Proceedings of the Royal Society of Edinburgh, page 1 of 27
DOI:10.1017/prm.2023.58

A revisit to "On BMO and Carleson measures on Riemannian manifolds"

Bo Li
College of Data Science, Jiaxing University, Jiaxing 314001, China (bli@zjxu.edu.cn)

Jinxia Li
School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, 454003, China (jinxiali@hpu.edu.cn)

Qingze Lin
School of Mathematics, Sun Yat-sen University, Guangzhou 510275, China (linqz@mail2.sysu.edu.cn)

Bolin Ma
College of Data Science, Jiaxing University, Jiaxing 314001, China (blma@zjxu.edu.cn)

Tianjun Shen*
Center for Applied Mathematics, Tianjin University, Tianjin 300072, China (shentj@tju.edu.cn)

(Received 18 February 2023; accepted 26 May 2023)

Let $\mathcal{M}$ be an Ahlfors n-regular Riemannian manifold such that either the Ricci curvature is non-negative or the Ricci curvature is bounded from below together with a bound on the gradient of the heat kernel. In the paper [IMRN, 2022, no. 2, 1245-1269] of Brazke–Schikorra–Sire, the authors characterised the BMO function $u : \mathcal{M} \to \mathbb{R}$ by a Carleson measure condition of its σ-harmonic extension $U : \mathcal{M} \times \mathbb{R}_+ \to \mathbb{R}$. This paper is concerned with the similar problem under a more general Dirichlet metric measure space setting, and the limiting behaviours of BMO & Carleson measure, where the heat kernel admits only the so-called diagonal upper estimate. More significantly, without the Ricci curvature condition, we relax the Ahlfors regularity to a doubling property, and remove the pointwise bound on the gradient of the heat kernel. Some similar results for the Lipschitz function are also given, and two open problems related to our main result are considered.

Keywords: Harmonic function; metric measure space; BMO; Carleson measure

2020 *Mathematics Subject Classification:* Primary: 31B25
Secondary: 43A85, 42B35

* Corresponding author.

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

1. Introduction

The Carleson measure ν on $\mathbb{R}^{n+1}_+$, meaning that

$$|||\nu||| = \sup_{B(x_B,r_B)\subset\mathbb{R}^n} \frac{\nu(B(x_B,r_B)\times(0,r_B))}{|B(x_B,r_B)|} < \infty,$$

as its name implies, was introduced by Carleson in the 1950s. This measure captures essential orthogonality properties and exploits properties of extension of a function on the upper half-space. There exists a natural and deep connection between Carleson measure and bounded mean oscillation function (BMO function for short); indeed, a measure derived from some function is Carleson if and only if the underlying function is in BMO. In his ground-breaking paper, Fefferman [14] announced that, a function u belongs to BMO space if and only if its Poisson integral (also called harmonic extension) $U(x,\,t) = \mathrm{e}^{-t\sqrt{-\Delta}}u(x)$ satisfies the following Carleson measure condition

$$\sup_{B(x_B,r_B)} \frac{1}{|B(x_B,r_B)|}\int_0^{r_B}\int_{B(x_B,r_B)} |t\nabla_{(x,t)}U(x,t)|^2\mathrm{d}x\frac{\mathrm{d}t}{t} \leqslant C < \infty, \tag{1.1}$$

where $\nabla_{(x,t)} = (\nabla_x,\,\partial_t)$ denotes the total gradient. This Poisson integral is effectively to solve the Dirichlet problem for the Laplace equation on $\mathbb{R}^{n+1}_+$, i.e., one obtains $U(x,\,t) = \mathrm{e}^{-t\sqrt{-\Delta}}u(x)$ as the solution to

$$\begin{cases} \Delta U(x,t) + \partial_t^2 U(x,t) = 0, & \forall\, x\in\mathbb{R}^n,\, t>0,\\ U(x,0) = u(x), & \forall\, x\in\mathbb{R}^n. \end{cases}$$

For more related works on BMO function and Carleson measure, we refer the reader to [5, 6, 10, 11, 13, 20, 22, 24, 27–29, 32, 33] and their references therein. On the other hand, to define the fractional powers of the Laplacian $(-\Delta)^\sigma$ in local form, $0<\sigma<1$, Caffarelli-Silvestre [3] introduced the σ-harmonic extension

$$\begin{cases} \Delta U(x,t) + \dfrac{1-2\sigma}{t}\partial_t U(x,t) + \partial_t^2 U(x,t) = 0, & \forall\, x\in\mathbb{R}^n,\, t>0,\\ U(x,0) = u(x), & \forall\, x\in\mathbb{R}^n. \end{cases}$$

They interpreted the function U satisfying the equation above as the σ-harmonic extension of u to a fractional dimension $2-2\sigma$, and proved that, up to a multiplicative constant,

$$(-\Delta)^\sigma u(x) = -\lim_{t\to 0} t^{1-2\sigma}\partial_t U(x,t).$$

This means any fractional power of the Laplacian can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via the above extension problem. Later on, Brazke–Schikorra–Sire [2] considered the boundary behaviour of the σ-harmonic extension. They characterised the BMO function $u:\mathcal{M}\to\mathbb{R}$ by a Carleson measure condition of its σ-harmonic extension $U:\mathcal{M}\times\mathbb{R}_+\to\mathbb{R}$, where $\mathcal{M}$ is a complete path connected and Ahlfors regular manifold without boundary, such that either the Ricci curvature is non-negative or the

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

Ricci curvature is bounded from below together with a pointwise bound on the gradient of the heat kernel. However, since this pointwise bound fails already for the elliptic operators (see [**8**, **23**] for instance), a basic question arising from the above discussions motivates our work:

- Question : Can we remove the pointwise bound on the gradient of the heat kernel or the non-negative Ricci curvature condition on $\mathcal{M}$?

In this paper, under a more general Dirichlet metric measure space setting, we will provide a positive answer to the above question; see Theorem 2.2. Moreover, we equip the solution U and the trace u with some limiting conditions to derive the relationship between vanishing Carleson measure and vanishing mean oscillation function (CMO function for short); see Theorem 6.1. It is remarkable that our method and technique in this paper are different from that of Brazke–Schikorra–Sire. Their proofs lean heavily on the Ricci curvature condition, the pointwise bound on the gradient of the heat kernel, the $T(b)$-theorem proved by Hofmann–Mitrea–Mitrea–Morris, and so on. However our main tools are the space-time conversion technology (see Lemma 3.2), the spectral theory (see Proposition 4.1), and the theory of the tent space (see the proof of Theorem 5.4). Therefore, our conclusions can get rid of the Ricci curvature condition and the pointwise bound on the gradient of the heat kernel, and are suitable for the Dirichlet metric measure space setting; see the proof of Theorem 2.2 for more details.

The remainder of this paper will be as follows. In § 2, we collect some preliminaries, state the main result (namely Theorem 2.2) that the relationship between BMO and Carleson measure without the pointwise bound on the gradient of the heat kernel, and pose two open problems related to the main result. In § 3, we first establish the time derivative estimate of the σ-Poisson kernel, and then control the spatial gradient of the σ-harmonic function by the time derivative part. § 4 provides the L^2-estimate of the square function by applying the spectral theory, and § 5 will be devoted to the proof of Theorem 2.2. In § 6, we continue the line of § 5 to study the σ-harmonic function with CMO trace. Some remarks are then presented in the final section.

The letter C (or c) will denote a positive constant that may vary from line to line but will remain independent of the main variables.

2. Main result

In this section, we first briefly describe our metric measure space settings; see [**15**, **19**, **31**] for more details, and then state the main result about BMO and Carleson measure; see Theorem 2.2. Finally, two open problems are considered.

2.1. Preliminaries

Suppose that $\mathcal{M}$ is a separable, connected, locally compact and metrisable space. Let μ be a Borel measure that is strictly positive on non-empty open sets and finite on compact sets. We consider a strongly local, closed, and regular Dirichlet form $\mathscr{E}$ on $L^2(\mathcal{M}, \mu)$ with dense domain $\mathscr{D} \subset L^2(\mathcal{M}, \mu)$ (see [**15**] or [**19**] for an accurate definition). Suppose that $\mathscr{E}$ admits a '*carré du champ*', which means that, for all

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

$f,\, g \in \mathscr{D}$, the measure-valued non-negative and symmetric bilinear form $\Gamma(f,\, g)$ is absolutely continuous with respect to the measure μ. In what follows, for simplicity of notation, let $\langle \nabla_x f,\, \nabla_x g\rangle$ denote the energy density $\mathrm{d}\Gamma(f,\, g)/\mathrm{d}\mu$ and $|\nabla_x f|$ denote the square root of $\mathrm{d}\Gamma(f,\, f)/\mathrm{d}\mu$. Assume that $(\mathcal{M},\, \mu,\, \mathscr{E})$ is endowed with the intrinsic (pseudo-)distance on $\mathcal{M}$ related to $\mathscr{E}$, which is defined by setting

$$d(x,y) = \sup\left\{f(x) - f(y) : f \in \mathscr{D}_{\mathrm{loc}} \cap C(\mathcal{M}),\ |\nabla_x f| \leqslant 1 \text{ a.e.}\right\},$$

where $C(\mathcal{M})$ is the space of continuous functions on $\mathcal{M}$. Suppose d is indeed a distance and induces a topology equivalent to the original topology on $\mathcal{M}$. As a summary of the above situation, we will say that $(\mathcal{M},\, d,\, \mu,\, \mathscr{E})$ is a complete Dirichlet metric measure space.

Corresponding to such a Dirichlet form $\mathscr{E}$, there exists an operator denoted by $\Delta_{\mathcal{M}}$ (similar to the Laplacian), acting on a dense domain $\mathscr{D}(\Delta_{\mathcal{M}})$ in $L^2(\mathcal{M},\, \mu)$, $\mathscr{D}(\Delta_{\mathcal{M}}) \subset W^{1,2}(\mathcal{M})$, such that for all $f \in \mathscr{D}(\Delta_{\mathcal{M}})$ and each $g \in W^{1,2}(\mathcal{M})$,

$$\int_X \Delta_{\mathcal{M}} f(x) g(x) \mathrm{d}\mu(x) = -\mathscr{E}(f,g),$$

where $W^{1,2}(\mathcal{M})$ is the Sobolev space equipped with the norm $([f]_{L^2}^2 + \mathscr{E}(f,\, f))^{1/2}$ on the domain $\mathscr{D}$. The 'Laplacian' $\Delta_{\mathcal{M}}$ is the infinitesimal generator of the heat semigroup $H_t = \mathrm{e}^{t\Delta_{\mathcal{M}}}$, $t > 0$. The heat semigroup $\{H_t\}_{t>0}$ has an integral kernel $h_t(x,\, y)$ (also called heat kernel), namely,

$$\begin{cases} (\partial_t - \Delta_{\mathcal{M}}) h_t(x,y) = 0, & \forall\, x \in \mathcal{M},\, t > 0, \\ h_0(x,y) = \delta_y(x), & \forall\, x \in \mathcal{M}, \end{cases}$$

which is a non-negative measureable function on $\mathbb{R}_+ \times \mathcal{M} \times \mathcal{M}$ such that

$$\mathrm{e}^{t\Delta_{\mathcal{M}}} f(x) = \int_{\mathcal{M}} h_t(x,y) f(y) \mathrm{d}\mu(y), \quad \forall\, f \in L^2(\mathcal{M}),\, t > 0;$$

see [**31**] for more details.

To state the main result, we impose some assumptions on the underlying space and the heat kernel, define the σ-harmonic extension and introduce some function classes.

Denote by $B(x,\, r)$ the open ball with centre x and radius r, by $V(x,\, r)$ its volume $\mu(B(x,\, r))$, and set $\lambda B(x,\, r) = B(x,\, \lambda r)$ for each $\lambda > 0$. We suppose that the measure μ is doubling, i.e., there exists a constant $C_D > 0$ such that

$$V(x,2r) \leqslant C_D V(x,r) < \infty, \quad \forall\, x \in \mathcal{M},\ r > 0. \tag{2.1}$$

Note that the doubling property of μ implies there exists a constant $n \geqslant 1$ such that

$$V(x,R) \leqslant C_D \left(\frac{R}{r}\right)^n V(x,r), \quad \forall\, x \in \mathcal{M},\, 0 < r < R < \infty,$$

and the reverse doubling property holds on a connected space (cf. [**21**, Remark 8.1.15]), i.e., there exists a constant $0 < \kappa \leqslant n$ such that

$$V(x,r) \leqslant C \left(\frac{r}{R}\right)^\kappa V(x,R), \quad \forall\, x \in \mathcal{M},\, 0 < r < R < \infty. \tag{2.2}$$

Obviously, the Ahlfors regular measure is doubling with $n = \kappa$.

Assume that the heat kernel $h_t(x,\, y)$ admits the so-called diagonal upper estimate

$$h_t(x,x) \leqslant \frac{C}{V(x,\sqrt{t})}, \quad \forall\, x \in \mathcal{M},\, t > 0. \tag{2.3}$$

The conjunction of (2.1) and (2.3) is well understood: it is equivalent to a relative Faber–Krahn inequality

$$\lambda_1(\Omega) \geqslant \frac{c}{r^2}\left(\frac{V(x,r)}{\mu(\Omega)}\right)^{\gamma}, \quad \exists\, \gamma > 0, \tag{2.4}$$

for all balls $B(x,\, r)$ in $\mathcal{M}$ and each open set $\Omega \subset B(x,\, r)$, where $\lambda_1(\Omega)$ is the smallest Dirichlet eigenvalue of $\Delta_{\mathcal{M}}$ in Ω; see [**18**] for more details. Under the doubling conditon (2.1), the diagonal upper estimate (2.3) self-improves into a Gaussian upper bound

$$h_t(x,y) \leqslant \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{ct}\right), \quad \forall\, x, y \in \mathcal{M},\, t > 0; \tag{2.5}$$

see [**17**, Theorem 1.1] for the Riemannian manifold case, and [**9**, Section 4.2] for the metric measure space case. Due to some technical reasons, we also assume the heat kernel $h_t(x,\, y)$ satisfies the Hölder condition of order θ $(0 < \theta \leqslant 1)$

$$|h_t(x,y) - h_t(x,z)| \leqslant C\left(\frac{d(y,z)}{\sqrt{t}}\right)^{\theta} h_t(x,y), \quad \forall\, d(y,z) < \sqrt{t}. \tag{2.6}$$

On the other hand, note that the doubling property (2.1) together with the L^2-Poincaré inequality

$$\left(\fint_B |f - f_B|^2 \mathrm{d}\mu\right)^{1/2} \leqslant C_P r_B \left(\fint_B |\nabla_x f|^2 \mathrm{d}\mu\right)^{1/2}, \quad \forall\, f \in W^{1,2}(B), \tag{2.7}$$

where f_B denotes the mean (or average) of f over B, implies that two sides Gaussian bounds for the heat kernel (also called Li-Yau's estimate), i.e., it holds

$$h_t(x,y) \approx \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{ct}\right), \quad \forall\, x, y \in \mathcal{M},\, t > 0; \tag{2.8}$$

see [**31**] for example. Moreover, the two-sided estimate above is equivalent to a parabolic Harnack inequality for positive solutions to the heat equation; see [**31**]. Therefore, under the validity of (2.1) and (2.7), the heat kernel is Hölder continuous.

2.2. Relationship between BMO and Carleson measure

The σ-harmonic extension is defined as follows.

Definition 2.1. *Let* $(\mathcal{M},\, d,\, \mu,\, \mathscr{E})$ *be a complete Dirichlet metric measure space. Given* $0 < \sigma < 1$*, for every* $u \in L^1(\mathcal{M},\, (1 + d(x,\, x_0))^{-\varepsilon} V(x_0,\, 1 + d(x,\, x_0))^{-1} \mathrm{d}\mu(x))$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

with some $x_0 \in \mathcal{M}$ *and* $0 < \varepsilon < \min\{2\sigma, \kappa/2\}$, *its* σ*-harmonic extension* $U : \mathcal{M} \times \mathbb{R}_+ \to \mathbb{R}$ *is the solution to*

$$\begin{cases} \Delta_{\mathcal{M}} U(x,t) + \dfrac{1-2\sigma}{t}\partial_t U(x,t) + \partial_t^2 U(x,t) = 0, & \forall\, x \in \mathcal{M},\, t > 0, \\ U(x,0) = u(x), & \forall\, x \in \mathcal{M}. \end{cases}$$

This solution is formally given by

$$\begin{aligned} U(x,t) = P_t^\sigma u(x) &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \left(\frac{t^2}{4s}\right)^\sigma \exp\left(-\frac{t^2}{4s}\right) \mathrm{e}^{s\Delta_{\mathcal{M}}} u(x) \frac{\mathrm{d}s}{s} \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty s^\sigma \mathrm{e}^{-s} \mathrm{e}^{\frac{t^2}{4s}\Delta_{\mathcal{M}}} u(x) \frac{\mathrm{d}s}{s}, \end{aligned}$$

and explicitly one has

$$\begin{aligned} U(x,t) &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \left(\frac{t^2}{4s}\right)^\sigma \exp\left(-\frac{t^2}{4s}\right) \int_{\mathcal{M}} h_s(x,y) u(y) \mathrm{d}\mu(y) \frac{\mathrm{d}s}{s} \\ &= \int_{\mathcal{M}} \left[\frac{1}{\Gamma(\sigma)} \int_0^\infty \left(\frac{t^2}{4s}\right)^\sigma \exp\left(-\frac{t^2}{4s}\right) h_s(x,y) \frac{\mathrm{d}s}{s}\right] u(y) \mathrm{d}\mu(y) \\ &= \int_{\mathcal{M}} p_t^\sigma(x,y) u(y) \mathrm{d}\mu(y), \end{aligned}$$

where P_t^σ *is the* σ*-Poisson operator and* $p_t^\sigma(x, y)$ *is its integral kernel.*

The previous definition is not explicitly stated in [30] (see also [16]) but it is easy to check that the semigroup approach automatically carries over to such a geometric setting under very weak assumptions on the metric measure space; see [1] for the Dirichlet case and [2] for the manifold case. Moreover, Brazke–Schikorra–Sire [2] assume the boundary value $u(x)$ is a smooth function with compact support to ensure that its σ-harmonic extension $U(x, t)$ is vanishing at infinity

$$\lim_{|(x,t)|\to\infty} U(x,t) = 0.$$

However, via a more elaborate analysis, we can substitute the integrability of u for the smoothness.

We define the following semi-norms. Denote the usual BMO norm as

$$[u]_{\mathrm{BMO}} = \sup_B \left(\fint_B |u - u_B|^2 \mathrm{d}\mu\right)^{1/2} < \infty.$$

Furthermore, let $U(x, t)$ be the σ-harmonic extension of a function $u(x)$ to the upper half-space $\mathcal{M} \times \mathbb{R}_+$, and we introduce a notion of the Carleson measure

$$[U]_{\mathrm{Car}} = \sup_B \left(\int_0^{r_B} \fint_B |t\nabla_{(x,t)} U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} < \infty.$$

Above the two supremum are taken over all balls B in $\mathcal{M}$.

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

The following Theorem 2.2 is the main result of this paper.

THEOREM 2.2. *Let $(\mathcal{M}, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition* (2.1). *Assume that the heat kernel $h_t(x, y)$ admits the diagonal upper estimate* (2.3) *and the Hölder continuity* (2.6). *If $0 < \sigma < 1$, then u is a BMO function if and only if U is a Carleson measure. Moreover, it holds*

$$[u]_{\mathrm{BMO}} \approx [U]_{\mathrm{Car}}.$$

REMARK 2.3. In 2017, Carron [**4**, Theorem 2.4] established that, if a manifold satisfies a volume comparison condition, the relatively connected to an end condition, and the Ricci curvature has quadratic decay, then the manifold satisfies the doubling condition (2.1) and the heat kernel admits the diagonal upper estimate (2.3).

REMARK 2.4. To the best of our knowledge, when $(\mathcal{M}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mathrm{d}x)$, Theorem 2.2 is new even for the second-order elliptic operator $L = -\mathrm{div}A\nabla$, where $A = A(x)$ is an $n \times n$ matrix of real, symmetric, bounded measurable coefficients, defined on $\mathbb{R}^n$, and satisfies the ellipticity condition, i.e., there exist constants $0 < \lambda \leqslant \Lambda < \infty$ such that, for all $\xi \in \mathbb{R}^n$,

$$\lambda|\xi|^2 \leqslant \langle A\xi, \xi\rangle \leqslant \Lambda|\xi|^2.$$

Moreover, as we do not need any curvature condition, our result can be applied to different settings, such as Euclidean space with Muckenhoupt weight, Lie group of polynomial growth, sub-Riemannian manifold; see [**8**, Section 7] for more details.

2.3. Two open problems

We pose two open problems related to Theorem 2.2.

Problem 1: As stated in [**14**], Fefferman in advance assumed the harmonic function can represented as the Poisson integral of an initial value. Later on, Fabes–Johnson–Neri [**12**] found that the Carleson measure condition actually characterises all harmonic functions $U(x, t)$ on $\mathbb{R}^{n+1}_+$ with boundary value in BMO space. They proved that a harmonic function U satisfies the Carleson measure condition if and only if its trace u is a BMO function.

Inspired by [**12**], rather than assuming $U = P^\sigma_t u$ in advance, can we seek the trace of the σ-harmonic function U? At this point we believe that the underlying space $(\mathcal{M}, d, \mu, \mathscr{E})$ needs to satisfy the doubling condition (2.1) and support the Poincaré inequality (2.7) (which is equivalent to Li-Yau' estimate (2.8)); see [**8**] and [**22**] for example.

Based on the arguments above, we pose that

CONJECTURE 2.5. *Let $(\mathcal{M}, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition* (2.1) *and supporting the Poincaré inequality* (2.7). *For every $0 < \sigma < 1$, a σ-harmonic function U satisfies the Carleson measure condition if and only if its trace u is a BMO function.*

The proof of the necessity relies heavily on the structure of the elliptic equation and the Poincaré inequality. The main difficulty lies in that the σ-harmonic function only satisfies a Hölder continuity condition (since $t^{1-2\sigma}$ is a Muckenhoupt

A_2-weight) instead of smoothness or Lipschitz continuity. When $\sigma = 1/2$, we can prove that such harmonic function enjoys Lipschitz regularity in the time direction; see [22] and [25] for example. This together with the Carleson measure condition enables us to derive suitable control of the growth of the harmonic function. Whereas $\sigma \neq 1/2$, we do not know that how to establish L^∞-estimate of $t\partial_t U(x,\, t)$. We look forward to proving the estimate

$$|t\partial_t U(x,t)| \leqslant C \sup_B \left(\int_0^{r_B} \fint_B |t\partial_t U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t} \right)^{1/2} \leqslant C[U]_{\mathrm{Car}}$$

holds for all $x \in \mathcal{M}$ and $t > 0$.

Problem 2: In paper [13] (see also [24]), Fabes-Neri solved the Dirichlet problem for the heat equation

$$-\Delta V(x,t) + \partial_t V(x,t) = 0$$

on the upper half-space. They characterised the caloric function (the solution to the heat equation) by a parabolic Carleson measure condition

$$[V]_{\mathrm{Car2}} = \sup_B \left(\int_0^{r_B^2} \fint_B |\sqrt{t}\nabla_x V|^2 \mathrm{d}x \frac{\mathrm{d}t}{t} \right)^{1/2} < \infty.$$

More precisely, they showed that a caloric function V satisfies the parabolic Carleson measure condition if and only if its trace v is a BMO function.

In the setting of this paper, how to define the σ-heat equation? Note that a $W^{1,2}$-function $U : \mathcal{M} \times \mathbb{R}_+ \to \mathbb{R}$ is said to be σ-harmonic if it is the solution to the elliptic equation

$$\Delta_{\mathcal{M}} U(x,t) + \frac{1-2\sigma}{t} \partial_t U(x,t) + \partial_t^2 U(x,t) = 0$$

in the weak sense, namely, it holds

$$\int_0^\infty \int_{\mathcal{M}} \langle \nabla_x U, \nabla_x \Phi \rangle t^{1-2\sigma} \mathrm{d}\mu \mathrm{d}t + \int_0^\infty \int_{\mathcal{M}} \partial_t U \partial_t \Phi t^{1-2\sigma} \mathrm{d}\mu \mathrm{d}t = 0$$

for all Lipschitz functions Φ on $\mathcal{M} \times \mathbb{R}_+$ with compact support, where the classical product measure $\mathrm{d}\mu \mathrm{d}t$ is replaced by the weighted measure $t^{1-2\sigma}\mathrm{d}\mu \mathrm{d}t$. Therefore the σ-caloric function (the solution to the σ-heat equation) $V : \mathcal{M} \times \mathbb{R}_+ \to \mathbb{R}$ should be also understood in the weighted product measure $t^{1-2\sigma}\mathrm{d}\mu \mathrm{d}t$, namely, it holds

$$\int_0^\infty \int_{\mathcal{M}} \langle \nabla_x V, \nabla_x \Psi \rangle t^{1-2\sigma} \mathrm{d}\mu \mathrm{d}t - \int_0^\infty \int_{\mathcal{M}} V \partial_t \Psi t^{1-2\sigma} \mathrm{d}\mu \mathrm{d}t = 0$$

for all Lipschitz functions Ψ on $\mathcal{M} \times \mathbb{R}_+$ with compact support.

Inspired by [13], can we consider the Dirichlet problem for the σ-heat equation

$$-\Delta_{\mathcal{M}} V(x,t) + \frac{1-2\sigma}{t} V(x,t) + \partial_t V(x,t) = 0$$

on the upper half-space, and study the relationship between the σ-caloric function and its trace?

Based on the above arguments, we pose a conjecture related to the σ-heat equation as follows.

CONJECTURE 2.6. *Let $(\mathcal{M}, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition (2.1) and supporting the Poincaré inequality (2.7). For every $0<\sigma<1$, a σ-caloric function V satisfies the parabolic Carleson measure condition*

$$[V]_{\mathrm{Car2}} = \sup_B \left(\int_0^{r_B^2} \fint_B (|\sqrt{t}\nabla_x V|^2 + |t\partial_t V|^2) \mathrm{d}\mu \frac{\mathrm{d}t}{t} \right)^{1/2} < \infty$$

if and only if its trace u is a BMO function.

The parabolic Carleson measure condition in Conjecture 2.6 is different from that in [**13**]. When $(\mathcal{M}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mathrm{d}x)$, these two parabolic Carleson measure conditions coincide with each other. The time derivative part $t\partial_t V$ in our parabolic Carleson measure condition is essential due to the complex structure of the parabolic equation; see [**26**] for example.

3. σ-Poisson kernel estimates

In this section, we estimate the time derivative of the σ-Poisson kernel, and the spatial gradient of the σ-harmonic function.

PROPOSITION 3.1. *For each integer $k \geqslant 0$, the σ-Poisson kernel satisfies*

$$|t^k \partial_t^k p_t^\sigma(x,y)| \leqslant C(\sigma,k) \left(\frac{t}{t+d(x,y)} \right)^{2\sigma} \frac{1}{V(x, t+d(x,y))}, \quad \forall\, x,y \in \mathcal{M},\, t>0.$$

Proof. It follows from the σ-Poisson formula that

$$p_t^\sigma(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \left(\frac{t^2}{4s}\right)^\sigma \exp\left(-\frac{t^2}{4s}\right) h_s(x,y) \frac{\mathrm{d}s}{s}.$$

Differentiating both sides of the identity with respect to t, using the basic inequality, and invoking the Gaussian upper bound (2.5), we deduce that

$$\begin{aligned}
|t^k \partial_t^k p_t^\sigma(x,y)| &= \frac{1}{\Gamma(\sigma)} \left| \int_0^\infty t^k \partial_t^k \left(\left(\frac{t^2}{4s}\right)^\sigma \exp\left(-\frac{t^2}{4s}\right) \right) h_s(x,y) \frac{\mathrm{d}s}{s} \right| \\
&\leqslant C(\sigma,k) \int_0^\infty \left(\frac{t^2}{s}\right)^\sigma \exp\left(-\frac{t^2}{5s}\right) h_s(x,y) \frac{\mathrm{d}s}{s} \\
&\leqslant C(\sigma,k) \int_0^\infty \left(\frac{t^2}{s}\right)^\sigma \exp\left(-\frac{t^2}{5s}\right) \frac{1}{V(x,\sqrt{s})} \exp\left(-\frac{d(x,y)^2}{cs}\right) \frac{\mathrm{d}s}{s} \\
&\leqslant C(\sigma,k) \int_0^\infty \left(\frac{t^2}{s}\right)^\sigma \frac{1}{V(x,\sqrt{s})} \exp\left(-\frac{t^2+d(x,y)^2}{cs}\right) \frac{\mathrm{d}s}{s} \\
&= C(\sigma,k) \left\{ \int_0^{t^2+d(x,y)^2} + \int_{t^2+d(x,y)^2}^\infty \right\} \cdots \frac{\mathrm{d}s}{s} = I_1 + I_2.
\end{aligned}$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

To estimate the local part I_1, we conclude by the doubling condition (2.1) that

$$\begin{aligned} I_1 &= \int_0^{t^2+d(x,y)^2} \left(\frac{t^2}{s}\right)^{\sigma} \frac{1}{V(x,\sqrt{s})} \exp\left(-\frac{t^2+d(x,y)^2}{cs}\right) \frac{\mathrm{d}s}{s} \\ &\leqslant C \left(\frac{t}{t+d(x,y)}\right)^{2\sigma} \frac{1}{V(x,t+d(x,y))} \int_0^1 s^{-\frac{n+2\sigma}{2}} \exp\left(-\frac{1}{cs}\right) \frac{\mathrm{d}s}{s} \\ &\leqslant C \left(\frac{t}{t+d(x,y)}\right)^{2\sigma} \frac{1}{V(x,t+d(x,y))}. \end{aligned}$$

For the global part I_2, there holds by the doubling condition (2.1) again that

$$\begin{aligned} I_2 &= \int_{t^2+d(x,y)^2}^{\infty} \left(\frac{t^2}{s}\right)^{\sigma} \frac{1}{V(x,\sqrt{s})} \exp\left(-\frac{t^2+d(x,y)^2}{cs}\right) \frac{\mathrm{d}s}{s} \\ &\leqslant C \frac{1}{V(x,t+d(x,y))} \int_{t^2+d(x,y)^2}^{\infty} \left(\frac{t^2}{s}\right)^{\sigma} \frac{\mathrm{d}s}{s} \\ &\leqslant C \left(\frac{t}{t+d(x,y)}\right)^{2\sigma} \frac{1}{V(x,t+d(x,y))}. \end{aligned}$$

Collecting the three inequalities above leads to the desired result. □

Lemma 3.2. *Given a σ-harmonic function U, then for each ball $B = B(x_B, r_B) \subset \mathcal{M}$, it holds that*

$$\int_{T(B)} |t\nabla_x U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t} \leqslant C \int_{T(2B)} (|U|^2 + |t\partial_t U|^2 + |t^2\partial_t^2 U|^2) \mathrm{d}\mu \frac{\mathrm{d}t}{t},$$

where $T(B)$ is the tent over the ball B, namely, $T(B) = B \times (0, r_B)$.

Proof. Take a Lipschitz function φ on $\mathcal{M}$ with $\operatorname{supp}\varphi \subset 2B$ such that $\varphi = 1$ on B and $|\nabla_x\varphi| \leqslant C/r_B$, and for each $\epsilon \in (0, r_B)$, take a smooth function $\phi_\epsilon(t)$ on $\mathbb{R}$ such that $\operatorname{supp}\phi_\epsilon \subset (\epsilon, 2r_B)$, $\phi_\epsilon(t) = 1$ on $(2\epsilon, r_B)$, $|\partial_t\phi_\epsilon(t)| \leqslant C/\epsilon$ for $t \in (\epsilon, 2\epsilon)$, $|\partial_t\phi_\epsilon(t)| \leqslant C/r_B$ for $t > r_B$. By repeating the Caccioppoli argument, we arrive at

$$\begin{aligned} &\int_{T(2B)} |\varphi\phi_\epsilon t\nabla_x U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t} \\ &\quad = \int_{T(2B)} \langle \nabla_x U, t\varphi^2\phi_\epsilon^2 \nabla_x U\rangle \mathrm{d}\mu \mathrm{d}t \end{aligned}$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

$$
\begin{aligned}
&= \int_{T(2B)} \langle \nabla_x U, \nabla_x (t\varphi^2\phi_\epsilon^2 U)\rangle \mathrm{d}\mu\mathrm{d}t - \int_{T(2B)} \langle \nabla_x U, U\nabla_x (t\varphi^2\phi_\epsilon^2)\rangle \mathrm{d}\mu\mathrm{d}t \\
&= \int_{T(2B)} \langle \nabla_x U, \nabla_x (t^{2\sigma}\varphi^2\phi_\epsilon^2 U)\rangle t^{1-2\sigma} \mathrm{d}\mu\mathrm{d}t - \int_{T(2B)} 2\langle \varphi\phi_\epsilon t\nabla_x U, \phi_\epsilon U t\nabla_x\varphi\rangle \mathrm{d}\mu\frac{\mathrm{d}t}{t} \\
&\leqslant \int_{T(2B)} \partial_t (t^{1-2\sigma}\partial_t U) t^{2\sigma}\varphi^2\phi_\epsilon^2 U \mathrm{d}\mu\mathrm{d}t \\
&\quad + \frac{1}{2}\int_{T(2B)} |\varphi\phi_\epsilon t\nabla_x U|^2 \mathrm{d}\mu\frac{\mathrm{d}t}{t} + 2\int_{T(2B)} |\phi_\epsilon U t\nabla_x\varphi|^2 \mathrm{d}\mu\frac{\mathrm{d}t}{t} \\
&= \int_{T(2B)} \left(\frac{1-2\sigma}{t}\partial_t U + \partial_t^2 U\right) t\varphi^2\phi_\epsilon^2 U \mathrm{d}\mu\mathrm{d}t \\
&\quad + \frac{1}{2}\int_{T(2B)} |\varphi\phi_\epsilon t\nabla_x U|^2 \mathrm{d}\mu\frac{\mathrm{d}t}{t} + 2\int_{T(2B)} |\phi_\epsilon U t\nabla_x\varphi|^2 \mathrm{d}\mu\frac{\mathrm{d}t}{t}.
\end{aligned}
$$

Using the fact $|\nabla_x\varphi| \leqslant C/r_B \leqslant C/t$ and employing the mean value inequality lead to

$$
\int_{T(2B)} |\varphi\phi_\epsilon t\nabla_x U|^2 \mathrm{d}\mu\frac{\mathrm{d}t}{t} \leqslant C\int_{T(2B)} (|U|^2 + |t\partial_t U|^2 + |t^2\partial_t^2 U|^2)\mathrm{d}\mu\frac{\mathrm{d}t}{t}.
$$

This combined with the monotonic converges theorem tells us

$$
\begin{aligned}
\int_{T(B)} |t\nabla_x U|^2 \mathrm{d}\mu\frac{\mathrm{d}t}{t} &= \lim_{\epsilon\to 0}\int_{2\epsilon}^{r_B}\int_B |t\nabla_x U|^2 \mathrm{d}\mu\frac{\mathrm{d}t}{t} \\
&\leqslant \lim_{\epsilon\to 0}\int_{T(2B)} |\varphi\phi_\epsilon t\nabla_x U|^2 \mathrm{d}\mu\frac{\mathrm{d}t}{t} \\
&\leqslant C\int_{T(2B)} (|U|^2 + |t\partial_t U|^2 + |t^2\partial_t^2 U|^2)\mathrm{d}\mu\frac{\mathrm{d}t}{t},
\end{aligned}
$$

which completes the proof. □

4. Square function estimate

By applying the spectral theory, we give the L^2-estimate of the square function in this section.

Proposition 4.1. *Let U be a σ-harmonic function U with trace u. Then it holds that*

$$
\left(\int_0^\infty\int_{\mathcal{M}} |t\nabla_{(x,t)} U|^2 \mathrm{d}\mu\frac{\mathrm{d}t}{t}\right)^{1/2} \leqslant \left(\frac{\Gamma(\sigma+1/2)}{\Gamma(\sigma)} + \frac{\sqrt{2}}{2}\right)\left(\int_{\mathcal{M}} |u|^2 \mathrm{d}\mu\right)^{1/2}.
$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

Proof. We only consider the spatial gradient part since the proof of the time part is similar. It follows from the σ-Poisson formula and Minkowski's inequality that

$$\begin{aligned}
&\left(\int_0^\infty \int_{\mathcal{M}} |t\nabla_x U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} \\
&= \left(\int_0^\infty \int_{\mathcal{M}} \left|\frac{2}{\Gamma(\sigma)}\int_0^\infty s^{\sigma+1/2}\mathrm{e}^{-s}\frac{t}{2\sqrt{s}}\nabla_x \mathrm{e}^{\frac{t^2}{4s}\Delta_{\mathcal{M}}}u\frac{\mathrm{d}s}{s}\right|^2 \mathrm{d}\mu\frac{\mathrm{d}t}{t}\right)^{1/2} \\
&\leqslant \frac{2}{\Gamma(\sigma)}\int_0^\infty s^{\sigma+1/2}\mathrm{e}^{-s}\left(\int_0^\infty\int_{\mathcal{M}}\left|\frac{t}{2\sqrt{s}}\nabla_x \mathrm{e}^{\frac{t^2}{4s}\Delta_{\mathcal{M}}}u\right|^2\mathrm{d}\mu\frac{\mathrm{d}t}{t}\right)^{1/2}\frac{\mathrm{d}s}{s} \\
&= \frac{\sqrt{2}\Gamma(\sigma+1/2)}{\Gamma(\sigma)}\left(\int_0^\infty\int_{\mathcal{M}}\left|\sqrt{t}\nabla_x\mathrm{e}^{t\Delta_{\mathcal{M}}}u\right|^2\mathrm{d}\mu\frac{\mathrm{d}t}{t}\right)^{1/2}.
\end{aligned}$$

By the self-adjointness of the heat semigroup, and the spectral theory, we arrive at

$$\begin{aligned}
\int_0^\infty\int_{\mathcal{M}}\left|\sqrt{t}\nabla_x\mathrm{e}^{t\Delta_{\mathcal{M}}}u\right|^2\mathrm{d}\mu\frac{\mathrm{d}t}{t} &= \int_0^\infty \langle t(-\Delta_{\mathcal{M}})\mathrm{e}^{2t\Delta_{\mathcal{M}}}u, u\rangle_{L^2}\frac{\mathrm{d}t}{t} \\
&= \int_0^\infty\left(\int_0^\infty t\lambda\mathrm{e}^{-2t\lambda}\mathrm{d}E_{u,u}(\lambda)\right)\frac{\mathrm{d}t}{t} \\
&= \int_0^\infty\left(\int_0^\infty t\lambda\mathrm{e}^{-2t\lambda}\frac{\mathrm{d}t}{t}\right)\mathrm{d}E_{u,u}(\lambda) \\
&= \frac{1}{2}\int_{\mathcal{M}}|u|^2\mathrm{d}\mu.
\end{aligned}$$

Collecting the two inequalities above leads to the desired result. □

5. BMO and Carleson measure: proof of Theorem 2.2

In this section, we give the proof of Theorem 2.2 stated in § 2.

5.1. From BMO to Carleson measure

LEMMA 5.1. *Let $(\mathcal{M}, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition (2.1). Assume that the heat kernel $h_t(x, y)$ admits the diagonal upper estimate (2.3). For every $0 < \sigma < 1$, if u is a BMO function, then its σ-harmonic extension $U : \mathcal{M} \times \mathbb{R}_+ \to \mathbb{R}$ is well-defined.*

Proof. By the conservation property of the heat semigroup

$$P_t^\sigma u_{B(x,t)} = \frac{1}{\Gamma(\sigma)}\int_0^\infty s^\sigma \mathrm{e}^{-s}\mathrm{e}^{\frac{t^2}{4s}\Delta_{\mathcal{M}}}u_{B(x,t)}\frac{\mathrm{d}s}{s} = u_{B(x,t)},$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

the σ-Poisson upper bound (Proposition 3.1), and Hölder's inequality, we arrive at

$$\begin{aligned}|U(x,t)| &\leqslant |U(x,t)-u_{B(x,t)}|+|u_{B(x,t)}|\\ &\leqslant C\int_{\mathcal{M}}\left(\frac{t}{t+d(x,y)}\right)^{2\sigma}\frac{|u(y)-u_{B(x,t)}|}{V(x,t+d(x,y))}\mathrm{d}\mu(y)+|u_{B(x,t)}|\\ &\leqslant C\int_{B(x,t)}\left(\frac{t}{t+d(x,y)}\right)^{2\sigma}\frac{|u(y)-u_{B(x,t)}|}{V(x,t+d(x,y))}\mathrm{d}\mu(y)\\ &\quad+\sum_{k=1}^{\infty}\int_{B(x,2^kt)\backslash B(x,2^{k-1}t)}\cdots\mathrm{d}\mu(y)+|u_{B(x,t)}|\\ &\leqslant C\sum_{k=0}^{\infty}\frac{1}{4^{k\sigma}}\fint_{B(x,2^kt)}|u(y)-u_{B(x,t)}|\mathrm{d}\mu(y)+|u_{B(x,t)}|\\ &\leqslant C\sum_{k=0}^{\infty}\frac{1}{4^{k\sigma}}\sum_{i=0}^{k}\fint_{B(x,2^it)}|u(y)-u_{B(x,2^it)}|\mathrm{d}\mu(y)+|u_{B(x,t)}|\\ &\leqslant C[u]_{\mathrm{BMO}}\sum_{k=0}^{\infty}\frac{k+1}{4^{k\sigma}}+|u_{B(x,t)}|\\ &\leqslant C[u]_{\mathrm{BMO}}+|u_{B(x,t)}|<\infty,\end{aligned}$$

which completes the proof. □

REMARK 5.2. From the argument used in the above lemma, we see that, if u is a BMO function, then

$$\int_{\mathcal{M}}\frac{|u(x)|}{(1+d(x,x_0))^{\varepsilon}V(x_0,1+d(x,x_0))}\mathrm{d}\mu(x)\leqslant C(x_0,\epsilon)<\infty$$

for all $x_0\in\mathcal{M}$ and $\varepsilon>0$.

THEOREM 5.3. *Let $(\mathcal{M},\, d,\, \mu,\, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition* (2.1)*. Assume that the heat kernel $h_t(x,\,y)$ admits the diagonal upper estimate* (2.3)*. For every $0<\sigma<1$, if u is a BMO function, then its σ-harmonic extension U is a Carleson measure. Moreover, there exists a constant $C>0$ such that*

$$[U]_{\mathrm{Car}}\leqslant C[u]_{\mathrm{BMO}}.$$

Proof. We follow the argument from [**2**, Proposition 4.1] without the Ahlfors regular condition and the pointwise bound on the gradient of the heat kernel.

Fixed a ball $B\subset\mathcal{M}$, we decompose

$$\nabla_{(x,t)}U=\nabla_{(x,t)}U_1+\nabla_{(x,t)}U_2+\nabla_{(x,t)}U_3,$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

where U_i is the σ-harmonic extension of u_i, respectively, given as

$$\begin{cases} u_1 = (u - u_B)\chi_{4B}, \\ u_2 = (u - u_B)(1 - \chi_{4B}), \\ u_3 = u_B. \end{cases}$$

It follows from the conservation property of the heat semigroup that

$$U_3(x,t) = \frac{1}{\Gamma(\sigma)} \int_0^\infty s^\sigma \mathrm{e}^{-s} \mathrm{e}^{\frac{t^2}{4s}\Delta_{\mathcal{M}}} u_B(x) \frac{\mathrm{d}s}{s} = u_B,$$

and thus $\nabla_{(x,t)} U_3 = 0$.

In view of Proposition 4.1, we have

$$\begin{aligned} \left(\int_0^{r_B} ⨍_B |t\nabla_{(x,t)} U_1|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t} \right)^{1/2} &\leqslant \left(\frac{1}{V(x_B, r_B)} \int_0^\infty \int_{\mathcal{M}} |t\nabla_{(x,t)} U_1|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t} \right)^{1/2} \\ &\leqslant C(\sigma) \left(\frac{1}{V(x_B, r_B)} \int_{\mathcal{M}} |u_1|^2 \mathrm{d}\mu \right)^{1/2} \\ &= C(\sigma) \left(\frac{1}{V(x_B, r_B)} \int_{4B} |u - u_B|^2 \mathrm{d}\mu \right)^{1/2} \\ &\leqslant C(\sigma) [u]_{\mathrm{BMO}}. \end{aligned}$$

It remains to estimate U_2. In view of Lemma 3.2, we arrive at

$$\left(\int_0^{r_B} ⨍_B |t\nabla_{(x,t)} U_2|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t} \right)^{1/2} \leqslant C \left(\int_0^{2r_B} ⨍_{2B} \max_{0 \leqslant m \leqslant 2} |t^m \partial_t^m U_2|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t} \right)^{1/2}.$$

Denote by B_k, $k \in \mathbb{N}$, the ball concentric around B but with radius 4^k times the radius of B. Since $\operatorname{supp} u_2 \subset \mathcal{M} \setminus 4B$, we deduce from the σ-Poisson upper bound (Proposition 3.1) that, for all $x \in 2B$,

$$\begin{aligned} \max_{0 \leqslant m \leqslant 2} |t^m \partial_t^m U_2(x,t)| &\leqslant C \int_{\mathcal{M} \setminus 4B} \left(\frac{t}{t + d(x,y)} \right)^{2\sigma} \frac{|u(y) - u_B|}{V(x, t + d(x,y))} \mathrm{d}\mu(y) \\ &\leqslant C \sum_{k=2}^\infty \int_{B_k \setminus B_{k-1}} \left(\frac{t}{t + d(x,y)} \right)^{2\sigma} \frac{|u(y) - u_B|}{V(x, t + d(x,y))} \mathrm{d}\mu(y) \\ &\leqslant C \sum_{k=2}^\infty \int_{B_k \setminus B_{k-1}} \left(\frac{t}{4^k r_B} \right)^{2\sigma} \frac{|u - u_B|}{V(x_B, 4^k r_B)} \mathrm{d}\mu \\ &\leqslant C \sum_{k=2}^\infty \left(\frac{t}{4^k r_B} \right)^{2\sigma} ⨍_{B_k} |u - u_B| \mathrm{d}\mu. \end{aligned}$$

By the triangle inequality and a telescoping sum, it holds that

$$⨍_{B_k} |u - u_B| \mathrm{d}\mu \leqslant ⨍_{B_k} |u - u_{B_k}| \mathrm{d}\mu + \sum_{i=1}^k |u_{B_{i-1}} - u_{B_i}| \leqslant C \sum_{i=1}^k ⨍_{B_i} |u - u_{B_i}| \mathrm{d}\mu.$$

Using Hölder's inequality and the definition of BMO function, we have

$$\fint_{B_k} |u - u_B| \mathrm{d}\mu \leqslant Ck[u]_{\mathrm{BMO}}.$$

Consequently one has

$$\max_{0\leqslant m\leqslant 2} |t^m \partial_t^m U_2(x,t)| \leqslant C\left(\frac{t}{r_B}\right)^{2\sigma} [u]_{\mathrm{BMO}} \sum_{k=2}^{\infty} \frac{k}{4^{2k\sigma}} = C\left(\frac{t}{r_B}\right)^{2\sigma} [u]_{\mathrm{BMO}}.$$

This implies

$$\begin{aligned}\left(\int_0^{r_B} \fint_B |t\nabla_{(x,t)} U_2|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} &\leqslant C\left(\int_0^{2r_B} \fint_{2B} \max_{0\leqslant m\leqslant 2} |t^m \partial_t^m U_2|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} \\ &\leqslant C\left(\int_0^{2r_B} \fint_{2B} \left(\frac{t}{r_B}\right)^{4\sigma} [u]_{\mathrm{BMO}}^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} \\ &\leqslant C[u]_{\mathrm{BMO}},\end{aligned}$$

which, combined with the estimates of U_1 and U_3, concludes the proof of Theorem 5.3. □

5.2. From Carleson measure to BMO

THEOREM 5.4. *Let* $(\mathcal{M}, d, \mu, \mathscr{E})$ *be a complete Dirichlet metric measure space satisfying the doubling condition* (2.1). *Assume that the heat kernel* $h_t(x, y)$ *admits the diagonal upper estimate* (2.3) *and the Hölder continuity* (2.6). *For every* $0 < \sigma < 1$, *if the* σ*-harmonic extension* U *is a Carleson measure, then its trace* u *is a BMO function. Moreover, there exists a constant* $C > 0$ *such that*

$$[u]_{\mathrm{BMO}} \leqslant C[U]_{\mathrm{Car}}.$$

To show Theorem 5.4, we need to introduce the definition of the H^1-atom. An L^∞-function $a(x)$ is called a H^1-atom associated with a ball $B = B(x_B, r_B) \subset \mathcal{M}$, if it satisfies

(i) $\operatorname{supp} a \subset B$;

(ii) $[a]_{L^\infty} \leqslant V(x_B, r_B)^{-1}$;

(iii) $\int_{\mathcal{M}} a \mathrm{d}\mu = 0$.

The following lemma establishes the Calderón reproducing formula with respect to the H^1-atom.

LEMMA 5.5. *Let* $U, A : \mathcal{M} \times \mathbb{R}_+ \to \mathbb{R}$ *be the* σ*-harmonic extension of* u *and* a, *respectively. Then it holds*

$$\int_{\mathcal{M}} ua \mathrm{d}\mu = \frac{1}{\sigma} \int_0^\infty \int_{\mathcal{M}} \langle t\nabla_{(x,t)} U, t\nabla_{(x,t)} A\rangle \mathrm{d}\mu \frac{\mathrm{d}t}{t}.$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

Proof. We first claim

$$\lim_{t\to\infty}|UA| = \lim_{t\to\infty}|Ut\partial_t A| = \lim_{t\to\infty}|At\partial_t U| = 0.$$

In fact, on the one hand, it follows from the σ-Poisson formula that

$$\begin{aligned}|U(x,t)| + |t\partial_t U(x,t)| &\leqslant C\int_{\mathcal{M}}\left(\frac{t}{t+d(x,y)}\right)^{2\sigma}\frac{|u(y)|}{V(x,t+d(x,y))}\mathrm{d}\mu(y)\\ &\leqslant C\int_{\mathcal{M}}\frac{t^{\min\{2\sigma,\kappa/2\}}}{(1+d(x,y))^{\min\{2\sigma,\kappa/2\}}}\frac{|u(y)|}{V(x,1+d(x,y))}\mathrm{d}\mu(y)\\ &\leqslant C(x,\sigma,\kappa)t^{\min\{2\sigma,\kappa/2\}}.\end{aligned}$$

On the other hand, by the σ-Poisson formula again, the definition of the H^1-atom, and the reverse doubling property (2.2), we arrive at

$$\begin{aligned}|A(x,t)| + |t\partial_t A(x,t)| &\leqslant C\int_{\mathcal{M}}\left(\frac{t}{t+d(x,y)}\right)^{2\sigma}\frac{|a(y)|}{V(x,t+d(x,y))}\mathrm{d}\mu(y)\\ &\leqslant \frac{C}{V(x,t)}\int_B|a(y)|\mathrm{d}\mu(y)\\ &\leqslant \frac{C}{V(x,1)}t^{-\kappa}\\ &\leqslant C(x)t^{-\kappa}.\end{aligned}$$

Therefore, one has

$$\lim_{t\to\infty}|UA| \leqslant C\lim_{t\to\infty}t^{\min\{2\sigma,\kappa/2\}-\kappa} = 0$$

and similarly

$$\lim_{t\to\infty}|Ut\partial_t A| = \lim_{t\to\infty}|At\partial_t U| = 0$$

as claimed.

Below we verify the Calderón reproducing formula. Employing the integration by parts and using the above decay as $t\to\infty$, we deduce that

$$\begin{aligned}\int_0^\infty t^{2\sigma}\partial_t(t^{1-2\sigma}\partial_t(UA))\mathrm{d}t &= t^{2\sigma}t^{1-2\sigma}\partial_t(UA)\big|_0^\infty - \int_0^\infty 2\sigma t^{2\sigma-1}t^{1-2\sigma}\partial_t(UA)\mathrm{d}t\\ &= (Ut\partial_t A + At\partial_t U))\big|_0^\infty - 2\sigma UA\big|_0^\infty\\ &= -\lim_{t\to 0}(Ut\partial_t A + At\partial_t U) + 2\sigma ua\\ &= -\lim_{t\to 0}t^{2\sigma}(Ut^{1-2\sigma}\partial_t A + At^{1-2\sigma}\partial_t U) + 2\sigma ua\\ &= C(\sigma)[u(-\Delta_{\mathcal{M}})^\sigma a + a(-\Delta_{\mathcal{M}})^\sigma u]\lim_{t\to 0}t^{2\sigma} + 2\sigma ua\\ &= 2\sigma ua,\end{aligned}$$

where the second line from the bottom is due to the extension problem for the fractional of $-\Delta_{\mathcal{M}}$; see [**1**] and [**30**] for example. One may use the above identity,

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

the integration by parts again, the vanishing properties of $Ut\partial_t A$ and $At\partial_t U$ at 0 and ∞, and the fact that U and A are σ-harmonic, to obtain

$$2\sigma\int_{\mathcal{M}} ua\mathrm{d}\mu = 2\int_0^\infty\int_{\mathcal{M}}\langle t\nabla_{(x,t)}U, t\nabla_{(x,t)}A\rangle \mathrm{d}\mu\frac{\mathrm{d}t}{t},$$

see also [**2**, Lemma 4.3]. This completes the proof. □

REMARK 5.6. It is worth noting the proof of the above lemma is analogous to that of [**2**, Lemma 4.3]. However, based on some new observations, we can provide another proof. Indeed note that the structure of the elliptic equation. One obtains

$$\begin{aligned}\int_0^\infty\int_{\mathcal{M}}\langle t\nabla_{(x,t)}U, t\nabla_{(x,t)}A\rangle \mathrm{d}\mu\frac{\mathrm{d}t}{t} &= \int_0^\infty\int_{\mathcal{M}}\langle t^{2\sigma}\nabla_{(x,t)}U, \nabla_{(x,t)}A\rangle t^{1-2\sigma}\mathrm{d}\mu\mathrm{d}t\\ &= \int_0^\infty\int_{\mathcal{M}}\langle \nabla_{(x,t)}(t^{2\sigma}U), \nabla_{(x,t)}A\rangle t^{1-2\sigma}\mathrm{d}\mu\mathrm{d}t\\ &\quad -\int_0^\infty\int_{\mathcal{M}} U\partial_t(t^{2\sigma})\partial_t A t^{1-2\sigma}\mathrm{d}\mu\mathrm{d}t\\ &= -2\sigma\int_0^\infty\int_{\mathcal{M}} U\partial_t A\mathrm{d}\mu\mathrm{d}t,\end{aligned}$$

and similarly

$$\int_0^\infty\int_{\mathcal{M}}\langle t\nabla_{(x,t)}U, t\nabla_{(x,t)}A\rangle \mathrm{d}\mu\frac{\mathrm{d}t}{t} = -2\sigma\int_0^\infty\int_{\mathcal{M}} A\partial_t U\mathrm{d}\mu\mathrm{d}t.$$

Therefore, we arrive at

$$\begin{aligned}&\int_0^\infty\int_{\mathcal{M}}\langle t\nabla_{(x,t)}U, t\nabla_{(x,t)}A\rangle \mathrm{d}\mu\frac{\mathrm{d}t}{t}\\ &\quad = -\sigma\left(\int_0^\infty\int_{\mathcal{M}} U\partial_t A\mathrm{d}\mu\mathrm{d}t + \int_0^\infty\int_{\mathcal{M}} A\partial_t U\mathrm{d}\mu\mathrm{d}t\right)\\ &\quad = -\sigma\int_{\mathcal{M}}\int_0^\infty \partial_t(UA)\mathrm{d}t\mathrm{d}\mu\\ &\quad = \sigma\int_{\mathcal{M}} ua\mathrm{d}\mu,\end{aligned}$$

as desired.

Proof of Theorem 5.4. **Step 1.** We verify that, if $a(x)$ is a H^1-atom associated with a ball $B = B(x_B, r_B) \subset \mathcal{M}$, then its σ-harmonic extension A satisfies

$$\left|\int_0^\infty\int_{\mathcal{M}}\langle t\nabla_{(x,t)}U, t\nabla_{(x,t)}A\rangle \mathrm{d}\mu\frac{\mathrm{d}t}{t}\right| \leqslant C[U]_{\mathrm{Car}}.$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

One can write

$$\begin{aligned}\left|\int_0^\infty \int_{\mathcal{M}} \langle t\nabla_{(x,t)}U, t\nabla_{(x,t)}A\rangle \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right| &\leqslant \int_0^\infty \int_{\mathcal{M}} |t\nabla_{(x,t)}U||t\nabla_{(x,t)}A| \mathrm{d}\mu \frac{\mathrm{d}t}{t} \\ &= \left\{\int_{T(4B)} + \sum_{k=3}^\infty \int_{T(2^kB)\setminus T(2^{k-1}B)}\right\} \cdots \mathrm{d}\mu \frac{\mathrm{d}t}{t} \\ &= \sum_{k=2}^\infty I_k.\end{aligned}$$

For the term I_2, it follows from Hölder's inequality, the square function estimate (Proposition 4.1) and the definition of the H^1-atom that

$$\begin{aligned}&\int_{T(4B)} |t\nabla_{(x,t)}U||t\nabla_{(x,t)}A| \mathrm{d}\mu \frac{\mathrm{d}t}{t} \\ &\leqslant \left(\int_{T(4B)} |t\nabla_{(x,t)}U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} \left(\int_{T(4B)} |t\nabla_{(x,t)}A|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} \\ &\leqslant CV(x_B, 4r_B)^{1/2}[U]_{\mathrm{Car}} \left(\int_B |a|^2 \mathrm{d}\mu\right)^{1/2} \\ &\leqslant CV(x_B, r_B)[U]_{\mathrm{Car}}[a]_{L^\infty} \\ &\leqslant C[U]_{\mathrm{Car}}.\end{aligned}$$

To estimate the term I_k, one may invoke the space–time conversion technology (see the proof of Lemma 3.2) to obtain

$$\begin{aligned}&\int_{T(2^kB)\setminus T(2^{k-1}B)} |t\nabla_x A|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t} \\ &\quad\leqslant C \int_{T(2^{k+1}B)\setminus T(2^{k-2}B)} \max_{0\leqslant m\leqslant 2} |t^m \partial_t^m A|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}.\end{aligned} \tag{5.1}$$

Below we claim that, for all $(x, t) \in T(2^{k+1}B) \setminus T(2^{k-2}B)$ with $k \geqslant 3$,

$$\max_{0\leqslant m\leqslant 2} |t^m \partial_t^m A(x,t)| \leqslant C2^{-k\theta} \left(\frac{t}{2^k r_B}\right)^{2\sigma} \frac{1}{V(x_B, 2^k r_B)},$$

where $0 < \theta \leqslant 1$ as in (2.6). Indeed, from the σ-Poisson formula, it holds

$$\begin{aligned}\max_{0\leqslant m\leqslant 2} |t^m \partial_t^m A(x,t)| &\leqslant C \int_0^\infty \left(\frac{t^2}{s}\right)^\sigma \exp\left(-\frac{t^2}{5s}\right) \left|\mathrm{e}^{s\Delta_{\mathcal{M}}} a(x)\right| \frac{\mathrm{d}s}{s} \\ &= C \int_0^\infty \left(\frac{t^2}{s}\right)^\sigma \exp\left(-\frac{t^2}{5s}\right) \left|\int_B h_s(x,y)a(y)\mathrm{d}\mu(y)\right| \frac{\mathrm{d}s}{s}.\end{aligned}$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

If $d(y, x_B) < \sqrt{s}$, then by the cancellation condition of $a(y)$

$$\left|\int_B h_s(x,y)a(y)\mathrm{d}\mu(y)\right| = \left|\int_B [h_s(x,y) - h_s(x,x_B)]a(y)\mathrm{d}\mu(y)\right|$$
$$\leqslant C\int_B \left(\frac{d(y,x_B)}{\sqrt{s}}\right)^\theta h_s(x,y)|a(y)|\mathrm{d}\mu(y),$$

where the last inequality is due to the Hölder continuity of the heat kernel (2.6). Otherwise $d(y, x_B) \geqslant \sqrt{s}$, there holds from the Gaussian upper bound (2.5) that

$$\left|\int_B h_s(x,y)a(y)\mathrm{d}\mu(y)\right| \leqslant C\int_B \left(\frac{d(y,x_B)}{\sqrt{s}}\right)^\theta h_s(x,y)|a(y)|\mathrm{d}\mu(y).$$

Therefore an argument similar to the one used in Proposition 3.1 shows that

$$\max_{0\leqslant m\leqslant 2}|t^m\partial_t^m A(x,t)|$$
$$\leqslant C\int_0^\infty \left(\frac{t^2}{s}\right)^\sigma \exp\left(-\frac{t^2}{5s}\right)\left|\int_B h_s(x,y)a(y)\mathrm{d}\mu(y)\right|\frac{\mathrm{d}s}{s}$$
$$\leqslant C\int_0^\infty \left(\frac{t^2}{s}\right)^\sigma \left(\frac{d(y,x_B)}{\sqrt{s}}\right)^\theta \exp\left(-\frac{t^2}{5s}\right)\int_B h_s(x,y)|a(y)|\mathrm{d}\mu(y)\frac{\mathrm{d}s}{s}$$
$$\leqslant C\int_B \left(\frac{d(y,x_B)}{t+d(x,y)}\right)^\theta \left(\frac{t}{t+d(x,y)}\right)^{2\sigma}\frac{1}{V(x,t+d(x,y))}|a(y)|\mathrm{d}\mu(y),$$

which, together with

$$t + d(x,y) \geqslant c2^k r_B$$

provided $y \in B$ and $(x, t) \in T(2^{k+1}B) \setminus T(2^{k-2}B)$ with $k \geqslant 3$, implies

$$\max_{0\leqslant m\leqslant 2}|t^m\partial_t^m A(x,t)| \leqslant C\int_B \left(\frac{d(y,x_B)}{2^k r_B}\right)^\theta \left(\frac{t}{2^k r_B}\right)^{2\sigma}\frac{1}{V(x,2^k r_B)}|a(y)|\mathrm{d}\mu(y)$$
$$\leqslant C2^{-k\theta}\left(\frac{t}{2^k r_B}\right)^{2\sigma}\frac{1}{V(x_B,2^k r_B)}\int_B |a(y)|\mathrm{d}\mu(y)$$
$$\leqslant C2^{-k\theta}\left(\frac{t}{2^k r_B}\right)^{2\sigma}\frac{1}{V(x_B,2^k r_B)}$$

as claimed. By substituting this estimate into (5.1), we obtain

$$\int_{T(2^kB)\setminus T(2^{k-1}B)} |t\nabla_{(x,t)}A|^2\mathrm{d}\mu\frac{\mathrm{d}t}{t}$$
$$\leqslant C\int_{T(2^{k+1}B)\setminus T(2^{k-2}B)} \max_{0\leqslant m\leqslant 2}|t^m\partial_t^m A|^2\mathrm{d}\mu\frac{\mathrm{d}t}{t}$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

$$\leqslant C \int_0^{2^{k+1}r_B} \int_{2^{k+1}B} 2^{-2k\theta} \left(\frac{t}{2^k r_B}\right)^{4\sigma} \frac{1}{V(x_B, 2^k r_B)^2} \mathrm{d}\mu \frac{\mathrm{d}t}{t}$$
$$\leqslant C 2^{-2k\theta} \frac{1}{V(x_B, 2^k r_B)}.$$

This combined with Hölder's inequality yields

$$\begin{aligned} I_k &\leqslant \int_{T(2^k B)\setminus T(2^{k-1}B)} |t\nabla_{(x,t)} U||t\nabla_{(x,t)} A| \mathrm{d}\mu \frac{\mathrm{d}t}{t} \\ &\leqslant \left(\int_{T(2^k B)\setminus T(2^{k-1}B)} |t\nabla_{(x,t)} U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} \\ &\quad \times \left(\int_{T(2^k B)\setminus T(2^{k-1}B)} |t\nabla_{(x,t)} A|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} \\ &\leqslant C 2^{-k\theta} \left(\int_0^{2^k r_B} \fint_{2^k B} |t\nabla_{(x,t)} U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} \\ &\leqslant C 2^{-k\theta} [U]_{\mathrm{Car}}. \end{aligned}$$

Summing over k, we arrive at

$$\left| \int_0^\infty \int_{\mathcal{M}} \langle t\nabla_{(x,t)} U, t\nabla_{(x,t)} A \rangle \mathrm{d}\mu \frac{\mathrm{d}t}{t} \right| \leqslant C [U]_{\mathrm{Car}}.$$

Step 2. Let f be a finite linear combinations of the H^1-atoms, namely,

$$f = \sum_{j=1}^N \lambda_j a_j.$$

From the Calderón reproducing formula (Lemma 5.5) and the conclusion of Step 1, it follows that

$$\begin{aligned} \left| \int_{\mathcal{M}} u f \mathrm{d}\mu \right| &= \left| \int_{\mathcal{M}} u \sum_{j=1}^N \lambda_j a_j \mathrm{d}\mu \right| \\ &\leqslant \sum_{j=1}^N |\lambda_j| \left| \int_{\mathcal{M}} u a_j \mathrm{d}\mu \right| \\ &= \sum_{j=1}^N |\lambda_j| \frac{1}{\sigma} \left| \int_0^\infty \int_{\mathcal{M}} \langle t\nabla_{(x,t)} U, t\nabla_{(x,t)} A_j \rangle \mathrm{d}\mu \frac{\mathrm{d}t}{t} \right| \\ &\leqslant C [U]_{\mathrm{Car}} \sum_{j=1}^N |\lambda_j|. \end{aligned}$$

By taking the infimum over all admissible decompositions of f on the both sides of the above inequality, and invoking the fact that the vector space of all finite

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

linear combinations of the H^1-atoms is dense in the Hardy space $H^1 = H^1(\mathcal{M})$, we conclude that, for any $g \in H^1(\mathcal{M})$,

$$\left|\int_{\mathcal{M}} ug\mathrm{d}\mu\right| \leqslant C[U]_{\mathrm{Car}}[g]_{H^1}.$$

From this and the H^1-BMO dual theorem of Fefferman–Stein (see [7] for instance), it follows that

$$[u]_{\mathrm{BMO}} = \sup_{[g]_{H^1}\leqslant 1}\left|\int_{\mathcal{M}} ug\mathrm{d}\mu\right| \leqslant C \sup_{[g]_{H^1}\leqslant 1}[U]_{\mathrm{Car}}[g]_{H^1} \leqslant C[U]_{\mathrm{Car}},$$

which completes the proof. □

6. Limiting behaviours of BMO and Carleson measure

In this section we set a fixed reference point x_0 in $\mathcal{M}$.

Let us introduce the CMO function and the vanishing Carleson measure. A BMO function u is said to be in CMO $=$ CMO$(\mathcal{M})$, the space of functions of vanishing mean oscillation, if u satisfies the limiting conditions

$$\gamma_1(u) = \gamma_2(u) = \gamma_3(u) = 0,$$

where

$$\begin{cases} \gamma_1(u) = \lim\limits_{a\to 0}\sup\limits_{B: r_B\leqslant a}\left(\fint_B |u-u_B|^2\mathrm{d}\mu\right)^{1/2}; \\ \gamma_2(u) = \lim\limits_{a\to \infty}\sup\limits_{B: r_B\geqslant a}\left(\fint_B |u-u_B|^2\mathrm{d}\mu\right)^{1/2}; \\ \gamma_3(u) = \lim\limits_{a\to \infty}\sup\limits_{B: B\subset B(x_0,a)^{\complement}}\left(\fint_B |u-u_B|^2\mathrm{d}\mu\right)^{1/2}. \end{cases}$$

A σ-harmonic function U with trace u is said to satisfy the vanishing Carleson measure condition, if U is a Carleson measure, and satisfies the limiting conditions

$$\beta_1(U) = \beta_2(U) = \beta_3(U) = 0,$$

where

$$\begin{cases} \beta_1(U) = \lim\limits_{a\to 0}\sup\limits_{B: r_B\leqslant a}\left(\int_0^{r_B}\fint_B |t\nabla_{(x,t)}U|^2\mathrm{d}\mu\frac{\mathrm{d}t}{t}\right)^{1/2}; \\ \beta_2(U) = \lim\limits_{a\to \infty}\sup\limits_{B: r_B\geqslant a}\left(\int_0^{r_B}\fint_B |t\nabla_{(x,t)}U|^2\mathrm{d}\mu\frac{\mathrm{d}t}{t}\right)^{1/2}; \\ \beta_3(U) = \lim\limits_{a\to \infty}\sup\limits_{B: B\subset B(x_0,a)^{\complement}}\left(\int_0^{r_B}\fint_B |t\nabla_{(x,t)}U|^2\mathrm{d}\mu\frac{\mathrm{d}t}{t}\right)^{1/2}. \end{cases}$$

Roughly speaking, the CMO function and the vanishing Carleson measure are the BMO function and the Carleson measure, which satisfy the vanishing conditions, respectively.

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

Theorem 6.1. *Let $(\mathcal{M}, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition (2.1). Assume that the heat kernel $h_t(x, y)$ admits the diagonal upper estimate (2.3) and the Hölder continuity (2.6). If $0 < \sigma < 1$, then u is a CMO function if and only if U is a vanishing Carleson measure. Moreover, it holds*

$$[u]_{\mathrm{BMO}} \approx [U]_{\mathrm{Car}}.$$

Remark 6.2.

(i) Evidently, $\mathrm{CMO}(\mathcal{M})$ is a proper subspace of $\mathrm{BMO}(\mathcal{M})$ since it is the predual of the Hardy space $H^1(\mathcal{M})$; see [**7**, Theorem 4.1].

(ii) When $(\mathcal{M}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mathrm{d}x)$, the space $\mathrm{CMO}(\mathbb{R}^n)$ is defined by the closure in the BMO norm of the smooth function with compact support.

6.1. From CMO to Carleson measure

Theorem 6.3. *Let $(\mathcal{M}, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition (2.1). Assume that the heat kernel $h_t(x, y)$ admits the diagonal upper estimate (2.3). For every $0 < \sigma < 1$, if u is a CMO function, then its σ-harmonic extension U is a vanishing Carleson measure. Moreover, there exists a constant $C > 0$ such that*

$$[U]_{\mathrm{Car}} \leqslant C[u]_{\mathrm{BMO}}.$$

Proof. By Theorem 5.3, we know

$$[U]_{\mathrm{Car}} \leqslant C[u]_{\mathrm{BMO}}.$$

It remains to verify that U satisfies the three limiting conditions. We only consider the last one since the proofs of the other two cases are similar. Fixed a ball $B \subset \mathcal{M}$, by the proof of Theorem 5.3, we see that

$$\begin{aligned}\left(\int_0^{r_B} \fint_B |t\nabla_{(x,t)} U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} &\leqslant C \sum_{k=2}^{\infty} 4^{-2k\sigma} \sum_{i=1}^{k} \left(\fint_{4^i B} |u - u_{4^i B}|^2 \mathrm{d}\mu\right)^{1/2} \\ &= C \sum_{k=2}^{\infty} 4^{-2k\sigma} \sum_{i=1}^{k} \eta_i(u, B).\end{aligned}$$

For any fixed integer i, it follows from $\gamma_2(u) = \gamma_3(u) = 0$ that

$$\sup_{B: B \subset B(x_0, a)^{\complement}} \eta_i(u, B) = \sup_{B: B \subset B(x_0, a)^{\complement}} \left(\fint_{4^i B} |u - u_{4^i B}|^2 \mathrm{d}\mu\right)^{1/2}$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

$$\leqslant \sup_{\substack{B:B\subset B(x_0,a)^{\complement}\\ 2^i r_B<a/2}} \left(\fint_{4^iB} |u-u_{4^iB}|^2 \mathrm{d}\mu\right)^{1/2} + \sup_{\substack{B:B\subset B(x_0,a)^{\complement}\\ 2^i r_B\geqslant a/2}} \left(\fint_{4^iB} |u-u_{4^iB}|^2 \mathrm{d}\mu\right)^{1/2}$$
$$\leqslant \sup_{B:B\subset B(x_0,a/2)^{\complement}} \left(\fint_{B} |u-u_{B}|^2 \mathrm{d}\mu\right)^{1/2} + \sup_{B:r_B\geqslant a/2} \left(\fint_{B} |u-u_{B}|^2 \mathrm{d}\mu\right)^{1/2} \to 0$$

as $a\to\infty$, and from the definition of BMO that

$$\eta_i(u,B)=\left(\fint_{4^iB} |u-u_{4^iB}|^2 \mathrm{d}\mu\right)^{1/2} \leqslant [u]_{\mathrm{BMO}}.$$

Therefore, we have

$$\begin{aligned}
\beta_3(U) &= \lim_{a\to\infty} \sup_{B:B\subset B(x_0,a)^{\complement}} \left(\int_0^{r_B}\fint_B |t\nabla_{(x,t)}U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2}\\
&\leqslant C \lim_{a\to\infty} \sup_{B:B\subset B(x_0,a)^{\complement}} \sum_{k=2}^{\infty} 4^{-2k\sigma} \sum_{i=1}^{k} \eta_i(u,B)\\
&= C \lim_{a\to\infty} \sup_{B:B\subset B(x_0,a)^{\complement}} \left\{\sum_{k=2}^{K} + \sum_{k=K+1}^{\infty}\right\} 4^{-2k\sigma} \sum_{i=1}^{k} \eta_i(u,B)\\
&\leqslant C \sum_{k=2}^{K} 4^{-2k\sigma} \sum_{i=1}^{k} \lim_{a\to\infty} \sup_{B:B\subset B(x_0,a)^{\complement}} \eta_i(u,B) + C \sum_{k=K+1}^{\infty} 4^{-k\sigma}[u]_{\mathrm{BMO}}\\
&\leqslant C 4^{-K\sigma}[u]_{\mathrm{BMO}},
\end{aligned}$$

where the constant C is independent of K. As the positive integer K is arbitrary, we derive that the last limiting condition $\beta_3(U)=0$ holds by letting $K\to\infty$. The proof is completed. □

6.2. From Carleson measure to CMO

THEOREM 6.4. *Let $(\mathcal{M}, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition (2.1). Assume that the heat kernel $h_t(x,y)$ admits the diagonal upper estimate (2.3) and the Hölder continuity (2.6). For every $0<\sigma<1$, if the σ-harmonic extension U is a vanishing Carleson measure, then its trace u is a CMO function. Moreover, there exists a constant $C>0$ such that*

$$[u]_{\mathrm{BMO}} \leqslant C[U]_{\mathrm{Car}}.$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

Proof. By Theorem 5.4, we know

$$[u]_{\mathrm{BMO}} \leqslant C[U]_{\mathrm{Car}}.$$

It remains to verify that u satisfies the three limiting conditions. Let f be a finite linear combinations of H^1-atoms, namely,

$$f = \sum_{j=1}^{N} \lambda_j a_j.$$

From the Calderón reproducing formula (Lemma 5.5), and the conclusion of Step 1 in Theorem 5.4, it follows that

$$\begin{aligned}
\left|\int_{\mathcal{M}} uf \mathrm{d}\mu\right| &= \left|\int_{\mathcal{M}} u \sum_{j=1}^{N} \lambda_j a_j \mathrm{d}\mu\right| \\
&\leqslant \sum_{j=1}^{N} |\lambda_j| \left|\int_{\mathcal{M}} ua_j \mathrm{d}\mu\right| \\
&= \sum_{j=1}^{N} |\lambda_j| \frac{1}{\sigma} \left|\int_0^\infty \int_{\mathcal{M}} \langle t\nabla_{(x,t)}U, t\nabla_{(x,t)}A_j\rangle \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right| \\
&\leqslant C \sum_{k=2}^{\infty} 2^{-k\theta} \left(\int_0^{2^k r_B} \fint_{2^k B} |t\nabla_{(x,t)}U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} \sum_{j=1}^{N} |\lambda_j|.
\end{aligned}$$

By taking the infimum over all admissible decompositions of f on the both sides of the above inequality, and invoking the fact that the vector space of all finite linear combinations of the H^1-atoms is dense in the Hardy space $H^1(\mathcal{M})$, we conclude that, for any $g \in H^1(\mathcal{M})$,

$$\left|\int_{\mathcal{M}} ug \mathrm{d}\mu\right| \leqslant C \sum_{k=2}^{\infty} 2^{-k\theta} \left(\int_0^{2^k r_B} \fint_{2^k B} |t\nabla_{(x,t)}U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2} [g]_{H^1}.$$

This, combined with the H^1-BMO dual theorem of Fefferman–Stein (see [7] for instance), tells us that

$$\begin{aligned}
\left(\fint_B |u-u_B|^2 \mathrm{d}\mu\right)^{1/2} &\leqslant \sup_{[g]_{H^1}\leqslant 1} \left|\int_{\mathcal{M}} ug \mathrm{d}\mu\right| \\
&\leqslant C \sum_{k=2}^{\infty} 2^{-k\theta} \left(\int_0^{2^k r_B} \fint_{2^k B} |t\nabla_{(x,t)}U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t}\right)^{1/2}.
\end{aligned}$$

Therefore an argument similar to the one used in Theorem 6.3 shows that the limiting conditions

$$\gamma_1(u) = \gamma_2(u) = \gamma_3(u) = 0$$

hold, which completes the proof. □

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

7. Final remarks

This paper will end by pointing out some further results. Since it is intended solely as a brief review and not as a rigorous development, the pertinent results are stated without proof.

For each $\alpha > 0$, a function u on $\mathcal{M}$ is said to be Lipschitz of order α if it satisfies

$$[u]_{\mathrm{Lip}(\alpha)} = \sup_{B \ni x,y} \frac{|u(x) - u(y)|}{[\mu(B)]^{\alpha}},$$

where the supremum is taken over all balls B containing x and y. In fact, our Theorem 2.2 is also valid for the Lipschitz function. To this end, let us introduce the corresponding α-Carleson measure of U as follows:

$$[U]_{\mathrm{Car}(\alpha)} = \sup_B \frac{1}{V(x_B, r_B)^{\alpha}} \left(\int_0^{r_B} \fint_B |t\nabla_{(x,t)} U|^2 \mathrm{d}\mu \frac{\mathrm{d}t}{t} \right)^{1/2}.$$

The following result is the Lipschitz version of Theorem 2.2.

Theorem 7.1. *Let $(\mathcal{M}, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition (2.1). Assume that the heat kernel $h_t(x, y)$ admits the diagonal upper estimate (2.3) and the Hölder continuity (2.6). If $0 < \alpha < \theta/n$ and $0 < \sigma < 1$, then u is a Lipschitz function of order α if and only if U is an α-Carleson measure. Moreover, it holds*

$$[u]_{\mathrm{Lip}(\alpha)} \approx [U]_{\mathrm{Car}(\alpha)}.$$

When $0 < \alpha < \theta/n$, the Lipschitz space $\mathrm{Lip}(\alpha) = \mathrm{Lip}(\alpha)(\mathcal{M})$ coincides with the Campanato space $\mathrm{Cam}(\alpha) = \mathrm{Cam}(\alpha)(\mathcal{M})$ as follows:

$$[u]_{\mathrm{Cam}(\alpha)} = \sup_B \frac{1}{V(x_B, r_B)^{\alpha}} \left(\fint_B |u - u_B|^2 \mathrm{d}\mu \right)^{1/2},$$

which is the dual of Hardy space $H^p(\mathcal{M})$ provided $\alpha = 1/p - 1$; see [7] for example. Note that $\mathrm{Cam}(0) = \mathrm{BMO}$. Therefore the Campanato norm $[\cdot]_{\mathrm{Cam}(\alpha)}$ unifies the BMO norm $[\cdot]_{\mathrm{BMO}}$ and the Lipschitz norm $[\cdot]_{\mathrm{Lip}(\alpha)}$ by assigning different values to α. Similar to the definition of CMO in § 6, we can introduce the vanishing $\mathrm{BMO}(\alpha)$ function and the vanishing α-Carleson measure, respectively. The details will be omitted.

The following result is the Campanato version of Theorems 2.2 and 6.1.

Theorem 7.2. *Let $(\mathcal{M}, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition (2.1). Assume that the heat kernel $h_t(x, y)$ admits the diagonal upper estimate (2.3) and the Hölder continuity (2.6). If $0 \leqslant \alpha < \theta/n$ and $0 < \sigma < 1$, then u is a (vanishing) $\mathrm{Cam}(\alpha)$ function if and only if U is a (vanishing) α-Carleson measure. Moreover, it holds*

$$[u]_{\mathrm{Cam}(\alpha)} \approx [U]_{\mathrm{Car}(\alpha)}.$$

https://doi.org/10.1017/prm.2023.58 Published online by Cambridge University Press

Acknowledgements

Bo Li thanks Denis Brazke for many helpful discussions during the preparation of this work. Bo Li is supported by NNSF of China (12201250) and NSF of Zhejiang Province (LQ23A010007). Jinxia Li is supported by NNSF of China (12101199), NSF of Henan Province (232300421142), and the Doctoral Fund of Henan Polytechnic University (B2021-54). Qingze Lin is supported by NNSF of China (11801094). Bolin Ma is supported by NNSF of China (11871452).

References

1 F. Baudoin, Q. Lang and Y. Sire, Powers of generators on Dirichlet spaces and applications to Harnack principles, arXiv: 2010.01036.

2 D. Brazke, A. Schikorra and Y. Sire. On BMO and Carleson measures on Riemannian manifolds. *Int. Math. Res. Not. IMRN* **2022** (2022), 1245–1269.

3 L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differ. Equ.* **32** (2007), 1245–1260.

4 G. Carron. Riesz transform on manifolds with quadratic curvature decay. *Rev. Mat. Iberoam.* **33** (2017), 749–788.

5 J. Chen. A representation theorem of harmonic functions and its application to BMO on manifolds. *Appl. Math. J. Chinese Univ. Ser. B* **16** (2001), 279–284.

6 P. Chen, X. Duong, J. Li, L. Song and L. Yan. Carleson measures, BMO spaces and balayages associated to Schrödinger operators. *Sci. China Math.* **60** (2017), 2077–2092.

7 R. Coifman and G. Weiss. Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.* **83** (1977), 569–645.

8 T. Coulhon, R. Jiang, P. Koskela and A. Sikora. Gradient estimates for heat kernels and harmonic functions. *J. Funct. Anal.* **278** (2020), 108398.

9 T. Coulhon and A. Sikora. Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. *Proc. Lond. Math. Soc. (3)* **96** (2008), 507–544.

10 Y. Ding, M. Lee and C. Lin. Carleson measure characterization of weighted BMO associated with a family of general sets. *J. Geom. Anal.* **27** (2017), 842–867.

11 X. Duong, L. Yan and C. Zhang. On characterization of Poisson integrals of Schrödinger operators with BMO traces. *J. Funct. Anal.* **266** (2014), 2053–2085.

12 E. Fabes, R. Johnson and U. Neri. Spaces of harmonic functions representable by Poisson integrals of functions in BMO and $L_{p,\lambda}$. *Indiana Univ. Math. J.* **25** (1976), 159–170.

13 E. Fabes and U. Neri. Characterization of temperatures with initial data in BMO. *Duke Math. J.* **42** (1975), 725–734.

14 C. Fefferman. Characterizations of bounded mean oscillation. *Bull. Amer. Math. Soc.* **77** (1971), 587–588.

15 M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, in: de Gruyter Studies in Mathematics, Vol. 19 (Walter de Gruyter & Co., Berlin, 1994).

16 J. Galé, P. Miana and P. Stinga. Extension problem and fractional operators: semigroups and wave equations. *J. Evol. Equ.* **13** (2013), 343–368.

17 A. Grigor'yan. Gaussian upper bounds for the heat kernel on arbitrary manifolds. *J. Differ. Geom.* **45** (1997), 33–52.

18 A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. *(Russian) Mat. Sb.* **182** (1991), 55–87. translation in Math. USSR-Sb. 72 (1992), 47–77.

19 P. Gyrya, L. Saloff-Coste: Neumann and Dirichlet heat kernels in inner uniform domains, *Astérisque*, **33** (2011), Soc. Math. France.

20 S. Hartzstein and O. Salinas. Weighted BMO and Carleson measures on spaces of homogeneous type. *J. Math. Anal. Appl.* **342** (2008), 950–969.

21 J. Heinonen, P. Koskela, N. Shanmugalingam and T. Tyson, *Sobolev spaces on metric measure spaces. An approach based on upper gradients*, New Mathematical Monographs, Vol. 27 (Cambridge University Press, Cambridge, 2015).

22 R. Jiang and B. Li. Dirichlet problem for Schrödinger equation with the boundary value in the BMO space. *Sci. China Math.* **65** (2022), 1431–1468.

23 R. Jiang and F. Lin. Riesz transform under perturbations via heat kernel regularity. *J. Math. Pures Appl. (9)* **133** (2020), 39–65.

24 R. Jiang, J. Xiao and D. Yang. Towards spaces of harmonic functions with traces in square Campanato spaces and their scaling invariants. *Anal. Appl. (Singap.)* **14** (2016), 679–703.

25 Y. Jin, B. Li and T. Shen, Harmonic functions with BMO traces and their limiting behaviors on metric measure spaces, preprint.

26 B. Li, B. Ma, T. Shen, X. Wu and C. Zhang. On the caloric functions with BMO traces and their limiting behaviors. *J. Geom. Anal.* **33** (2023), 215, 42 pp.

27 C. Lin and H. Liu. $\mathrm{BMO}_L(\mathbb{H}^n)$ spaces and Carleson measures for Schrödinger operators. *Adv. Math.* **228** (2011), 1631–1688.

28 L. Song, X. Tian and L. Yan. On characterization of Poisson integrals of Schrödinger operators with Morrey traces. *Acta Math. Sin. (Engl. Ser.)* **34** (2018), 787–800.

29 L. Song and L. Wu. The CMO-Dirichlet problem for the Schrödinger equation in the upper half-space and characterizations of CMO. *J. Geom. Anal.* **32** (2022), 130, 37 pp.

30 P. Stinga and J. Torrea. Extension problem and Harnack's inequality for some fractional operators. *Comm. Partial Differ. Equ.* **35** (2010), 2092–2122.

31 K. Sturm. Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality. *J. Math. Pures Appl. (9)* **75** (1996), 273–297.

32 N. Trong and N. Tung. Weighted BMO type spaces associated to admissible functions and applications. *Acta Math. Vietnam.* **41** (2016), 209–241.

33 Y. Wang and J. Xiao. Homogeneous Campanato–Sobolev classes. *Appl. Comput. Harmon. Anal.* **39** (2015), 214–247.