_{\overline{\mathbb{R}}_0} g(x)\, \mu(dx)$. A sequence of random elements ν_n in $\mathcal{M}(\overline{\mathbb{R}}_0)$ converges weakly to ν, denoted as $\nu_n \stackrel{d}{\to} \nu$, if and only if, for all $g \in C_c^+(\overline{\mathbb{R}}_0)$, $\nu_n(g)$ converges weakly to $\nu(g)$.

We claim that there exists a non-decreasing function h_t with $h_t \uparrow \infty$ such that

$$\lim_{t\to\infty} e^{\lambda t} h_t^{-\alpha} L(h_t) = 1, \tag{1.7}$$

where λ is defined by (1.2). In fact, using [11, Theorem 1.5.4], there exists a non-increasing function g such that $g(x) \sim x^{-\alpha} L(x)$ as $x \to \infty$. Then $g(x) \to 0$ as $x \to \infty$. Define $h_t := \inf\{x > 0 : g(x) \le e^{-\lambda t}\}$. It is clear that h_t is non-decreasing and $h_t \uparrow \infty$. By the definition of h_t, for any $\varepsilon > 0$, $g(h_t/(1+\varepsilon)) \ge e^{-\lambda t} \ge g(h_t(1+\varepsilon))$, which implies that

$$\begin{aligned}(1+\varepsilon)^{-\alpha} &= (1+\varepsilon)^{-\alpha} \lim_{t\to\infty} \frac{L(h_t)}{L(h_t/(1+\varepsilon))} = \lim_{t\to\infty} \frac{g(h_t)}{g(h_t/(1+\varepsilon))} \\ &\le \liminf_{t\to\infty} e^{\lambda t} g(h_t) \le \limsup_{t\to\infty} e^{\lambda t} g(h_t) \\ &\le \lim_{t\to\infty} \frac{g(h_t)}{g(h_t(1+\varepsilon))} = (1+\varepsilon)^{\alpha} \lim_{t\to\infty} \frac{L(h_t)}{L(h_t(1+\varepsilon))} = (1+\varepsilon)^{\alpha}.\end{aligned}$$

Since ε is arbitrary, we get $\lim_{t\to\infty} e^{\lambda t} h_t^{-\alpha} L(h_t) = \lim_{t\to\infty} e^{\lambda t} g(h_t) = 1$. In particular, $h_t = e^{\lambda t/\alpha}$ if $L = 1$. In Lemma 2.1 we prove that $e^{\lambda t} \mathrm{P}(h_t^{-1} \xi_s \in \cdot) \stackrel{v}{\to} s\nu_\alpha(\cdot)$, where

$$\nu_\alpha(dx) = q_1 x^{-1-\alpha} \mathbf{1}_{(0,\infty)}(x)\, dx + q_2 |x|^{-1-\alpha} \mathbf{1}_{(-\infty,0)}(x)\, dx, \tag{1.8}$$

with q_1 and q_2 being non-negative numbers, uniquely determined by $c_* = \alpha\Gamma(1-\alpha)(q_1 e^{-i\pi\alpha/2} + q_2 e^{i\pi\alpha/2})$ if $\alpha \ne 1$ and $q_1 = q_2 = \mathrm{Re}(c_*)/\pi$ if $\alpha = 1$.

Now we are ready to state our main result. Define a renormalized version of $\mathbb{X}_t$ by

$$\mathcal{N}_t := \sum_{v\in\mathcal{L}_t} \delta_{h_t^{-1}\xi_t^v}. \tag{1.9}$$

In this paper we will investigate the limit of $\mathcal{N}_t$ as $t \to \infty$.

Theorem 1.1. *Under $\mathbb{P}$, $\mathcal{N}_t$ converges weakly to a random measure $\mathcal{N}_\infty \in \mathcal{M}(\overline{\mathbb{R}}_0)$ defined on some extension $(\Omega, \mathcal{G}, P)$ of the probability space on which the branching Lévy process is defined, with Laplace transform given by*

$$E\left(e^{-\mathcal{N}_\infty(g)}\right) = \mathbb{E}\left(\exp\left\{-W \int_0^\infty e^{-\lambda r} \int_{\mathbb{R}_0} \mathbb{E}\left(1 - e^{-Z_r g(x)}\right) \nu_\alpha(dx)\, dr\right\}\right), \quad g \in C_c^+(\overline{\mathbb{R}}_0),$$

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where λ *is defined in* (1.2) *and* W *is the martingale limit defined in* (1.3)*. Moreover,* $\mathcal{N}_\infty = \sum_j T_j \delta_{e_j}$*, where, given* W*,* $\sum_j \delta_{e_j}$ *is a Poisson random measure with intensity* $\vartheta W \nu_\alpha(\mathrm{d}x)$*,* $\{T_j, j \geq 1\}$ *is a sequence of i.i.d. random variables with common law*

$$P(T_j = k) = \vartheta^{-1} \int_0^\infty \mathrm{e}^{-\lambda r} \mathbb{P}(Z_r = k)\,\mathrm{d}r, \qquad k \geq 1, \tag{1.10}$$

where $\nu_\alpha(\mathrm{d}x)$ *is given by* (1.8)*,* Z_r *is the number of particles alive at time* r*,* $\vartheta = \int_0^\infty \mathrm{e}^{-\lambda r} \mathbb{P}(Z_r > 0)\,\mathrm{d}r$*, and* $\sum_j \delta_{e_j}$ *and* $\{T_j, j \geq 1\}$ *are independent.*

Theorem 1.1 says that, given W, $\mathcal{N}_\infty$ is an integer-valued random measure with the locations of the atoms being a Poisson random measure with intensity $\vartheta W \nu_\alpha(\mathrm{d}x)$ and with weights being i.i.d. with common distribution given by (1.10).

The proof of Theorem 1.1 consists of two steps. First, we use the idea of 'one large jump', which has been used in [8, 9, 23] for branching random walks, to deduce that $\mathcal{N}_t$ has the same limit as the family of random measures defined by

$$\widetilde{\mathcal{N}}_t := \sum_{v \in \mathcal{L}_t} \sum_{u \in I_v} \delta_{h_t^{-1} X_{u,t}}.$$

By 'one large jump' we mean that with large probability, for all $v \in \mathcal{L}_t$, at most one of the ancestors of v has a large enough movement. Then we prove that with large probability, for all u born before $t - s$, $\left|h_t^{-1} X_{u,t}\right|$ is small. Thus, the main contribution to $\widetilde{\mathcal{N}}_t$ is

$$\widetilde{\mathcal{N}}_{s,t} := \sum_{v \in \mathcal{L}_t} \sum_{u \in I_v, b_u > t-s} \delta_{h_t^{-1} X_{u,t}}.$$

Remark 1.1. Given a function f, we use D_f to denote its set of discontinuity points. Then, by Theorem 1.1, $\mathcal{N}_t(f) \xrightarrow{\mathrm{d}} \mathcal{N}_\infty(f)$ for any bounded measurable function f on $\overline{\mathbb{R}}_0$ with compact support satisfying $\mathcal{N}_\infty(D_f) = 0$ P-a.s. Furthermore, for any $k \geq 1$,

$$(\mathcal{N}_t(B_1), \mathcal{N}_t(B_2), \ldots, \mathcal{N}_t(B_k)) \xrightarrow{\mathrm{d}} (\mathcal{N}_\infty(B_1), \mathcal{N}_\infty(B_2), \ldots, \mathcal{N}_\infty(B_k)),$$

where $\{B_j\}$ are relatively compact subsets of $\overline{\mathbb{R}}_0$ satisfying $\mathcal{N}_\infty(\partial B_j) = 0$, $j = 1, \ldots, k$, P-a.s. See [28, Theorem 4.4] for a proof.

Now we list the positions of all particles alive at time t in decreasing order, $M_{t,1} \geq M_{t,2} \geq \cdots \geq M_{t,Z_t}$, and for $n > Z_t$ define $M_{t,n} := -\infty$. In particular, $M_{t,1} = \max_{v \in \mathcal{L}_t} \xi_t^v$ is the rightmost position of the particles alive at time t. Note that $\nu_\alpha(0, \infty) = \infty$ if and only if $q_1 > 0$. By the definition of $\mathcal{N}_\infty$ in Theorem 1.1, we have that if $q_1 = 0$ then $P(\mathcal{N}_\infty(0, \infty) = 0) = P\big(\sum_j \mathbf{1}_{e_j > 0} = 0\big) = 1$. If $q_1 > 0$ then

$$P(\mathcal{N}_\infty(0, \infty) = \infty \mid \mathcal{S}) = P\big(\textstyle\sum_j \mathbf{1}_{e_j > 0} = \infty \mid \mathcal{S}\big) = 1,$$

and since, for any $x > 0$, $\nu_\alpha(x, \infty) < \infty$, we have

$$P(\mathcal{N}_\infty(x, \infty) < \infty \mid \mathcal{S}) = P\Bigg(\sum_j \mathbf{1}_{e_j > x} < \infty \mid \mathcal{S}\Bigg) = 1.$$

Thus, on the set $\mathcal{S}$, we can order the atoms of $\mathcal{N}_\infty$ on $(0, \infty)$ in decreasing order: $M_{(1)} \geq M_{(2)} \geq \cdots \geq M_{(k)} \geq \cdots \to 0$. On the set $\mathcal{S}^\mathrm{c}$, $\mathcal{N}_\infty$ is null; then we define $M_{(k)} = -\infty$ for $k \geq 1$.

Define $\mathbb{P}^*(\cdot) := \mathbb{P}(\cdot \mid \mathcal{S})$ ($P^*(\cdot) := P(\cdot \mid \mathcal{S})$) and let $\mathbb{E}^*$ (E^*) be the corresponding expectation. As a consequence of Theorem 1.1, we have the following corollary.

Corollary 1.1. *If $q_1 > 0$ then, for any $n \geq 1$,*

$$\left(h_t^{-1}M_{t,1}, h_t^{-1}M_{t,2}, \ldots, h_t^{-1}M_{t,n}; \mathbb{P}^*\right) \xrightarrow{\mathrm{d}} \left(M_{(1)}, M_{(2)}, \ldots, M_{(n)}; P^*\right).$$

Moreover, $M_{(k)} > 0$, $k \geq 1$, P^*-a.s.

In particular, for the rightmost position $R_t := M_{t,1} = \max_{v \in \mathcal{L}_t} \xi_t^v$, we have the following result.

Corollary 1.2. *If $q_1 > 0$ then $\left(h_t^{-1}R_t; \mathbb{P}^*\right) \xrightarrow{\mathrm{d}} \left(M_{(1)}; P^*\right)$, where the law of $\left(M_{(1)}; P^*\right)$ is given by*

$$P^*\left(M_{(1)} \leq x\right) = \begin{cases} \mathbb{E}^*\left(\mathrm{e}^{-\alpha^{-1}q_1\vartheta W x^{-\alpha}}\right), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Proof. Using Corollary 1.1, $\left(h_t^{-1}R_t; \mathbb{P}^*\right) \xrightarrow{\mathrm{d}} \left(M_{(1)}; P^*\right)$, and $M_{(1)} > 0$ P^*-a.s. For any $x > 0$, $P^*(M_{(1)} \leq x) = P^*(\mathcal{N}_\infty(x, \infty) = 0) = P^*\left(\sum_j \mathbf{1}_{(x,\infty)}(e_j) = 0\right) = \mathbb{E}^*\left(\mathrm{e}^{-\vartheta W \nu_\alpha(x,\infty)}\right) = \mathbb{E}^*\left(\mathrm{e}^{-\alpha^{-1}q_1\vartheta W x^{-\alpha}}\right)$. The proof is now complete. □

Remark 1.2. Similarly, we can order the particles alive at time t in an increasing order: $L_{t,1} \leq L_{t,2} \leq \cdots \leq L_{t,Z_t}$. When $q_2 = 0$, $P(\mathcal{N}_\infty(-\infty, 0) = 0) = P\left(\sum_j \mathbf{1}_{e_j<0} = 0\right) = 1$. When $q_2 > 0$, on the set $\mathcal{S}$, we can order the atoms of $\mathcal{N}_\infty$ on $(-\infty, 0)$ as $L_{(1)} \leq L_{(2)} \leq \cdots \leq L_{(k)} \leq \cdots \to 0$. Note that $\{M_{(k)}, k \geq 1\}$ and $\{L_{(k)}, k \geq 1\}$ cover all the atoms of $\mathcal{N}_\infty$. Similar to Corollaries 1.1 and 1.2, we have the following weak convergence of $(L_{t,1}, L_{t,2}, \ldots, L_{t,n})$: if $q_2 > 0$ then, for any $n \geq 1$,

$$\left(h_t^{-1}L_{t,1}, h_t^{-1}L_{t,2}, \ldots, h_t^{-1}L_{t,n}; \mathbb{P}^*\right) \xrightarrow{\mathrm{d}} \left(L_{(1)}, L_{(2)}, \ldots, L_{(n)}; P^*\right);$$

and the distribution of $L_{(1)}$ under P^* is as follows: for any $x < 0$, $P^*(L_{(1)} \leq x) = P^*(\mathcal{N}_\infty(-\infty, x] > 0) = P^*\left(\sum_j \mathbf{1}_{(-\infty,x]}(e_j) > 0\right) = 1 - \mathbb{E}^*\left(\mathrm{e}^{-\vartheta W \nu_\alpha(-\infty,x]}\right) = 1 - \mathbb{E}^*\left(\mathrm{e}^{-\alpha^{-1}q_2\vartheta W|x|^{-\alpha}}\right)$.

The rest of the paper is organized as follows. In Section 2 we introduce the one large jump principle and give the proof of Theorem 1.1 based on Proposition 2.1, which will be proved in Section 2.3. The proof of Corollary 1.1 is given in Section 3. In Section 4 we give more examples satisfying condition (H2) and conditions which are weaker than (H2), but sufficient for the main result of this paper. We discuss the front position of the Fisher–KPP equation (1.6) in Section 5.

2. Proof of Theorem 1.1

2.1. Preliminaries

Recall that h_t is a function satisfying (1.7). Let $C_{\mathrm{b}}^0(\mathbb{R})$ be the set of all bounded continuous functions vanishing in a neighborhood of 0. It is clear that if $g \in C_{\mathrm{c}}^+\left(\overline{\mathbb{R}}_0\right)$ then $g^*(x) := \mathbf{1}_{\mathbb{R}_0}(x)g(x) \in C_{\mathrm{b}}^0(\mathbb{R})$.

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Lemma 2.1. *For any $g \in C_b^0(\mathbb{R})$ and $s > 0$, $\lim_{t\to\infty} e^{\lambda t} E\big(g\big(h_t^{-1}\xi_s\big)\big) = s \int_{\mathbb{R}_0} g(x)\, \nu_\alpha(dx)$.*

Proof. Let ν_t be the law of $h_t^{-1}\xi_s$. Then, by (H2), as $t \to \infty$,

$$\exp\left\{e^{\lambda t}\int_{\mathbb{R}}\left(e^{i\theta x}-1\right)\nu_t(dx)\right\} = \exp\left\{e^{\lambda t}\left(e^{s\psi(h_t^{-1}\theta)}-1\right)\right\} \to \exp\left\{s\widetilde{\psi}(\theta)\right\}, \tag{2.1}$$

where

$$\widetilde{\psi}(\theta) = \begin{cases} -c_*\theta^\alpha, & \theta > 0; \\ -\overline{c_*}|\theta|^\alpha, & \theta \le 0. \end{cases}$$

Note that the left-hand side of (2.1) is the characteristic function of an infinitely divisible random variable Y_t with Lévy measure $e^{\lambda t}\nu_t$, and, by (1.4), $e^{s\widetilde{\psi}(\theta)}$ is the characteristic function of a strictly α-stable random variable Y with Lévy measure $s\nu_\alpha(dx)$. Thus, Y_t weakly converges to Y. The desired result follows immediately from [36, Theorem 8.7 (1)]. □

It is well known (see [11, Theorem 1.5.6], for instance) that, for any $\varepsilon > 0$, there exists $a_\varepsilon > 0$ such that, for any $y > a_\varepsilon$ and $x > a_\varepsilon$,

$$\frac{L(y)}{L(x)} \le (1-\varepsilon)^{-1} \max\left\{(y/x)^\varepsilon, (y/x)^{-\varepsilon}\right\}, \tag{2.2}$$

which is occasionally called Potter's bound.

Lemma 2.2. *There exists $c_0 > 0$ such that, for any $s > 0$ and $x > 2 + 2a_{0.5}$, $G_s(x) := P(|\xi_s| > x) \le c_0 s x^{-\alpha} L(x)$.*

Proof. By [24, (3.3.1)], for any $x > 2$,

$$P(|\xi_s| > x) \le \frac{x}{2}\int_{-2x^{-1}}^{2x^{-1}}\left(1-e^{s\psi(\theta)}\right)d\theta \le s\frac{x}{2}\int_{-2x^{-1}}^{2x^{-1}}\|\psi(\theta)\|\,d\theta = s\int_0^2\|\psi(\theta/x)\|\,d\theta,$$

where in the last equality we used the symmetry of $\|\psi(\theta)\|$. By (H2), it is clear that there exists $c_1 > 0$ such that $\|\psi(\theta)\| \le c_1\theta^\alpha L\big(\theta^{-1}\big)$, $|\theta| \le 1$. Thus, for $x > 2 + 2a_{0.5}$, using (2.2) with $\varepsilon = 0.5$, we get

$$P(|\xi_s| > x) \le c_1 s x^{-\alpha}\int_0^2 \theta^\alpha L(x/\theta)\,d\theta \le 2c_1 s x^{-\alpha} L(x)\int_0^2 \theta^\alpha\left(\theta^{-1/2}+\theta^{1/2}\right)d\theta.$$

The proof is now complete. □

Remark 2.1. It follows from Lemma 2.1 that $\lim_{t\to\infty} e^{\lambda t} P(|\xi_s| \ge h_t) = s(q_1+q_2)/\alpha$, which implies that $P(|\xi_s| \ge x) \sim ((q_1+q_2)/\alpha)s x^{-\alpha}L(x)$, $x \to \infty$.

Now we recall the many-to-one formula which is useful in computing expectations. We only list some special cases that we use here; see [26, Theorem 8.5] for general cases.

Recall that, for any $u \in \mathbb{T}$, n^u is the number of particles in $I_u \setminus \{o\}$.

Lemma 2.3. *(Many-to-one formula.) Let $\{n_t\}$ be a Poisson process with parameter β on some probability space $(\Omega, \mathcal{G}, P)$. Then, for any $g \in \mathcal{B}_b^+(\mathbb{R})$, $\mathbb{E}\big(\sum_{v\in\mathcal{L}_t} g(n^v)\big) = e^{\lambda t}E(g(n_t))$ and, for any $0 \le s < t$, $\mathbb{E}\big(\sum_{v\in\mathcal{L}_t}\mathbf{1}_{b_v\le t-s}\big) = e^{\lambda t}P(n_t - n_{t-s} = 0) = e^{\lambda t}e^{-\beta s}$.*

2.2. Proof of the Theorem 1.1

Recall that, on some extension $(\Omega, \mathcal{G}, P)$ of the probability space on which the branching Lévy process is defined, given W, $\sum_j \delta_{e_j}$ is a Poisson random measure with intensity $\vartheta W \nu_\alpha(\mathrm{d}x)$, $\{T_j, j \geq 1\}$ is a sequence of i.i.d. random variables with common law

$$P(T_j = k) = \vartheta^{-1} \int_0^\infty \mathrm{e}^{-\lambda r} \mathbb{P}(Z_r = k)\, \mathrm{d}r, \qquad k \geq 1,$$

where $\vartheta = \int_0^\infty \mathrm{e}^{-\lambda r} \mathbb{P}(Z_r > 0)\, \mathrm{d}r$, and $\sum_j \delta_{e_j}$ and $\{T_j, j \geq 1\}$ are independent.

Lemma 2.4. *Let $\mathcal{N}_\infty = \sum_j T_j \delta_{e_j}$. Then $\mathcal{N}_\infty \in \mathcal{M}(\overline{\mathbb{R}}_0)$ and the Laplace transform of $\mathcal{N}_\infty$ is given by*

$$E\big(\mathrm{e}^{-\mathcal{N}_\infty(g)}\big) = \mathbb{E}\bigg(\exp\bigg\{ -W \int_0^\infty \mathrm{e}^{-\lambda r} \int_{\mathbb{R}_0} \mathbb{E}\big(1 - \mathrm{e}^{-Z_r g(x)}\big)\, \nu_\alpha(\mathrm{d}x)\, \mathrm{d}r \bigg\} \bigg), \qquad g \in C_\mathrm{c}^+\big(\overline{\mathbb{R}}_0\big).$$

Proof. First note that, for any $a > 0$, $\vartheta W \nu_\alpha([-\infty, -a] \cup [a, \infty]) < \infty$, $\mathbb{P}$-a.s. Thus, given W, $\sum_j \mathbf{1}_{|e_j| \geq a}$ is Poisson distributed with parameter $\vartheta W \nu_\alpha([-\infty, -a] \cup [a, \infty])$, which implies that $\sum_j \mathbf{1}_{|e_j| \geq a} < \infty$, a.s. Thus, by the definition of $\mathcal{N}_\infty$,

$$P(\mathcal{N}_\infty([-\infty, -a] \cup [a, \infty]) < \infty) = P\Bigg(\sum_j \mathbf{1}_{|e_j| \geq a} < \infty \Bigg) = 1.$$

So $\mathcal{N}_\infty \in \mathcal{M}(\overline{\mathbb{R}}_0)$. Note that

$$\begin{aligned}
\phi(\theta) := E\big(\mathrm{e}^{-\theta T_j}\big) &= \vartheta^{-1} \sum_{k \geq 1} \mathrm{e}^{-\theta k} \int_0^\infty \mathrm{e}^{-\lambda r} \mathbb{P}(Z_r = k)\, \mathrm{d}r \\
&= \vartheta^{-1} \int_0^\infty \mathrm{e}^{-\lambda r} \mathbb{E}\big(\mathrm{e}^{-\theta Z_r}, Z_r > 0\big)\, \mathrm{d}r \\
&= 1 - \vartheta^{-1} \int_0^\infty \mathrm{e}^{-\lambda r} \mathbb{E}\big(1 - \mathrm{e}^{-\theta Z_r}\big)\, \mathrm{d}r.
\end{aligned}$$

Thus, for any $g \in C_\mathrm{c}^+\big(\overline{\mathbb{R}}_0\big)$,

$$\begin{aligned}
E\big(\mathrm{e}^{-\mathcal{N}_\infty(g)}\big) = E\big(\mathrm{e}^{-\sum_j T_j g(e_j)}\big) &= E\Bigg(\prod_j \phi(g(e_j)) \Bigg) \\
&= \mathbb{E}\bigg(\exp\bigg\{ -\vartheta W \int_{\mathbb{R}_0} (1 - \phi(g(x)))\, \nu_\alpha(\mathrm{d}x) \bigg\} \bigg) \\
&= \mathbb{E}\bigg(\exp\bigg\{ -W \int_0^\infty \mathrm{e}^{-\lambda r} \int_{\mathbb{R}_0} \mathbb{E}\big(1 - \mathrm{e}^{-Z_r g(x)}\big)\, \nu_\alpha(\mathrm{d}x)\, \mathrm{d}r \bigg\} \bigg).
\end{aligned}$$

The proof is now complete. □

To prove Theorem 1.1 we use the idea of 'one large jump', which has been used in [8, 9, 23] for branching random walks. By 'one large jump' we mean that with large probability, for all $v \in \mathcal{L}_t$, at most one of the random variables $\{|X_{u,t}| : u \in I_v\}$ is bigger than $h_t \theta / t$ ($\theta > 0$). Thus,

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by (1.1), to investigate the limit property of $\mathcal{N}_t$ defined by (1.9), we consider the limit of the point process defined by $\widetilde{\mathcal{N}}_t := \sum_{v\in\mathcal{L}_t}\sum_{u\in I_v}\delta_{h_t^{-1}X_{u,t}}$.

Proposition 2.1. *Under $\mathbb{P}$, as $t\to\infty$, $\widetilde{\mathcal{N}}_t \xrightarrow{\mathrm{d}} \mathcal{N}_\infty$.*

The proof of this proposition is postponed to the next subsection. The following lemma formalizes the well-known one large jump principle (see, e.g., Steps 3 and 4 in [23, Section 2]) at the level of point processes. Because of Lemma 2.5, it suffices to investigate the weak convergence of $\widetilde{\mathcal{N}}_t$, which is much easier compared to that of $\mathcal{N}_t$.

Lemma 2.5. *Assume $g\in C_{\mathrm{c}}^{+}\big(\overline{\mathbb{R}}_0\big)$. For any $\varepsilon>0$, $\lim_{t\to\infty}\mathbb{P}(|\mathcal{N}_t(g)-\widetilde{\mathcal{N}}_t(g)|>\varepsilon)=0$.*

Proof. Since $g\in C_{\mathrm{c}}^{+}\big(\overline{\mathbb{R}}_0\big)$, we have $\mathrm{Supp}(g)\subset\{x:|x|>\delta\}$ for some $\delta>0$.

Step 1: For any $\theta>0$, let $A_t(\theta)$ denote the event that, for all $v\in\mathcal{L}_t$, at most one of the random variables $\{|X_{u,t}|:u\in I_v\}$ is bigger than $h_t\theta/t$. We claim that

$$\mathbb{P}\big(A_t(\theta)^{\mathrm{c}}\big)\to 0. \tag{2.3}$$

Note that

$$\mathbb{P}\Big(A_t(\theta)^{\mathrm{c}}\mid\mathcal{F}_t^{\mathbb{T}}\Big)\le\sum_{v\in\mathcal{L}_t}\mathbb{P}\Bigg(\sum_{u\in I_v}\mathbf{1}_{\{|X_{u,t}|>h_t\theta/t\}}\ge 2\mid\mathcal{F}_t^{\mathbb{T}}\Bigg). \tag{2.4}$$

By Lemma 2.2 and (2.2) with $\varepsilon=0.5$, we have, for $h_t\theta/t>2+2a_{0.5}$ and $h_t>a_{0.5}$,

$$\begin{aligned}\mathbb{P}\Big(|X_{u,t}|>h_t\theta/t\mid\mathcal{F}_t^{\mathbb{T}}\Big)&=\mathrm{P}(|\xi_s|>h_t\theta/t)|_{s=\tau_{u,t}}\\&\le c_0\tau_{u,t}h_t^{-\alpha}t^{\alpha}\theta^{-\alpha}L(h_t\theta/t)\\&\le 2c_0\theta^{-\alpha}t^{1+\alpha}h_t^{-\alpha}L(h_t)\big[(\theta/t)^{1/2}+(\theta/t)^{-1/2}\big]:=p_t.\end{aligned} \tag{2.5}$$

Recall that the number of elements in I_v is n^v+1. Since they are conditioned on $\mathcal{F}_t^{\mathbb{T}}$, the $\{X_{u,t},\,u\in I_v\}$ are independent, and by (2.5) we get

$$\begin{aligned}\mathbb{P}\Bigg(\sum_{u\in I_v}\mathbf{1}_{\{|X_{u,t}|>h_t\theta/t\}}\ge 2\mid\mathcal{F}_t^{\mathbb{T}}\Bigg)&\le\sum_{m=2}^{n^v+1}\binom{n^v+1}{m}p_t^m\\&=p_t^2\sum_{m=0}^{n^v-1}\binom{n^v+1}{m+2}p_t^m\\&\le p_t^2\sum_{m=0}^{n^v-1}n^v(n^v+1)\binom{n^v-1}{m}p_t^m\\&=p_t^2n^v(n^v+1)(1+p_t)^{n^v-1}.\end{aligned}$$

Thus, by (2.4) and the many-to-one formula (Lemma 2.3),

$$\begin{aligned}\mathbb{P}\big(A_t(\theta)^{\mathrm{c}}\big)=\mathbb{E}\big(\mathbb{P}\big(A_t(\theta)^{\mathrm{c}}\mid\mathcal{F}_t^{\mathbb{T}}\big)\big)&\le \mathrm{e}^{\lambda t}p_t^2E\big(n_t(n_t+1)(1+p_t)^{n_t-1}\big)\\&=\mathrm{e}^{\lambda t}p_t^2\big(2\beta+(1+p_t)\beta^2\big)\mathrm{e}^{\beta p_t},\end{aligned} \tag{2.6}$$

where n_t is a Poisson process with parameter β on some probability space $(\Omega,\mathcal{G},P)$. Since $\mathrm{e}^{\lambda t}h_t^{-\alpha}L(h_t)\to 1$, (2.3) follows immediately from (2.5) and (2.6).

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Step 2: Let $\varrho > \beta + 1$, to be chosen later. Let $B_t(\varrho)$ be the event that, for all $v \in \mathcal{L}_t$, $n_t^v \le \varrho t$. Using the many-to-one formula,

$$\begin{aligned}\mathbb{P}\big(B_t(\varrho)^{\rm c}\big) \le \mathbb{E}\left(\sum_{v\in\mathcal{L}_t} \mathbf{1}_{n^v > \varrho t}\right) &= {\rm e}^{\lambda t} P(n_t > \varrho t)\\ &\le {\rm e}^{\lambda t} \inf_{r>0} {\rm e}^{-r\varrho t} E\big({\rm e}^{rn_t}\big)\\ &= {\rm e}^{\lambda t} \inf_{r>0} \exp\big\{\big(({\rm e}^r - 1)\beta - r\varrho\big)t\big\}\\ &= {\rm e}^{\lambda t} \exp\{-(\varrho(\log\varrho - \log\beta) - \varrho + \beta)t\}.\end{aligned}$$

Choose ϱ large enough that $\varrho(\log\varrho - \log\beta) - \varrho + \beta > \lambda$; then $\lim_{t\to\infty} \mathbb{P}\big(B_t(\varrho)^{\rm c}\big) = 0$.

Step 3: Since $g \in C_{\rm c}^+\big(\overline{\mathbb{R}}_0\big)$, g is uniformly continuous, i.e. for any $a > 0$ there exists $\eta > 0$ such that $|g(x_1) - g(x_2)| \le a$ whenever $|x_1 - x_2| < \eta$.

Now consider θ small enough that $\varrho\theta < \eta \wedge (\delta/2)$. Let $v' \in I_v$ be such that $|X_{v',t}| = \max_{u\in I_v}\{|X_{u,t}|\}$. We note that, on the event $A_t(\theta)$, $|X_{u,t}| \le \theta h_t/t \le h_t\delta/2$ for any $u \in I_v \setminus \{v'\}$ and $t > 1$, and thus $g(X_{u,t}/h_t) = 0$, which implies that

$$\widetilde{\mathcal{N}}_t(g) = \sum_{v\in\mathcal{L}_t}\sum_{u\in I_v} g(X_{u,t}/h_t) = \sum_{v\in\mathcal{L}_t} g\big(X_{v',t}/h_t\big).$$

Thus it follows that, on the event $A_t(\theta)$,

$$\big|\mathcal{N}_t(g) - \widetilde{\mathcal{N}}_t(g)\big| = \left|\sum_{v\in\mathcal{L}_t}\big[g\big(\xi_t^v/h_t\big) - g\big(X_{v',t}/h_t\big)\big]\right|. \tag{2.7}$$

Since $\xi_t^v = \sum_{u\in I_v} X_{u,t}$, on the event $A_t(\theta) \cap B_t(\varrho)$ we have

$$h_t^{-1}|\xi_t^v - X_{v',t}| = h_t^{-1}\left|\sum_{u\in I_v\setminus\{v'\}} X_{u,t}\right| \le \theta t^{-1} n^v \le \varrho\theta < \eta \wedge (\delta/2).$$

Note that if $|X_{v',t}/h_t| \le \delta/2$, then $|\xi_t^v|/h_t < \delta$, which implies that $g\big(\xi_t^v/h_t\big) - g\big(X_{v',t}/h_t\big) = 0$. Thus,

$$\big|g\big(\xi_t^v/h_t\big) - g\big(X_{v',t}/h_t\big)\big| = \big|g\big(\xi_t^v/h_t\big) - g\big(X_{v',t}/h_t\big)\big|\mathbf{1}_{\{|X_{v',t}|>h_t\delta/2\}} \le a\mathbf{1}_{\{|X_{v',t}|>h_t\delta/2\}}.$$

It follows from this and (2.7) that, on the event $A_t(\theta) \cap B_t(\varrho)$,

$$\begin{aligned}\big|\mathcal{N}_t(g) - \widetilde{\mathcal{N}}_t(g)\big| &\le a\sum_{v\in\mathcal{L}_t}\mathbf{1}_{\{|X_{v',t}|>h_t\delta/2\}}\\ &\le a\sum_{v\in\mathcal{L}_t}\sum_{u\in I_v}\mathbf{1}_{\{|X_{u,t}|>h_t\delta/2\}} = a\widetilde{\mathcal{N}}_t\{[-\infty, -\delta/2) \cup (\delta/2, \infty]\}.\end{aligned}$$

Let $f \in C_{\rm c}^+\big(\overline{\mathbb{R}}_0\big)$ satisfy $f(x) = 1$ for $|x| \ge \delta/2$. Then $|\mathcal{N}_t(g) - \widetilde{\mathcal{N}}_t(g)| \le a\widetilde{\mathcal{N}}_t(f)$.

Combining Steps 1–3, we get

$$\begin{aligned}\limsup_{t\to\infty}\mathbb{P}\big(\big|\mathcal{N}_t(g) - \widetilde{\mathcal{N}}_t(g)\big| > \varepsilon\big) &\le \limsup_{t\to\infty}\mathbb{P}\big(A_t(\theta)^{\rm c}\big) + \mathbb{P}\big(B_t(\varrho)^{\rm c}\big) + \mathbb{P}\big(\widetilde{\mathcal{N}}_t(f) > a^{-1}\varepsilon\big)\\ &= \limsup_{t\to\infty}\mathbb{P}\big(\widetilde{\mathcal{N}}_t(f) > a^{-1}\varepsilon\big) = P\big(\mathcal{N}_\infty(f) > a^{-1}\varepsilon\big),\end{aligned}$$

where the final equality follows from Proposition 2.1 (the proof of Proposition 2.1 does not use the result in this lemma). Then, letting $a \to 0$, we get the desired result. □

Proof of Theorem 1.1. Using Lemma 2.4, Proposition 2.1, and Lemma 2.5, the results of Theorem 1.1 follow immediately. □

2.3. Proof of Proposition 2.1

To prove the weak convergence of $\widetilde{\mathcal{N}}_t$, we first cut the tree at time $t-s$. We divide the particles born before time t into two parts: the particles born before time $t-s$ and after $t-s$. Define

$$\widetilde{\mathcal{N}}_{s,t} := \sum_{v\in\mathcal{L}_t} \sum_{u\in I_v, b_u>t-s} \delta_{h_t^{-1}X_{u,t}}. \tag{2.8}$$

Lemma 2.6. *For any $\varepsilon>0$ and $g\in C_c^+(\overline{\mathbb{R}}_0)$, $\lim_{s\to\infty}\limsup_{t\to\infty}\mathbb{P}\big(\big|\widetilde{\mathcal{N}}_t(g)-\widetilde{\mathcal{N}}_{s,t}(g)\big|>\varepsilon\big)=0$.*

Proof. Since $g\in C_c^+(\overline{\mathbb{R}}_0)$, we have $\mathrm{Supp}(g)\subset\{x:|x|>\delta\}$ for some $\delta>0$.

Let $J_{s,t}$ be the event that, for all u with $b_u\le t-s$, $|X_{u,t}|\le h_t\delta/2$. On $J_{s,t}$, $\widetilde{\mathcal{N}}_t(g)-\widetilde{\mathcal{N}}_{s,t}(g)=0$, and thus we only need to show that

$$\lim_{s\to\infty}\limsup_{t\to\infty}\mathbb{P}\big(J_{s,t}^{\mathrm{c}}\big)=0. \tag{2.9}$$

Recall that $G_s(x):=\mathrm{P}(|\xi_s|>x)$. By Lemma 2.2, for t large enough that $h_t\delta/2\ge 2+2a_{0.5}$,

$$\begin{aligned}\mathbb{P}\big(J_{s,t}^{\mathrm{c}}\big)&=1-\mathbb{P}(J_{s,t})=1-\mathbb{E}\Bigg(\prod_{u:b_u\le t-s}\big(1-G_{\tau_{u,t}}(h_t\delta/2)\big)\Bigg)\\&\le\mathbb{E}\Bigg(\sum_{u:b_u\le t-s}G_{\tau_{u,t}}(h_t\delta/2)\Bigg)\\&\le c_0h_t^{-\alpha}(\delta/2)^{-\alpha}L(h_t\delta/2)\mathbb{E}\Bigg(\sum_{u:b_u\le t-s}\tau_{u,t}\Bigg).\end{aligned} \tag{2.10}$$

In the first inequality we used $1-\prod_{i=1}^n(1-x_i)\le\sum_{i=1}^n x_i$, $x_i\in(0,1)$. By the definition of $\tau_{u,t}$,

$$\begin{aligned}\sum_{u:b_u\le t-s}\tau_{u,t}&=\sum_{u:b_u\le t-s}\int_0^t\mathbf{1}_{(b_u,\sigma_u)}(r)\,\mathrm{d}r\\&=\int_0^{t-s}\sum_u\mathbf{1}_{(b_u,\sigma_u)}(r)\,\mathrm{d}r+\int_{t-s}^t\sum_u\mathbf{1}_{b_u<t-s,\sigma_u>r}\,\mathrm{d}r.\end{aligned} \tag{2.11}$$

For the first part, noting that $r\in(b_u,\sigma_u)$ is equivalent to $u\in\mathcal{L}_r$, we get

$$\mathbb{E}\int_0^{t-s}\sum_u\mathbf{1}_{(b_u,\sigma_u)}(r)\,\mathrm{d}r=\mathbb{E}\int_0^{t-s}Z_r\,\mathrm{d}r=\int_0^{t-s}\mathrm{e}^{\lambda r}\,\mathrm{d}r=\lambda^{-1}\big(\mathrm{e}^{\lambda(t-s)}-1\big). \tag{2.12}$$

For the second part, using the many-to-one formula we have

$$\mathbb{E}\Bigg(\sum_u\mathbf{1}_{b_u<t-s,\sigma_u>r}\Bigg)=\mathbb{E}\Bigg(\sum_{u\in\mathcal{L}_r}\mathbf{1}_{b_u<t-s}\Bigg)=\mathrm{e}^{\lambda r}\mathrm{e}^{-\beta(r+s-t)}.$$

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Thus,

$$\mathbb{E}\int_{t-s}^{t}\sum_{u}\mathbf{1}_{b_u<t-s,\sigma_u>r}\,\mathrm{d}r=\int_{t-s}^{t}\mathrm{e}^{\lambda r}\mathrm{e}^{-\beta(r-t+s)}\,\mathrm{d}r=\mathrm{e}^{\lambda t}\frac{\mathrm{e}^{-\beta s}-\mathrm{e}^{-\lambda s}}{\lambda-\beta}. \tag{2.13}$$

Combining (2.11), (2.12), and (2.13),

$$\mathbb{E}\sum_{u:b_u\le t-s}\tau_{u,t}\le \mathrm{e}^{\lambda t}\left(\lambda^{-1}\mathrm{e}^{-\lambda s}+\frac{\mathrm{e}^{-\beta s}-\mathrm{e}^{-\lambda s}}{\lambda-\beta}\right).$$

Therefore, by (2.10),

$$\mathbb{P}\big(J_{s,t}^{\mathrm{c}}\big)\le c_0(\delta/2)^{-\alpha}\mathrm{e}^{\lambda t}h_t^{-\alpha}L(h_t\delta/2)\left(\lambda^{-1}\mathrm{e}^{-\lambda s}+\frac{\mathrm{e}^{-\beta s}-\mathrm{e}^{-\lambda s}}{\lambda-\beta}\right). \tag{2.14}$$

It follows from (1.7) that $\lim_{t\to\infty}\mathrm{e}^{\lambda t}h_t^{-\alpha}L(h_t\delta/2)=1$. First letting $t\to\infty$ and then $s\to\infty$ in (2.14), we get (2.9) immediately. The proof is now complete. □

Now we consider the weak convergence of $\widetilde{\mathcal{N}}_{s,t}$. Recall the definition of $\widetilde{\mathcal{N}}_{s,t}$ in (2.8). Note that the atoms of $\widetilde{\mathcal{N}}_{s,t}$ are $\big\{h_t^{-1}X_{u,t},\ t-s<b_u\le t\big\}$. Thus $\widetilde{\mathcal{N}}_{s,t}=\sum_{u:t-s<b_u<t}Z_t^u\delta_{h_t^{-1}X_{u,t}}$, where Z_t^u is the number of offspring of u alive at time t. Using the tree structure, we can split all the particles born after $t-s$ according to the branches generated by the particles alive at $t-s$. More precisely,

$$\widetilde{\mathcal{N}}_{s,t}=\sum_{w\in\mathcal{L}_{t-s}}\sum_{u\in D_t^w}Z_t^u\delta_{h_t^{-1}X_{u,t}}=:\sum_{w\in\mathcal{L}_{t-s}}M_{s,t}^w, \tag{2.15}$$

where, for $w\in\mathcal{L}_{t-s}$, $D_t^w:=\{u: w\in I_u,\ t-s<b_u\le t\}$ is the set of all the offspring of w before time t. By the branching property, $M_{s,t}^w$ are i.i.d. with a common law which is the same as that of $M_{s,t}:=\sum_{u\in D_s}Z_s^u\delta_{h_t^{-1}X_{u,s}}$, where $D_s=\{u:0<b_u\le s\}$.

Lemma 2.7. *For any* $j=1,\ldots,n$, *let* $\gamma_j(t)$ *be a* $(0,1]$*-valued function on* $(0,\infty)$. *Suppose* a_t *is a positive function with* $\lim_{t\to\infty}a_t=\infty$ *such that* $\lim_{t\to\infty}a_t(1-\gamma_j(t))=c_j<\infty$. *Then* $\lim_{t\to\infty}a_t\big(1-\prod_{j=1}^n\gamma_j(t)\big)=\sum_{j=1}^n c_j$.

Proof. Note that $1-\prod_{j=1}^n\gamma_j(t)=\sum_{j=1}^n\prod_{k=1}^{j-1}\gamma_k(t)(1-\gamma_j(t))$. Since $\gamma_j(t)\to 1$ we get that, as $t\to\infty$,

$$a_t\left(1-\prod_{j=1}^n\gamma_j(t)\right)=\sum_{j=1}^n\prod_{k=1}^{j-1}\gamma_k(t)a_t(1-\gamma_j(t))\to\sum_{j=1}^n c_j.$$ □

Proof of Proposition 2.1. By Lemma 2.6, we only need to consider the convergence of $\widetilde{\mathcal{N}}_{s,t}$. Assume that $\mathrm{Supp}(g)\subset\{x:|x|>\delta\}$ for some $\delta>0$. Using the Markov property and the decomposition of $\widetilde{\mathcal{N}}_{s,t}$ in (2.15), we have

$$\mathbb{E}\Big(\mathrm{e}^{-\widetilde{\mathcal{N}}_{s,t}(g)}\Big)=\mathbb{E}\Big(\big[\mathbb{E}\big(\mathrm{e}^{-M_{s,t}(g)}\big)\big]^{Z_{t-s}}\Big). \tag{2.16}$$

We claim that

$$\lim_{t\to\infty}\big(1-\mathbb{E}\big(\mathrm{e}^{-M_{s,t}(g)}\big)\big)\mathrm{e}^{\lambda t}=\int_{\mathbb{R}_0}\mathbb{E}\left[\sum_{u\in D_s}\tau_{u,s}1-\mathrm{e}^{-Z_s^u g(x)}\right]\nu_\alpha(\mathrm{d}x). \tag{2.17}$$

By the definition of $M_{s,t}$, we have

$$\Big(1-\mathbb{E}\Big(\mathrm{e}^{-M_{s,t}(g)}\mid \mathcal{F}_s^{\mathbb{T}}\Big)\Big)\mathrm{e}^{\lambda t}=\mathrm{e}^{\lambda t}\Bigg(1-\prod_{u\in D_s}\mathbb{E}\Big(\mathrm{e}^{-Z_s^u g\left(h_t^{-1}X_{u,s}\right)}\mid \mathcal{F}_s^{\mathbb{T}}\Big)\Bigg).$$

Note that, given $\mathcal{F}_s^{\mathbb{T}}$, $X_{u,s}\overset{\mathrm{d}}{=}\xi_{\tau_{u,s}}$. Thus, by Lemma 2.1 (with s replaced by $\tau_{u,s}$ and g replaced by $1-\mathrm{e}^{-Z_s^u g(x)}$),

$$\mathrm{e}^{\lambda t}\Big(1-\mathbb{E}\Big[\mathrm{e}^{-Z_s^u g\left(h_t^{-1}X_{u,s}\right)}\mid \mathcal{F}_s^{\mathbb{T}}\Big]\Big)\to\tau_{u,s}\int_{\mathbb{R}_0}1-\mathrm{e}^{-Z_s^u g(x)}\,\nu_\alpha(\mathrm{d}x)\quad\text{as } t\to\infty.$$

Hence, it follows from Lemma 2.7 that

$$\lim_{t\to\infty}\mathrm{e}^{\lambda t}\big(1-\mathbb{E}\big[\mathrm{e}^{-M_{s,t}(g)}\mid \mathcal{F}_s^{\mathbb{T}}\big]\big)=\int_{\mathbb{R}_0}\sum_{u\in D_s}\tau_{u,s}\big[1-\mathrm{e}^{-Z_s^u g(x)}\big]\,\nu_\alpha(dx). \tag{2.18}$$

Moreover, for $h_t\delta\geq 2+2a_{0.5}$,

$$\begin{aligned}\mathrm{e}^{\lambda t}\big(1-\mathbb{E}\big[\mathrm{e}^{-M_{s,t}(g)}\mid \mathcal{F}_s^{\mathbb{T}}\big]\big)&\leq \mathrm{e}^{\lambda t}\mathbb{E}\big(M_{s,t}(g)\mid \mathcal{F}_s^{\mathbb{T}}\big)\\&\leq \|g\|_\infty \mathrm{e}^{\lambda t}\sum_{u\in D_s}Z_s^u G_{\tau_{u,s}}(h_t\delta)\\&\leq c_0\|g\|_\infty\delta^{-\alpha}\mathrm{e}^{\lambda t}h_t^{-\alpha}L(h_t\delta)\sum_{u\in D_s}\tau_{u,s}Z_s^u\\&\leq C\sum_{u\in D_s}\tau_{u,s}Z_s^u,\end{aligned} \tag{2.19}$$

where C is a constant not depending on t. The third inequality follows from Lemma 2.2, and the final inequality from the fact that $\mathrm{e}^{\lambda t}h_t^{-\alpha}L(h_t\delta)\to 1$. Since $\tau_{u,s}=\int_0^s \mathbf{1}_{(b_u,\sigma_u)}(r)\,\mathrm{d}r$,

$$\begin{aligned}\mathbb{E}\Bigg(\sum_{u\in D_s}\tau_{u,s}Z_s^u\Bigg)&=\int_0^s\mathbb{E}\Bigg(\sum_{u\in D_s}\mathbf{1}_{(b_u,\sigma_u)}(r)Z_s^u\Bigg)\,\mathrm{d}r\\&=\int_0^s\mathbb{E}\Bigg(\sum_{u\in\mathcal{L}_r-\{o\}}Z_s^u\Bigg)\,\mathrm{d}r\leq\int_0^s\mathbb{E}(Z_s)\,\mathrm{d}r=s\mathrm{e}^{\lambda s}<\infty.\end{aligned}$$

Thus, by (2.18), (2.19), and the dominated convergence theorem, the claim (2.17) holds.

By (2.17) and the fact that $\lim_{t\to\infty}\mathrm{e}^{-\lambda t}Z_{t-s}=\mathrm{e}^{-\lambda s}W$, we have

$$\lim_{t\to\infty}\big[\mathbb{E}\big(\mathrm{e}^{-M_{s,t}(g)}\big)\big]^{Z_{t-s}}=\exp\Bigg\{-\mathrm{e}^{-\lambda s}W\int_{\mathbb{R}_0}\mathbb{E}\Bigg[\sum_{u\in D_s}\tau_{u,s}\big(1-\mathrm{e}^{-Z_s^u g(x)}\big)\Bigg]\nu_\alpha(\mathrm{d}x)\Bigg\}.$$

Thus, by (2.16) and the bounded convergence theorem,

$$\lim_{t\to\infty}\mathbb{E}\big(\mathrm{e}^{-\widetilde{\mathcal{N}}_{s,t}(g)}\big)=\mathbb{E}\Bigg(\exp\Bigg\{-\mathrm{e}^{-\lambda s}W\int_{\mathbb{R}_0}\mathbb{E}\Bigg[\sum_{u\in D_s}\tau_{u,s}\big(1-\mathrm{e}^{-Z_s^u g(x)}\big)\Bigg]\nu_\alpha(\mathrm{d}x)\Bigg\}\Bigg).$$

By the definition of $\tau_{u,s}$, we have

$$\sum_{u\in D_s}\tau_{u,s}\big(1-\mathrm{e}^{-Z_s^u g(x)}\big)=\sum_{u\in D_s}\int_0^s\mathbf{1}_{(b_u,\sigma_u)}(r)\,\mathrm{d}r\big(1-\mathrm{e}^{-Z_s^u g(x)}\big)=\int_0^s\sum_{u\in\mathcal{L}_r\setminus\{o\}}\big(1-\mathrm{e}^{-Z_s^u g(x)}\big)\,\mathrm{d}r.$$

Using the Markov property and the branching property, the Z_s^u, $u\in\mathcal{L}_r$, are i.i.d. with the same distribution as Z_{s-r}, and independent of $\mathcal{L}_r$. Thus,

$$\mathbb{E}\sum_{u\in D_s}\tau_{u,s}\big(1-\mathrm{e}^{-Z_s^u g(x)}\big)=\int_0^s\mathbb{E}\big(Z_r-\mathbf{1}_{\{o\in\mathcal{L}_r\}}\big)\mathbb{E}\big(1-\mathrm{e}^{-Z_{s-r}g(x)}\big)\,\mathrm{d}r$$
$$=\int_0^s(\mathrm{e}^{\lambda r}-\mathrm{e}^{-\beta r})\mathbb{E}\big(1-\mathrm{e}^{-Z_{s-r}g(x)}\big)\,\mathrm{d}r,$$

which implies that

$$\mathrm{e}^{-\lambda s}\mathbb{E}\sum_{u\in D_s}\tau_{u,s}\big(1-\mathrm{e}^{-Z_s^u g(x)}\big)\to\int_0^\infty\mathrm{e}^{-\lambda r}\mathbb{E}\big(1-\mathrm{e}^{-Z_r g(x)}\big)\,\mathrm{d}r$$

and

$$\mathrm{e}^{-\lambda s}\mathbb{E}\sum_{u\in D_s}\tau_{u,s}\big(1-\mathrm{e}^{-Z_s^u g(x)}\big)\le\int_0^\infty\mathrm{e}^{-\lambda r}\mathbb{E}\big(1-\mathrm{e}^{-Z_r g(x)}\big)\,\mathrm{d}r\le\lambda^{-1}\mathbf{1}_{\{|x|>\delta\}}.$$

The final inequality follows from the fact that $\mathrm{Supp}(g)\subset\{x:|x|>\delta\}$. Since $\nu_\alpha(\mathbf{1}_{\{|x|>\delta\}})<\infty$, using the dominated convergence theorem we get

$$\lim_{s\to\infty}\mathrm{e}^{-\lambda s}\int_{\mathbb{R}_0}\mathbb{E}\left[\sum_{u\in D_s}\tau_{u,s}\big(1-\mathrm{e}^{-Z_s^u g(x)}\big)\right]\nu_\alpha(\mathrm{d}x)=\int_0^\infty\mathrm{e}^{-\lambda r}\int_{\mathbb{R}_0}\mathbb{E}\big(1-\mathrm{e}^{-Z_r g(x)}\big)\,\nu_\alpha(\mathrm{d}x)\,\mathrm{d}r,$$

which implies that

$$\lim_{s\to\infty}\lim_{t\to\infty}\mathbb{E}\big(\mathrm{e}^{-\widetilde{\mathcal{N}}_{s,t}(g)}\big)=\mathbb{E}\left(\exp\left\{-W\int_0^\infty\mathrm{e}^{-\lambda r}\int_{\mathbb{R}_0}\mathbb{E}\big(1-\mathrm{e}^{-Z_r g(x)}\big)\,\nu_\alpha(\mathrm{d}x)\,\mathrm{d}r\right\}\right).$$

By Lemmas 2.6 and 2.4, $\lim_{t\to\infty}\mathbb{E}\Big(\mathrm{e}^{-\widetilde{\mathcal{N}}_t(g)}\Big)=E\big(\mathrm{e}^{-\mathcal{N}_\infty(g)}\big)$. The proof is now complete. □

3. Joint convergence of the order statistics

Proof of Corollary 1.1. Since $q_1>0$, we have, for all $k\ge1$, $M_{(k)}>0$, P^*-a.s.

Note that, for any $x\in\overline{\mathbb{R}}_0$, $\mathcal{N}_\infty(\{x\})=0$, a.s. Since $\{M_{t,k}\le h_t x\}=\{\mathcal{N}_t(x,\infty)\le k-1\}$ for any $x>0$, by Remark 1.1 with $B_k=(x_k,\infty)$, we have, for any $n\ge1$ and $x_1,x_2,x_3,\ldots,x_n>0$,

$$\begin{aligned}
&\mathbb{P}(M_{t,1}\le h_t x_1, M_{t,2}\le h_t x_2, M_{t,3}\le h_t x_3,\ldots,M_{t,n}\le h_t x_n)\\
&\quad=\mathbb{P}(\mathcal{N}_t(x_k,\infty)\le k-1, k=1,\ldots,n)\\
&\quad\to P(\mathcal{N}_\infty(x_k,\infty)\le k-1, k=1,\ldots,n)\\
&\quad=P\big(M_{(1)}\le x_1, M_{(2)}\le x_2, M_{(3)}\le x_3,\ldots,M_{(n)}\le x_n\big)\quad\text{as } t\to\infty.
\end{aligned}$$

Thus, as $t\to\infty$,

$$\begin{aligned}
&\mathbb{P}^*\big(M_{t,1}\le h_t x_1, M_{t,2}\le h_t x_2,\ldots,M_{t,n}\le h_t x_n\big)\\
&\quad=\mathbb{P}(\mathcal{S})^{-1}\big[\mathbb{P}\big(M_{t,k}\le h_t x_k, k=1,\ldots,n\big)-\mathbb{P}\big(M_{t,k}\le h_t x_k, k=1,\ldots,n,\mathcal{S}^c\big)\big]\\
&\quad\to\mathbb{P}(\mathcal{S})^{-1}\big[P\big(M_{(k)}\le x_k, k=1,\ldots,n\big)-\mathbb{P}\big(\mathcal{S}^c\big)\big]\\
&\quad=P^*\big(M_{(k)}\le x_k, k=1,\ldots,n\big),
\end{aligned}\tag{3.1}$$

where in the final equality we used the fact that on the event of extinction, $M_{(k)}=-\infty$, $k\ge1$.

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Now we consider the case $x_1, \ldots, x_n \in \mathbb{R}$ with $x_i \le 0$ for some i and $x_j > 0, j \ne i$. By (3.1), for any $\varepsilon > 0$,

$$\limsup_{t\to\infty} \mathbb{P}^*\big(M_{t,1} \le h_t x_1, M_{t,2} \le h_t x_2, \ldots, M_{t,n} \le h_t x_n\big)$$
$$\le \lim_{t\to\infty} \mathbb{P}^*\big(M_{t,j} \le h_t x_j, j \ne i, M_{t,i} \le h_t \varepsilon\big) = P^*\big(M_{(j)} \le x_j, j \ne i, M_{(i)} \le \varepsilon\big).$$

The right-hand side of the display above tends to 0 as $\varepsilon \to 0$ since $M_{(i)} > 0$ a.s. Thus,

$$\lim_{t\to\infty} \mathbb{P}^*(M_{t,k} \le h_t x_k, k = 1, \ldots, n) = 0 = P^*(M_{(k)} \le x_k, k = 1, \ldots, n).$$

Similarly, this can be shown to hold for any $x_1, \ldots, x_n \in \mathbb{R}$.

The proof is now complete. □

4. Examples and an extension

This section provides more examples satisfying (H2) and an extension.

Lemma 4.1. *Assume that L^* is a positive function on $(0, \infty)$ slowly varying at ∞ such that $l_\varepsilon(x) := \sup_{y\in(0,x]} y^\varepsilon L^*(y) < \infty$ for any $\varepsilon > 0$ and $x > 0$. Then, for any $\varepsilon > 0$, there exist $c_\varepsilon, C_\varepsilon > 0$ such that, for any $y > 0$ and $a > c_\varepsilon$,*

$$\frac{L^*(ay)}{L^*(a)} \le C_\varepsilon\big(y^\varepsilon + y^{-\varepsilon}\big).$$

Proof. By [11, Theorem 1.5.6], for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that, for any $a \ge c_\varepsilon$ and $y \ge a^{-1}c_\varepsilon$,

$$\frac{L^*(ay)}{L^*(a)} \le (1-\varepsilon)^{-1} \max\big\{y^\varepsilon, y^{-\varepsilon}\big\}. \tag{4.1}$$

Thus, for any $a > c_\varepsilon$,

$$\frac{L^*(c_\varepsilon)}{L^*(a)} \le (1-\varepsilon)^{-1}(a/c_\varepsilon)^\varepsilon.$$

Hence, for $a > c_\varepsilon$ and $0 < y \le a^{-1}c_\varepsilon$,

$$\frac{L^*(ay)}{L^*(a)} \le \frac{l_\varepsilon(c_\varepsilon)(ay)^{-\varepsilon}}{L^*(a)} \le \frac{l_\varepsilon(c_\varepsilon)}{L^*(c_\varepsilon)(1-\varepsilon)c_\varepsilon^\varepsilon} y^{-\varepsilon}. \tag{4.2}$$

Combining (4.1) and (4.2), there exists $C_\varepsilon > 0$ such that, for any $y > 0$ and $a > c_\varepsilon$,

$$\frac{L^*(ay)}{L^*(a)} \le C_\varepsilon\big(y^\varepsilon + y^{-\varepsilon}\big).$$ □

Example 4.1. Let $n(\mathrm{d}y) = c_1 x^{-(1+\alpha)} L^*(x)\mathbf{1}_{(0,\infty)}(x)\mathrm{d}x + c_2|x|^{-(1+\alpha)}L^*(|x|)\mathbf{1}_{(-\infty,0)}(x)\mathrm{d}x$, where $\alpha \in (0, 2)$, $c_1, c_2 \ge 0$, $c_1 + c_2 > 0$, and L^* is a positive function on $(0, \infty)$ slowly varying at ∞ such that $\sup_{y\in(0,x]} y^\varepsilon L^*(y) < \infty$ for any $\varepsilon > 0$ and $x > 0$.

(i) For $\alpha \in (0, 1)$, assume that the Lévy exponent of ξ has the form

$$\psi(\theta) = \mathrm{i}a\theta - b^2\theta^2 + \int \big(\mathrm{e}^{\mathrm{i}\theta y} - 1\big)\, n(\mathrm{d}y),$$

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where $a \in \mathbb{R}$, $b \geq 0$. Using Lemma 4.1 with $\varepsilon \in (0, (1-\alpha) \wedge \alpha)$ we have, by the dominated convergence theorem, as $\theta \to 0_+$,

$$\int_0^\infty \left(e^{i\theta y} - 1\right) n(dy) = \theta^\alpha \int_0^\infty \left(e^{iy} - 1\right) y^{-1-\alpha} L^*\left(\theta^{-1} y\right) dy$$
$$\sim \theta^\alpha L^*\left(\theta^{-1}\right) \int_0^\infty \left(e^{iy} - 1\right) y^{-1-\alpha}\, dy = -\alpha\Gamma(1-\alpha) e^{-i\pi\alpha/2} \theta^\alpha L^*\left(\theta^{-1}\right),$$
$$\int_{-\infty}^0 \left(e^{i\theta y} - 1\right) n(dy) = \theta^\alpha \int_0^\infty \left(e^{-iy} - 1\right) y^{-1-\alpha} L^*\left(\theta^{-1} y\right) dy$$
$$\sim \theta^\alpha L^*\left(\theta^{-1}\right) \int_0^\infty \left(e^{-iy} - 1\right) y^{-1-\alpha}\, dy = -\alpha\Gamma(1-\alpha) e^{i\pi\alpha/2} \theta^\alpha L^*\left(\theta^{-1}\right).$$

Thus as $\theta \to 0_+$, $\psi(\theta) \sim -\alpha\Gamma(1-\alpha)\left(e^{-i\pi\alpha/2} c_1 + e^{i\pi\alpha/2} c_2\right) \theta^\alpha L^*\left(\theta^{-1}\right)$.

(ii) For $\alpha \in (1, 2)$, assume that the Lévy exponent of ξ has the form

$$\psi(\theta) = -b^2\theta^2 + \int \left(e^{i\theta y} - 1 - i\theta y\right) n(dy),$$

where $b \geq 0$. Using Lemma 4.1 with $\varepsilon \in (0, (2-\alpha) \wedge (\alpha - 1))$, we have, by the dominated convergence theorem, as $\theta \to 0_+$,

$$\int_0^\infty \left(e^{i\theta y} - 1 - i\theta y\right) n(dy) = \theta^\alpha \int_0^\infty \left(e^{iy} - 1 - iy\right) y^{-1-\alpha} L^*\left(\theta^{-1} y\right) dy$$
$$\sim \theta^\alpha L^*\left(\theta^{-1}\right) \int_0^\infty \left(e^{iy} - 1 + iy\right) y^{-1-\alpha}\, dy$$
$$= -\alpha\Gamma(1-\alpha) e^{-i\pi\alpha/2} \theta^\alpha L^*\left(\theta^{-1}\right),$$
$$\int_{-\infty}^0 \left(e^{i\theta y} - 1 - i\theta y\right) n(dy) = \theta^\alpha \int_0^\infty \left(e^{-iy} - 1 + iy\right) y^{-1-\alpha} L^*\left(\theta^{-1} y\right) dy$$
$$\sim \theta^\alpha L^*\left(\theta^{-1}\right) \int_0^\infty \left(e^{-iy} - 1 + iy\right) y^{-1-\alpha}\, dy$$
$$= -\alpha\Gamma(1-\alpha) e^{i\pi\alpha/2} \theta^\alpha L^*\left(\theta^{-1}\right).$$

Thus, as $\theta \to 0_+$, $\psi(\theta) \sim -\alpha\Gamma(1-\alpha)\left(e^{-i\pi\alpha/2} c_1 + e^{i\pi\alpha/2} c_2\right) \theta^\alpha L^*\left(\theta^{-1}\right)$.

(iii) For $\alpha = 1$, assume that $c_1 = c_2$ and the Lévy exponent of ξ has the form

$$\psi(\theta) = ia\theta - b^2\theta^2 + \int \left(e^{i\theta y} - 1 - i\theta y \mathbf{1}_{|y| \leq 1}\right) n(dy),$$

where $a \in \mathbb{R}$, $b \geq 0$. Since $c_1 = c_2$, we have

$$\int_{-\infty}^\infty \left(e^{i\theta y} - 1 - i\theta y \mathbf{1}_{|y| \leq 1}\right) n(dy) = -2c_1\theta \int_0^\infty (1 - \cos y) y^{-2} L^*\left(\theta^{-1} y\right) dy.$$

Using Lemma 4.1 with $\varepsilon \in (0, 1)$, we have, by the dominated convergence theorem,

$$\lim_{\theta \to 0_+} L^*\left(\theta^{-1}\right)^{-1} \int_0^\infty (1 - \cos y) y^{-2} L^*\left(\theta^{-1} y\right) dy = \int_0^\infty (1 - \cos y) y^{-2}\, dy = \pi/2,$$

which implies that as $\theta \to 0_+$, $\psi(\theta) \sim -(c_1\pi - ia)\theta L^*\left(\theta^{-1}\right)$.

Remark 4.1. (*An extension.*) Checking the proof of Theorem 1.1, we see that Theorem 1.1 holds for more general branching Lévy processes with spatial motions satisfying the following assumptions:

(A1) There exist a non-increasing function h_t with $h_t \uparrow \infty$ and a measure $\pi(\mathrm{d}x) \in \mathcal{M}(\overline{\mathbb{R}}_0)$ such that

$$\lim_{t\to\infty} \mathrm{e}^{\lambda t}\mathrm{E}\big(g\big(h_t^{-1}\xi_s\big)\big) = s\int_{\mathbb{R}_0} g(x)\,\pi(\mathrm{d}x), \qquad g \in C_{\mathrm{c}}^{+}\big(\overline{\mathbb{R}}_0\big).$$

(A2) $\mathrm{e}^{\lambda t}p_t^2 \to 0$, where $p_t := \sup_{s\le t} \mathrm{P}(|\xi_s| > h_t\theta/t)$.

(A3) For any $\theta > 0$, $\sup_{t>1}\sup_{s\le t} s^{-1}\mathrm{e}^{\lambda t}\mathrm{P}(|\xi_s| > h_t\theta) < \infty$.

First, (H2) implies (A1)–(A3). Next, we explain that Theorem 1.1 holds under assumptions (A1)–(A3). Checking the proof of Lemma 2.5, we see that Lemma 2.5 holds under conditions (A1)–(A3). In fact, we may replace Lemma 2.2 by (A2) to get (2.3) (see (2.5) and (2.6)). For the proof of Lemma 2.6, using (A3) we get that $\mathbb{P}\big(J_{s,t}^{\mathrm{c}}\big) \le C\mathrm{e}^{-\lambda t}\mathbb{E}\sum_{u:b_u\le t-s}\tau_{u,t}$, which says that (2.10) holds. Thus, (2.9) holds using the same arguments as in Lemma 2.6. Replacing Lemma 2.1 by (A1), we see that Proposition 2.1 holds with ν_α replaced by $\pi(\mathrm{d}x)$. So, under (A1)–(A3), Theorem 1.1 holds with ν_α replaced by $\pi(\mathrm{d}x)$.

An easy example which satisfies (A1)–(A3) but not (H2) is the non-symmetric 1-stable process. Assume ξ is a non-symmetric 1-stable process with Lévy measure $n(\mathrm{d}x) = c_1x^{-2}\mathbf{1}_{(0,\infty)}(x)\,\mathrm{d}x + c_2|x|^{-2}\mathbf{1}_{(-\infty,0)}(x)\,\mathrm{d}x$, where $c_1, c_2 \ge 0$, $c_1 + c_2 > 0$, and $c_1 \ne c_2$. The Lévy exponent of ξ is given, for $\theta > 0$, by

$$\psi(\theta) = -\frac{\pi}{2}(c_1+c_2)\theta - \mathrm{i}(c_1-c_2)\theta\log\theta + \mathrm{i}a(c_1-c_2)\theta \sim -\mathrm{i}(c_1-c_2)\theta\log\theta, \qquad \theta \to 0+,$$

where a is constant. Thus, $c_* = \mathrm{i}(c_1 - c_2)$. So $\psi(\theta)$ does not satisfy (H2) since $\mathrm{Re}(c_*) = 0$.

By [7, Section 1.5, Exercise 1], $(1/t)\mathrm{P}(\xi_t \in \cdot) \xrightarrow{\mathrm{v}} n(\mathrm{d}x)$ as $t \to 0$. Since $\mathrm{e}^{-\lambda t}\xi_s \overset{\mathrm{d}}{=} \xi_{s\mathrm{e}^{-\lambda t}} + (c_1 - c_2)s\lambda t\mathrm{e}^{-\lambda t}$ for $s, t > 0$, we have $\mathrm{e}^{\lambda t}\mathrm{P}(\mathrm{e}^{-\lambda t}\xi_s \in \cdot) \xrightarrow{\mathrm{v}} s\,n(\mathrm{d}x)$ as $t \to \infty$. So (A1) holds with $h_t = \mathrm{e}^{\lambda t}$. We claim that, for any $x > 0$ and $s > 0$,

$$\mathrm{P}(|\xi_s| > x) \le c\big(sx^{-1} + s^2x^{-2} + s^2x^{-2}(\log x)^2\big), \tag{4.3}$$

where c is a constant. Thus it is easy to prove that (A2) and (A3) hold.

In fact, for any $x > 0$,

$$\mathrm{P}(|\xi_s| > x) \le \frac{x}{2}\int_{-2x^{-1}}^{2x^{-1}} \big(1 - \mathrm{e}^{s\psi(\theta)}\big)\,\mathrm{d}\theta = x\int_0^{2x^{-1}} \big(1 - \mathrm{Re}\big(\mathrm{e}^{s\psi(\theta)}\big)\big)\,\mathrm{d}\theta.$$

Note that

$$\begin{aligned} 1 - \mathrm{Re}\big(\mathrm{e}^{s\psi(\theta)}\big) &= 1 - \mathrm{e}^{s\mathrm{Re}(\psi(\theta))}\cos[s\mathrm{Im}(\psi(\theta))] \\ &= 1 - \mathrm{e}^{s\mathrm{Re}(\psi(\theta))} + \mathrm{e}^{s\mathrm{Re}(\psi(\theta))}(1 - \cos[s\mathrm{Im}(\psi(\theta))]) \\ &\le -s\mathrm{Re}(\psi(\theta)) + s^2[\mathrm{Im}(\psi(\theta))]^2 \\ &= \frac{\pi}{2}(c_1+c_2)s\theta + (c_1-c_2)^2s^2(a - \log\theta)^2\theta^2. \end{aligned}$$

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Thus, we have

$$\begin{aligned}
\mathrm{P}(|\xi_s| > x) &\le \pi(c_1+c_2)sx^{-1} + (c_1-c_2)^2 s^2 x^{-2} \int_0^2 (a - \log\theta + \log x)^2 \theta^2 \,\mathrm{d}\theta \\
&\le \pi(c_1+c_2)sx^{-1} + 2(c_1-c_2)^2 s^2 x^{-2} \int_0^2 [(a-\log\theta)^2 + (\log x)^2]\theta^2 \,\mathrm{d}\theta \\
&\le c\big(sx^{-1} + s^2x^{-2} + s^2x^{-2}(\log x)^2\big),
\end{aligned}$$

which proves the claim (4.3).

5. Frontal position of Fisher–KPP equation

The Fisher-KPP equation related to our branching Lévy process is given by

$$\begin{cases} \partial_t u - \mathcal{A}u = -\varphi(1-u) \text{ in } (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x), \qquad\qquad x \in \mathbb{R}, \end{cases} \tag{5.1}$$

where $\mathcal{A}$ is the generator of the Lévy process $\{(\xi_t)_{t\ge 0}, \mathrm{P}\}$, $\varphi(s) = \beta\big(\sum_k s^k p_k - s\big)$, $u_0(x) \in [0, 1]$, $x \in \mathbb{R}$; see, for instance, [17].

Recall that, for any $g \in C_{\mathrm{b}}^{+}(\mathbb{R})$, $u_g(t,x) = \mathbb{E}\big(\exp\big\{-\sum_{v\in\mathcal{L}_t} g\big(\xi_t^v + x\big)\big\}\big)$ satisfies (1.5), and thus is a mild solution of the following Cauchy problem:

$$\begin{cases} \partial_t u - \mathcal{A}u = \varphi(u) \text{ in } (0,\infty)\times\mathbb{R}, \\ u(0,x) = \mathrm{e}^{-g(x)}, \ \ x \in \mathbb{R}. \end{cases}$$

Hence $1-u_g(t,x)$ is a mild solution to (5.1) with $u_0(x) = 1-\mathrm{e}^{-g(x)}$.

We are interested in the large-time behavior of $1-u_g(t,x)$. For $\theta \in (0, 1)$, the level set $\{x \in \mathbb{R} : 1-u_g(t,x) = \theta\}$ is also called the front of $1-u_g$. The evolution of the front of $1-u_g$ as time goes to ∞ is of considerable interest. Using analytic methods, it was shown in [5] that, if ξ is a standard Brownian motion, the frontal position of branching Brownian motion is $\sqrt{2\lambda}t$, with λ given by (1.2). More precisely, under the condition that g is compactly supported, if $c > \sqrt{2\lambda}$, then $1-u_g(t,x) \to 0$ uniformly in $\{|x| \ge ct\}$ as $t\to\infty$; if $c < \sqrt{2\lambda}$, then $1-u_g(t,x)\to 1$ uniformly in $\{|x| \le ct\}$ as $t\to\infty$. But if the density of ξ is comparable to that of a symmetric α-stable process, [17, Theorem 1.5] proved that the frontal position is exponential in time; see Remark 5.1 for the precise meaning. In this paper we provide a probabilistic proof of [17, Theorem 1.5] using Corollary 1.2, and also partially generalize it.

Proposition 5.1. *Assume that* $q_1 > 0$.

(i) *Assume that* a_t *satisfies* $a_t/h_t \to \infty$ *as* $t\to\infty$, *and that* g *is a non-negative function satisfying*

$$\mathrm{e}^{\lambda t} \sup_{x \le -a_t/2} g(x) \to 0 \qquad \text{as } t \to \infty. \tag{5.2}$$

Then $\lim_{t\to\infty} \sup_{x\le -a_t} (1-u_g(t,x)) = 0$.

(ii) *Assume that* c_t *satisfies* $c_t/h_t \to 0$ *as* $t\to\infty$, *and that* g *is a non-negative function satisfying* $a_0 := \liminf_{x\to\infty} g(x) > 0$. *Then*

$$\lim_{t\to\infty} \sup_{x \ge -c_t} |u_g(t,x) - \mathbb{P}(\mathcal{S}^{\mathrm{c}})| = 0.$$

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Proof. (i) Let $g^*(x) = \sup_{y\le -x} g(y)$. Note that, for $x \le -a_t$,

$$\begin{aligned}1 - u_g(t, x) &= \mathbb{E}\left(1 - \exp\left\{-\sum_{v\in\mathcal{L}_t} g\big(\xi_t^v + x\big)\right\}\right)\\ &\le \mathbb{P}(R_t \ge a_t/2) + \mathbb{E}\left(1 - \exp\left\{-\sum_{v\in\mathcal{L}_t} g\big(\xi_t^v + x\big)\right\}; R_t < a_t/2\right)\\ &\le \mathbb{P}(R_t \ge a_t/2) + \mathbb{E}\big(1 - \mathrm{e}^{-g^*(a_t/2)Z_t}\big)\\ &\le \mathbb{P}(R_t \ge a_t/2) + \mathrm{e}^{\lambda t} g^*(a_t/2), \end{aligned} \tag{5.3}$$

where in the second inequality we used the fact that, on the event $\{R_t < a_t/2\}$, $\xi_t^v + x < a_t/2 - a_t = -a_t/2$ and $g\big(\xi_t^v + x\big) \le g^*(a_t/2)$. By the assumption (5.2), $\mathrm{e}^{\lambda t} g^*(a_t/2) \to 0$. By Corollary 1.2, $\mathbb{P}^*(R_t \ge a_t/2) \to 0$. Thus

$$\mathbb{P}(R_t \ge a_t/2) \le \mathbb{P}^*(R_t \ge a_t/2)\mathbb{P}(\mathcal{S}) + \mathbb{P}(\|X_t\| > 0, \mathcal{S}^c) \to 0$$

as $t \to \infty$. Thus, by (5.3), $\lim_{t\to\infty} \sup_{x\le -a_t} (1 - u_g(t, x)) = 0$.

(ii) Note that

$$|u_g(t, x) - \mathbb{P}(\mathcal{S}^c)| \le \mathbb{E}\left(\exp\left\{-\sum_{v\in\mathcal{L}_t} g\big(\xi_t^v + x\big)\right\}; \mathcal{S}\right) + \mathbb{E}\left(1 - \exp\left\{-\sum_{v\in\mathcal{L}_t} g\big(\xi_t^v + x\big)\right\}; \mathcal{S}^c\right).$$

Noticing that on the event $Z_t = 0$, $1 - \exp\big\{-\sum_{v\in\mathcal{L}_t} g\big(\xi_t^v + x\big)\big\} = 0$, we get, for any $x \in \mathbb{R}$, $\mathbb{E}\big(1 - \exp\big\{-\sum_{v\in\mathcal{L}_t} g\big(\xi_t^v + x\big)\big\}; \mathcal{S}^c\big) \le \mathbb{P}(Z_t > 0; \mathcal{S}^c) \to 0$ as $t \to \infty$. Let $g_*(x) = \inf_{y\ge x} g(y)$. Since $c_t/h_t \to 0$ for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $c_t \le \varepsilon h_t$ for $t > t_\varepsilon$. For any $t > t_\varepsilon$ and $x \ge -c_t$,

$$\begin{aligned}\mathbb{E}\left(\exp\left\{-\sum_{v\in\mathcal{L}_t} g\big(\xi_t^v + x\big)\right\}; \mathcal{S}\right) &\le \mathbb{E}\left(\exp\left\{-g_*(c_t)\sum_{v\in\mathcal{L}_t} \mathbf{1}_{\xi_t^v > 2c_t}\right\}; \mathcal{S}\right)\\ &\le \mathbb{E}\left(\exp\left\{-g_*(c_t)\sum_{v\in\mathcal{L}_t} \mathbf{1}_{\xi_t^v > 2\varepsilon h_t}\right\}; \mathcal{S}\right)\\ &= \mathbb{E}\big(\mathrm{e}^{-g_*(c_t)\mathcal{N}_t(2\varepsilon,\infty)}; \mathcal{S}\big).\end{aligned}$$

Thus

$$\limsup_{t\to\infty} \sup_{x\ge -c_t} |u_g(t, x) - \mathbb{P}(\mathcal{S}^c)| \le E(\mathrm{e}^{-a_0\mathcal{N}_\infty(2\varepsilon,\infty)}, \mathcal{S}). \tag{5.4}$$

Since on the event $\mathcal{S}$, $\vartheta W \nu_\alpha(0, \infty) = \infty$, we have $\mathcal{N}_\infty(0, \infty) = \infty$. Now letting $\varepsilon \to 0$ in (5.4) we get the desired result. □

Remark 5.1. Proposition 5.1 is a slight generalization of [17, Theorem 1.5]. Assume that $p_0 = 0$, which ensures that $\mathbb{P}(\mathcal{S}^c) = 0$. If $L = 1$, then $h_t = \mathrm{e}^{\lambda t/\alpha}$, and we have the following results:

(i) Let g be a non-negative measurable function satisfying

$$g(x) \le C|x|^{-\alpha}, \qquad x < 0. \tag{5.5}$$

Then, for any $\gamma > \lambda/\alpha$, $\mathrm{e}^{\lambda t} g^*(-\mathrm{e}^{\gamma t}/2) \le C2^\alpha \mathrm{e}^{\lambda t}\mathrm{e}^{-\alpha\gamma t} \to 0$. Thus, by Proposition 5.1, $\lim_{t\to\infty} \sup_{x\le -\mathrm{e}^{\gamma t}} (1 - u_g(t, x)) = 0$.

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(ii) Assume that g is a non-negative function satisfying $a_0 := \liminf_{x\to\infty} g(x) > 0$. For any $\gamma < \lambda/\alpha$, by Proposition 5.1 we have $\lim_{t\to\infty} \sup_{x\geq -e^{\gamma t}} u_g(t, x) = 0$.

Note that in the notation of [17], $\sigma^{**} = \lambda/\alpha$, and our condition (5.5) is equivalent to $1 - e^{-g(x)} \leq C|x|^{-\alpha}$, $x < 0$, for some constant C. If g is non-decreasing, it is clear that $\liminf_{x\to\infty} g(x) > 0$. Thus, when the Lévy process ξ satisfies (H2) with $L = 1$, we can get that the conclusion of [17, Theorem 1.5] holds from Proposition 5.1. Note that the independent sum of Brownian motion and a symmetric α-stable process satisfies (H2) with $L = 1$, but its transition density is not comparable with that of the symmetric α-stable process, see [22, 39]. Note also that the independent sum of a symmetric α-stable process and a symmetric β-stable process, $0 < \alpha < \beta < 2$, also satisfies (H2) with $L = 1$, but its transition density is not comparable with that of the symmetric α-stable process, see [21]. Note that in this paper we do not need to assume that g is non-decreasing. Thus Proposition 5.1 partially generalizes [17, Theorem 1.5].

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There were no competing interests to declare which arose during the preparation or publication process of this article.

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