Can. J. Math. Vol. 43 (2), 1991 pp. 313–321

THE HÖLDER EXPONENT FOR RADIALLY SYMMETRIC SOLUTIONS OF POROUS MEDIUM TYPE EQUATIONS

GASTON E. HERNANDEZ AND IOANNIS M. ROUSSOS

1. **Introduction.** The density $u(x,t)$ of an ideal gas flowing through a homogeneous porous media satisfies the equation

$$u_t = \Delta u^m \text{ in } \Omega_T = \mathbb{R}^N \times (0,T). \tag{1}$$

Here $m > 1$ is a physical constant and u also satisfies the initial condition

$$u(x,0) = u_0(x) \geq 0 \text{ for } x \in \mathbb{R}^N. \tag{2}$$

If the initial data is not strictly positive it is necessary to work with generalized solutions of the Cauchy problem (1), (2) (see [1]). By a *weak solution* we shall mean a function $u(x,t)$ such that for $T < \infty$, $u \in L^2(\Omega_T)$, $\nabla u^m \in L^2(\Omega_T)$ (in the sense of distributions) and

$$\int_{\Omega_T}\int (u\varphi_t - \nabla u^m \nabla \varphi)\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\,dx = 0 \tag{3}$$

for any continuously differentiable function $\varphi(x,t)$ with compact support in $\mathbb{R}^N \times (0,T)$.

We assume here that $0 \leq u_0(x) \leq M_0$, u_0^{m-1} is Lipschitz and $u_0 \in L^2(\mathbb{R}^N)$. Then (see [8]) there exists a unique solution $u(x,t)$ of (1) and (2). This solution is obtained as the limit of classical solutions $u_\varepsilon(x,t)$ of

$$\begin{aligned} (u_\varepsilon)_t &= \Delta u_\varepsilon^m \\ u_\varepsilon(x,0) &= u_0(x) + \varepsilon. \end{aligned} \tag{4}$$

We let $v(x,t) = \frac{m}{m-1}u^{m-1}(x,t)$ (This is the pressure of the gas up to a multiple), then (1) and (2) becomes

$$\begin{aligned} v_t &= (m-1)v\Delta v + |\nabla v|^2 \\ v(x,0) &= v_0(x) = \frac{m}{m-1}u_0^{m-1}(x). \end{aligned} \tag{5}$$

In the one dimensional case, Aronson [2], proved that if v_0 is a Lipschitz continuous function then v is also Lipschitz continuous with respect to x in $\mathbb{R} \times (0,T)$. Aronson and Caffarelli [3] proved that v is also Lipschitz continuous with respect to t in the same domain. In particular $u(x,t)$ is α-Hölder continuous for any $\alpha \in (0, \frac{1}{m-1})$, i.e., the quotient

$$\frac{|u(x,t) - u(y,\tau)|}{|x-y|^\alpha + |t-\tau|^{\alpha/2}} \tag{6}$$

Received by the editors January 30, 1990.

is bounded in $\mathbb{R} \times (0, T)$ by a constant K that depends only in u_0, m and T.

In higher dimensions Caffarelli and Friedman [5] proved that $u(x, t)$ is continuous with modulus of continuity

$$W(\rho) = C|\log \rho|^{-\epsilon}, \quad N \geq 3, \ 0 < \epsilon < \frac{2}{N}$$

and

$$W(\rho) = 2^{-c}|\log \rho|^{1/2}, \quad N = 2$$

where $\rho = (|x - y|^2 + |t - \tau|)^{1/2}$ is the parabolic distance between (x, t) and (y, τ). Thus if u_0 is α-Hölder continuous for some $\alpha \in (0, 1)$, then $|u(x, t) - u(y, \tau)| \leq W(\rho)$ uniformly in $\mathbb{R}^N \times (0, T)$. The same authors in [6] proved that $u(x, t)$ is actually α-Hölder continuous for some $\alpha \in (0, 1)$, but α is completely unknown.

The more general porous medium equation

$$\begin{aligned} u_t &= \Delta u^m + h(x, t, u)u \\ u(x, 0) &= u_0(x) \geq 0 \end{aligned} \tag{7}$$

was treated by the author in [7] in the case $N = 1$. It is shown there that the corresponding $v(x, t)$ is α-Hölder continuous for any $\alpha \in (0, 1)$ provided v_0 is α-Hölder continuous and h is bounded. (In particular the bound K in (6) does not depend on the modulus of continuity of h.)

Let $r = |x| = (\sum x_i^2)^{1/2}$ be the Euclidean distance in $\mathbb{R}^N$. If $v(r, t)$ is a radially symmetric solution of (5), it satisfies

$$\begin{aligned} v_t &= (m-1)vv_{rr} + v_r^2 + (m-1)(N-1)\frac{vv_r}{r}, \quad r > 0 \\ v(r, 0) &= v_0(r) \geq 0. \end{aligned} \tag{8}$$

We shall first consider the spatial dimension $N = 3$. We are interested in the "bad" case, when $v_0(r)$ has compact support and is possibly 0 at $r = 0$. By using only elementary considerations we show that for a solution $v(r, t)$ of (8), $r^\alpha v(r, t)$ is α-Hölder continuous for $\alpha \in (0, \frac{m-1}{m}]$ in a domain $[0, R] \times [0, T]$. As an application of this result it follows that $v(r, t)$ is α-Hölder continuous for $r \geq r_0 > 0$. Also if $v_0(0) > 0$ then v is α-Hölder continuous in the whole domain Ω_T for $0 < \alpha \leq \frac{m-1}{m}$.

Further we prove the same result for the more general equation

$$\begin{aligned} v_t &= (m-1)vv_{rr} + v_r^2 + (m-1)(N-1)\frac{vv_r}{r} + h(r, t, v)v \\ v(r, 0) &= v_0(r) \end{aligned} \tag{9}$$

that corresponds to $u_t = \Delta u^m + h(x, t, u)u$. Here the bound is also independent of the modulus of continuity of h.

Through this work we shall assume the following:

A1. $v_0(r)$ is a nonnegative Lipschitz continuous function (contant M_0), with compact support in $[0, R_1]$, $v_0(r) \leq M_0$.

A2. $L_1[v_0] = (m-1)v_0v_0'' + (v_0')^2 + 2(m-1)\frac{v_0v_0'}{r}$ is bounded by M_0 for $r > 0$.
Under assumption (A1) there exists a unique classical solution $v = v_\varepsilon$ of the problem

$$(12)\qquad \begin{aligned} v_t &= (m-1)vv_{rr} + v_r^2 + (m-1)(N-1)\frac{vv_r}{r}, \quad r > 0 \\ v(r,0) &= v_0(r) + \varepsilon \end{aligned}$$

$v(r,t)$ has bounded derivatives (depending on ε) and $\varepsilon \le v(r,t) \le M$ (M depends only on v_0 and m). Also $v_\varepsilon(r,t) \longrightarrow v(r,t)$ as $\varepsilon \longrightarrow 0$.

We will consider $\alpha \in (0, \frac{m-1}{m}]$ to be fixed, $R_2 \in [0,R]$ is a point at which v_0 is strictly positive, $v_0(R_2) = \eta > 0$.

2. **Main results.**

THEOREM 1. *Let $v(r,t)$ be a (weak) solution of (8), where $v_0(r)$ satisfies A1, A2. Let $R_2 \in [0,R_1]$, be a point at which $v_0(R_2) = \eta > 0$. Then $r^\alpha v(r,t)$ is α-Hölder continuous in $\bar{\Omega}_{R_2} = [0,R_2] \times [0,T]$.*

PROOF. We shall prove that for a solution $v(r,t)$ of (12), the α-quotient (6) corresponding to $r^\alpha v$ can be estimated independently of ε.

Let $B = (0,R_2) \times (0,R_2) \times (0,T) \times (0,T)$. Since v_r, v_t are bounded (in terms of ε) we can choose $\delta > 0$ small, such that $|v_r|, |v_t| \le \delta^{\alpha-1}$ in $\bar{\Omega}_T = [0,\infty] \times [0,T]$. Define $B_\delta = \{(r,s,t,\tau) \in B \mid |r-s| > \delta \text{ or } |t-\tau| > \delta\}$ and

$$h(r,s,t,\tau) = \frac{|r^\alpha v(r,t) - s^\alpha v(s,\tau)|^\lambda}{|r-s|^2 + A|t-\tau|}$$

where $\lambda = \frac{2}{\alpha}$ and $A = 6mM + 1$.

We also put $w(r,t) = r^\alpha v(r,t)$. Then for $\alpha = \frac{m-1}{m}$, $w(r,t)$ satisfies

$$(13)\qquad \begin{aligned} r^\alpha w_t &= (m-1)ww_{rr} + w_r^2, \quad r > 0 \\ w(r,0) &= r^\alpha v_0(r) \end{aligned}$$

We shall prove that in B_δ, $h(r,s,t,\tau) \le K_1$ independently of δ. Then, since $\lambda = \frac{2}{\alpha}$

$$K_1^{\alpha/2} \ge \frac{|r^\alpha v(r,t) - s^\alpha v(s,\tau)|}{(|r-s|^2 + A|t-\tau|)^{\alpha/2}} \ge \frac{|r^\alpha v(r,t) - s^\alpha v(s,\tau)|}{|r-s|^\alpha + A^{\alpha/2}|t-\tau|^{\alpha/2}}$$

and since $A > 1$, we obtain that $r^\alpha v(r,t)$ is α-Hölder continuous with constant $A^{\alpha/2}K^{\alpha/2}$.

Clearly h is continuous in B_δ. Let us assume that $\max h$ occurs at a point $Q_0 = (r_0, s_0, t_0, \tau_0) \in \bar{B}_\delta$. We look first at the case $t_0 = \tau_0$.

Let $g(r,s,t) = h(r,s,t,t)$. We will use the abbreviations

$$g(r,s,t) = \frac{|w(r,t) - w(s,t)|^\lambda}{(r-s)^2} = \frac{|w_1 - w_2|^\lambda}{(r-s)^2} = |S|^\lambda R^{-2}.$$

LEMMA 1. *$g(r,s,t)$ is bounded independently of δ in $\bar{B}_{\delta,1}$ where* $B_{\delta,1} = \{(r,s,t) \mid 0 < r < s < R_2, 0 < t \leq T, |r-s| > \delta\}$.

PROOF. Clearly g is continuous in $\bar{B}_{\delta,1}$. Let us assume that $\max g$ occurs at a point $Q_1 = (r,s,t) \in \bar{B}_{\delta,1}$. Then either Q_1 is an interior point at which g is differentiable or Q_1 is a boundary point of $\bar{B}_{\delta,1}$ (g is not differentiable only when $g = 0$). We begin with the former case. At Q_1 we have

$$g_r = g_s = 0, g_{rr}, g_{ss} \leq 0 \text{ and } g_t \geq 0. \tag{14}$$

The first derivatives are:

$$\begin{aligned} g_r &= \lambda|S|^{\lambda-1}\sigma w_{1r}R^{-2} - 2|S|^{\lambda}R^{-3}, \quad \sigma = \operatorname{sgn} S \\ g_s &= -\lambda|S|^{\lambda y}\sigma w_{2s}R^{-2} + 2|S|^{\lambda}R^{-3} \\ g_t &= \lambda|S|^{\lambda-1}\sigma(w_{1t} - w_{2t})R^{-2} \end{aligned} \tag{15}$$

Thus, (14) implies

$$w_{1r} = \frac{2\sigma}{\lambda}|S|R^{-1} = w_{2s} \tag{16}$$

and

$$\begin{aligned} g_{rr} &= 2|S|^{\lambda}R^{-4}(1 - \frac{2}{\lambda}) + \lambda|S|^{\lambda-1}\sigma R^{-2}w_{rr} \\ g_{ss} &= 2|S|^{\lambda}R^{-4}(1 - \frac{2}{\lambda}) - \lambda|S|^{\lambda-1}\sigma R^{-2}w_{ss} \end{aligned} \tag{17}$$

Let

$$E = (m-1)r^{-2}w_1g_{rs} + (m-1)s^{-2}w_2g_{ss} - g_t. \tag{18}$$

Then $E \leq 0$ at Q_1. Replacing g_{rr}, g_{ss} and g_t in E we get

(19)
$$\begin{aligned} &2(m-1)|S|^{\lambda}R^{-4}\Big(1 - \frac{2}{\lambda}\Big)\Big(\frac{w_1}{r^{\alpha}} + \frac{w_2}{s^{\alpha}}\Big) \\ &\quad + \lambda|S|^{\lambda-1}\sigma R^{-2}\Big[\Big((m-1)r^{-\alpha}w_1w_{1rr} - w_{1t}\Big) - \Big((m-1)s^{-\alpha}w_2w_{2ss} - w_{2t}\Big)\Big] \leq 0. \end{aligned}$$

ii) If $s = 0$ we have

$$g(r,0,t) = \Big[\frac{r^{\alpha}v(r,t)}{r^{\alpha}}\Big]^{\lambda} \leq M^{\lambda}.$$

iii) If $|r-s| = \delta$ we use the mean value theorem and the fact that the first derivatives of v are bounded by $\delta^{\alpha-1}$ to get

$$\begin{aligned} g(r,s,t) &\leq \Big[r^{\alpha}\frac{|v(r,t) - v(s,t)|}{|r-s|^{\alpha}} + v(s,t)\frac{|r^{\alpha} - s^{\alpha}|}{|r-s|^{\alpha}}\Big]^{\lambda} \\ &\leq \Big[R_2^{\alpha}\frac{\delta^{\alpha-1}\cdot\delta}{\delta^{\alpha}} + M\Big]^{\lambda} \leq \big[R_2^{\alpha} + M\big]^{\lambda}. \end{aligned} \tag{22}$$

iv) Finally assume $r = R_2$. Since $v_0(R_2) = \eta > 0$ there exists $\delta_1 > 0$ such that $v_0(r) > \frac{v_0(R_2)}{2}$ for $R_2 - \delta_1 < r < R_2 + \delta_1$.

In this case there exists $N_1 > 0$ such that $v(r,t) \geq N_1$ in $\Omega_2 = [R_2 - \delta_1, R_2 + \delta_1] \times [0,T]$. Thus (see [8]) $|v_r^\varepsilon|$, $|v_s^\varepsilon|$ are bounded by constant K_3 independently of ε and δ in Ω_2. Without loss of generality we assume $K_3 > 1$, $\delta_1 < 1$. Then by (19) if $|R - s| \leq \delta_1$ we have

$$g(R,s,t) \leq \left[R_2^\alpha K_3 |R-s|^{1-\alpha} + M\right]^\lambda \leq \left[R_2^\alpha K_3 + M\right]^\lambda.$$

Otherwise

$$g(R,s,t) \leq \frac{|r^\alpha v_1 - s^\alpha v_2|^\lambda}{(R-s)^2} \leq \frac{(2R_2^\alpha M)^\lambda}{\delta_1^2}.$$

We conclude that $g(r,s,t) \leq K_2$, where

$$K_2 = \max\left\{\left(R_2^\alpha K_3 + M\right)^\lambda, M^\lambda, \left(R_2^\alpha + M\right)^\lambda, \frac{2^\lambda R_2^2 M^\lambda}{\delta_1^2}\right\} \text{ for any point in } B_{1,\delta}.$$

From this lemma we obtain that if $t_0 = \tau_0$, then $h(r_0, s_0, t_0, \tau_0) = g(r_0, s_0, t_0) \leq K_1$.

Let us assume next that Q is an interior point of B and h is differentiable at Q, (i.e., $h(Q) \neq 0$) then

$$h_r = h_s = 0 \text{ and } h_{rr}, h_{ss}, -h_t, -h_\tau \leq 0 \text{ at } Q. \tag{23}$$

Assume $t_0 > \tau_0$. This time instead of (18) we take

$$E = 2(m-1)r^{-\alpha} w_1 h_{rr} + (m-1)s^{-\alpha} w_2 h_{ss} - 2h_1 - h_s. \tag{24}$$

Then $E \leq 0$ at Q.

We write

$$h(r,s,t,\tau) = \frac{|w(r,t) - w(s,\tau)|^\lambda}{(r-s)^2 + A|t-\tau|} = \frac{|w_1 - w_2|^\lambda}{R} = |S|^\lambda R^{-1}.$$

Then using (21) we get

$$\begin{aligned} E = {} & 2(m-1)|S|^\lambda R^{-2}\big((2-\alpha)R^{-1}(r-s)^2 - 1\big)(2R^{-\alpha} w_1 + s^{-\alpha} w_2) \\ & + \lambda |S|^{\lambda-1} \sigma R^{-1}\Big[\big(2(m-1)r^{-\alpha} w_1 w_{1rr} - 2w_t\big) - \big((m-1)s^{-\alpha} w_2 w_{2ss} - w_2\big)\Big] \\ & + A|S|^\lambda R^{-2} \leq 0 \end{aligned}$$

We use the differential equation in (r,t) and (s,τ) in the second term to get

$$\begin{aligned} E = {} & 2(m-1)|S|^\lambda R^{-2}\big((2-\alpha)R^{-1}(r-s)^2 - 1\big)(2r^{-\alpha} w_1 + s^{-\alpha} w_2) \\ & + \lambda |S|^{\lambda-1} \sigma R^{-1}(s^{-\alpha} w_s^2 - 2r^{-\alpha} w_r^2) + A|S|^\lambda R^{-2} \leq 0 \end{aligned}$$

Now, $h_r = h_s = 0$ implies $w_r = \frac{2\sigma}{\lambda}|S|R^{-1}(r-s) = w_s$, so

$$\begin{aligned} E = {} & 2(m-1)|S|^\lambda R^{-2}\big((2-\alpha)R^{-1}(r-s)^2 - 1\big)(2v_1 + v_2) \\ & + \frac{4}{\lambda}|S|^{\lambda+1} \sigma R^{-3}(r-s)^2(s^{-\alpha} - 2r^{-\alpha}) + A|S|^\lambda R^{-2} \leq 0 \end{aligned}$$

https://doi.org/10.4153/CJM-1991-017-1 Published online by Cambridge University Press

Also

$$\sigma|S|(s^{-\alpha} - 2r^{\alpha}) = (w_1 - w_2)(s^{-\alpha} - 2r^{-\alpha})$$
$$= \left(\frac{R}{s}\right)^{\alpha} v_1 + 2\left(\frac{s}{r}\right)^{\alpha} v_2 - (2v_1 + v_2).$$

Thus, dividing by $|S|^{\lambda}R^{-2}$, we have:

$$2(m-1)\big((2-\alpha)R^{-1}(r-s)^2 - 1\big)(2v_1 + v_2)$$
$$+ \frac{4}{\lambda}R^{-1}(r-s)^2\left[\left(\frac{s}{r}\right)^{\alpha} v_1 + 2\left(\frac{s}{r}\right)^{\alpha} v_2\right] - \frac{4}{\lambda}R^{-1}(r-s)^2(2v_1+v_2) + A \le 0$$

Dropping the positive term containing $\frac{r}{s}$ and $\frac{s}{r}$ we get

$$(2v_1 + v_2)\left[R^{-1}(r-s)^2\left(2(m-1)(2-\alpha) - \frac{4}{\lambda}\right) - 2(m-1)\right] + A \le 0.$$

But $2(m-1)(2-\alpha) - \frac{4}{\lambda} = 2(m-1)$.

Thus $-2(m-1)(2v_1 + v_2) + A \le 0$, in contradiction with the choice of $A\big(A \ge 6(m-1)M + 1\big)$.

Therefore $\max h$ does not occur at an interior point.

If the maximum of h occurs at a lateral point $r_0 = R_2$ or $s_0 = R_2$ or at an *interior* boundary point ($|r_0 - s_0| = \delta$) and ($|t_0 - s_0| \le \delta$) or ($|r_0 - s_0| \le \delta$) and ($|t_0 - s_0| = \delta$), we use the same arguments as in Lemma 1 to conclude that h is uniformly bounded in these cases.

Finally, if the maximum of h occurs at $t_0 = 0$ or $\tau_0 = 0$ we use the following:

LEMMA 2. *Let* $f(r,s,t) = \frac{|w(r,t)-w(s,0)|^{\lambda}}{(r-s)^2+At}$. *Then f is uniformly bounded on the set* $\Omega_{3,\delta} = \{(r,s,t) \mid 0 < s \le R_2,\ 0 \le t \le T,\ |r-s| \ge \delta \text{ or } t \ge \delta\}$.

PROOF. We test the boundary points as in the previous cases. For an interior boundary point (at which $f \ne 0$) we choose $E = (m-1)r^{-\alpha}w_1 f_{rr} + (m-1)s^{-\alpha}w_2 f_{ss} - f_t$. Then $E \le 0$ at a point of maximum.

Replacing the derivatives, the differential equation in (r,t), and using the condition $f_r = f_s = 0$, we get

$$E = 2(m-1)|S|^{\lambda}R^{-2}\big((2-\alpha)R^{-1}(r-s)^2 - 1\big)(v_1 + v_2)$$
$$+ \lambda\sigma|S|^{\lambda-1}R^{-1}\left(\frac{-4}{\lambda^2}r^{-\alpha}|S|^2R^{-2}(r-s)^2 - (m-1)s^{-\alpha}w_2 w_{2rs}\right)$$
$$+ A|S|^{\lambda}R^{-2} \le 0.$$

In the second term we add and subtract $s^{-\alpha}W_2^2\sigma = \frac{4}{\lambda^2}s^{-\alpha}|S|^2R^{-2}(r-s)^2$ this term, I_2, becomes

$$I_2 = \lambda\sigma|S|^{\lambda-1}R^{-1}2\left(\frac{4}{\lambda^2}|S|^2R^{-2}(r-s)^2(s^{-\alpha} - r^{-\alpha}) - s^{-\alpha}\big((m-1)W_2W_{2ss} + W_{2s}^2\big)\right)$$
$$= \frac{4}{\lambda}|S|^{\lambda+1}\sigma R^{-3}(r-s)^2(s^{-\alpha} - r^{-\alpha}) - \lambda|S|^{\lambda-1}\sigma R^{-1}s^{-\alpha}(m-1)W_2W_{2ss} + W_{2s}^2.$$

https://doi.org/10.4153/CJM-1991-017-1 Published online by Cambridge University Press

As before $\sigma|S| = W_1 - W_2$, so the first term in I_2 is $\frac{4}{\lambda}|S|^\lambda R^{-3}(r-s)^2\big((\frac{r}{s})^\alpha v_1 + (\frac{s}{r})^\alpha v_2 - (v_1+v_2)\big)$ thus we get

$$E = |S|^\lambda R^{-2}\Big[(v_1 + v_2\{2(m-1)\big((2-\alpha)R^{-1}(r-s)^2 - 1\big) - \frac{4}{\lambda}R^{-1}(r-s)^2\} + \frac{4}{\lambda}|S|^2R^{-3}(r-s)^2\Big(\big(\frac{r}{s}\big)^\alpha v_1 + \big(\frac{s}{r}\big)^\alpha v_2\Big) + A\Big] - \lambda|S|^{\lambda-1}\sigma R^{-1}s^{-\alpha}\big((m-1)W_2W_{2ss} + W_{2s}^2\big) \le 0.$$

Now $2(m-1)(2-\alpha) - \frac{4}{\lambda} = 2(m-1)$. We drop the positive term in $\frac{r}{s}, \frac{s}{r}$ and get $|S|^\lambda R^{-1}\big[A - 2(m-1)(v_1+v_2)\big] \le \lambda|S|^{\lambda-1}\sigma s^{-\alpha}\big((m-1)W_2W_{2ss} + W_{2s}^2\big)$.

By the choice of A the coefficient on the left hand side is larger than 1. Also, in terms of the function v we have

$$(m-1)ww_{rs} + w_s^2 = s^{2\alpha}\Big((m-1)vv_{ss} + v_s^2 + 2\alpha m\frac{vv_s}{s}\Big).$$

Since $\alpha = \frac{m-1}{m}$, the right hand side is $s^{2\alpha}[v_0]$, so

$$\frac{|S|^\lambda}{R} \le \lambda(2R_2^\lambda M)^{\lambda-1}R_2^\alpha \quad [v_0] \le \lambda(2R_2^\alpha M)^{\lambda-1}R_2^{-\alpha}M_0.$$

We conclude that h is uniformly bounded independently of δ, on B_δ. Thus letting $\delta \longrightarrow 0$ we obtain that

$$h(r,s,t,\tau) = \frac{|r^\alpha v(r,t) - s^\alpha v(s,\tau)|^\lambda}{|r-s|^2 + A|t-\tau|} \le K_1 \text{ on } [0,R_2]^2 \times [0,T]^2.$$

Therefore $r^\alpha v(r,t)$ is α-Hölder continuous on $[0,R_2] \times [0,T]$, with constant $K_2 = (AK_1)^{\alpha/2}$.

COROLLARY 1. *$v(r,t)$ is α-Hölder continuous for $r \ge r_0 > 0$.*

PROOF. We have

$$K_2 \ge \frac{|r^\alpha v(r,t) - s^\alpha v(s,\tau)|}{|r-s|^\alpha + |t-\tau|^{\alpha/2}} \ge r^\alpha\frac{|v(r,t) - v(s,\tau)|}{|r-s|^\alpha + |t-\tau|^{\alpha/2}} - v(s,\tau)\frac{|r^\alpha - s^\alpha|}{|r-s|^\alpha + |t-\tau|^{\alpha/2}}$$

thus

$$\frac{v(r,t) - v(s,\tau)|}{|r-s|^\alpha + |t-\tau|^{\alpha/2}} \le \frac{K_2 + M}{r_0^\alpha} \text{ for } r_0 \le r,s \le R_2, \quad t,\tau \in [0,T].$$

COROLLARY 2. *If $v_0(0) = \eta > 0$ then $v(r,t)$ is α-Hölder continuous on the whole domain Ω_{R_2} for $0 < \alpha \le \frac{m-1}{m}$.*

PROOF. In this case, there exist $\eta_1, \delta_1 > 0$ such that $v(r,t) \ge \eta_1 > 0$ for $0 \le r \le \delta_1$. Then v is a classical solution in $[0,\delta_1] \times [0,T]$ with bounded derivative independent of ε. Thus v is Lipschitz in $[0,\delta_1]$, and by Corollary 1 it is Hölder in $[0,R_2] \times [0,T]$.

THEOREM 2. *Let $v(r,t)$ be a radially symmetric solution of (9). Then $r^\alpha v(r,t)$ is α-Hölder continuous in Ω_{R_2}, for $\alpha \in (0, \frac{m-1}{m}]$.*

PROOF. $v(r,t)$ satisfies

$$v_t = (m-1)vv_{rr} + v_r^2 + 2(m-1)\frac{vv_r}{r} + h(r,t,v)v \qquad (25)$$
$$v(r,0) = v_0(r) \geq 0.$$

We assume $|h| \leq M_0$.

Again we put $w(r,t) = r^\alpha v(r,t)$ and consider the same functions $h(r,s,t,\tau)$, $g(r,s,t)$ and $f(r,s,t)$ over their corresponding domains. Here $w(r,t)$ satisfies

$$r^\alpha w_t = (m-1)ww_{rr} + w_r^2 + r^\alpha h(r,t,v)w. \qquad (26)$$

We study first the function

$$g(r,s,t) = \frac{|w(r,t) - w(r,t)|^\lambda}{(r-s)^2} = |S|^\lambda R^{-1}.$$

This time at an interior point of maximum of g we get like in (19)

(27)
$$2(m-1)|S|^\lambda R^{-\alpha}\Big(1 - \frac{2}{\lambda}\Big)\Big(\frac{w_1}{r^\alpha} + \frac{w_2}{s^\alpha}\Big)$$
$$+ \lambda|S|^{\lambda-1}\sigma R^{-2}\Big[\Big((m-1)r^{-\alpha}w_1w_{rr} - w_{1t}\Big) - \Big((m-1)s^{-\alpha}w_2w_{2ss} - w_{2t}\Big)\Big] \leq 0.$$

When we use the differential equation (26), and the condition $w_{1r} = \frac{2\sigma}{\lambda}|S|R^{-1} = W_{2s}$ we get

$$2(m-1)|S|^\lambda R^{-4}\Big(1 - \frac{2}{\lambda}\Big)\Big(\frac{w_1}{r^\alpha} + \frac{w_2}{s^\alpha}\Big) \qquad (28)$$
$$+ \lambda|S|^{\lambda-1}\sigma R^{-2}\Big[\frac{4}{\lambda^2}|S|^2R^{-2}\Big(\frac{1}{s^\alpha} - \frac{1}{r^\alpha}\Big) + h(s,t,v_2)w_2 - h(r,t,v_1)w_1\Big] \leq 0$$

this is, like in Theorem 1,

$$2\alpha|S|^\lambda R^{-4}\Big(\frac{w_1}{s^\alpha} + \frac{w_2}{r^\alpha}\Big) + \lambda|S|^{\lambda-1}\sigma R^{-2}(h_2w_2 - h_1w_1) \leq 0,$$

factoring R^{-2}, transposing the second term, taking absolute value and replacing w, we get

$$2\alpha\frac{|S|^\lambda}{R^2}(v_1 + v_2) \leq \lambda|S|^{\lambda-1}|h_2r^\alpha v_2 - h_1s^\alpha v_1|.$$

Thus

$$\frac{|S|^\lambda}{R^2} \leq \frac{1}{\alpha^2}(2R_2^\alpha M)^{\lambda-1}\Big[(M_0R_2^\alpha)\Big[\frac{v_2}{v_2+v_2}\Big] + (M_0R_2^\alpha)\Big[\frac{v_1}{v_1+v_2}\Big]\Big],$$

i.e., $\frac{|S|^\lambda}{R^2} \leq 2\frac{M_0R_2^\alpha}{\alpha^2}(2R_2^\alpha M)^{\lambda-1}$ in this case.

The boundary points of the domain of $g(r,s,t)$, the function $h(r,s,t,\tau)$ and the function $f(r,s,t)$ are treated with the same techniques of Theorem 1.

Now we turn to the case $N > 3$.

THEOREM 3. *Theorem 1 and its corollaries, and Theorem 2, remain valid for* $N > 3$. *In this case,* $r^{\alpha/\beta}v(r,t)$ *is* $\alpha\beta$*-Hölder continuous for* $\alpha \in (0, \frac{m-1}{m}]$, $\beta = \frac{1}{N-2}$.

PROOF. We let $W(r,t) = r^\alpha v(r^\beta, t)$. With $\alpha = \frac{m-1}{m}$, $W(r,t)$ satisfies:

$$\begin{aligned} &\beta^2 r^p W_t = (m-1)WW_{rr} + W_r^2, \quad r > 0 \\ &W(r,0) = r^\alpha v_0(r^\beta), \quad p = \alpha + 2\beta - 2 \end{aligned} \tag{30}$$

This is like equation (13) with p instead of α. We define the same domains and functions as in Theorem 1, with $\lambda = \frac{2}{\alpha\beta}$, and $|v_r|, |v_s| \le \delta^{\beta(\alpha-1)}$. Then all the previous arguments can be extended to this case obtaining that $W(r,t)$ is $\alpha\beta$-Hölder continuous with constant, say, K_5.

Next, let $r_1 = r^{1/\beta}$, $s_1 = s^{1/\beta}$, then

$$\begin{aligned} \frac{|r^{\alpha/\beta}v(r,t) - s^{\alpha/\beta}v(s,t)|}{|r-s|^{\alpha\beta}} &= \frac{|r_1^\alpha v(r_1^\beta,t) - s_1^\alpha v(s_1^\beta,t)|}{|r-s|^{\alpha\beta}} \\ &\le \frac{|W(r_1,t) - W(s_1,t)|}{|r_1 - s_1|^{\alpha\beta}} \frac{|r_1 - s_1|^{\alpha\beta}}{|r-s|^{\alpha\beta}} \\ &\le K_5\left(\frac{r^{1/\beta} - s^{1/\beta}}{|r-s|}\right)^{\alpha\beta} \le K_5\left(\frac{1}{\beta}\right)^{\alpha\beta} R_2^{\alpha(1-\beta)}. \end{aligned}$$

This shows that $r^{\alpha/\beta}v(r,t)$ is $\alpha\beta$-Hölder continuous (with respect to r).

All other lemmas and corollaries follow in a similar way.

REFERENCES

1. D. G. Aronson, *Regularity properties of flows through porous media: a counterexample*, SIAM J. Appl. Math. **19**(1970), 299–307.
2. ———, *Regularity properties of gas through porous media*, SIAM J. Appl. Math. **17**(1969), 461–467.
3. D. G. Aronson and L. C. Cafarelli, *Optimal regularity for one dimensional porous medium*, Revista Mat. Iberoamericana (4)**2**(1986), 357–366.
4. P. N. Benilam, *A strong regularity L^p for solutions of the porous media equation*, (in Contributions to Nonlinear Partial Differential Equations, C. Bardos et al. editors), Research notes in Math, **89**, Pitman, London, 1983, 39–58.
5. L. A. Caffarelli and A. Friedman, *Continuity of the density of a gas flow in a porous medium*, Trans. Amer. Math. Soc. **252**(1979), 99–113.
6. ———, *Regularity of the free boundary of a gas flow in a n-dimensional porous medium*, Indiana Univ. Math. J. **29**(1980), 361–391.
7. G. Hernandez, *the Hölder property in some degenerate parabolic problems*, J. Diff. Eq. **65**(1986), 240–249.
8. E. S. Sabinina, *On the Cauchy problem for the equation of nonstationary gas filtration in several space variables*, Dokl. Akad. Nauk SSSR **136**(1961), 1034–1037.

Department of Mathematics
University of Connecticut
Storrs, Connecticut 06269-3009
USA

Department of Mathematics
Hamline University
St. Paul, Minnesota 55104-1284
USA