8

The conformal Einstein field equations

To use conformal rescalings to analyse the global existence of asymptotically simple spacetimes one requires a suitable *conformal representation* of the Einstein field equations. The naive direct approach to this problem is to make use of the transformation law of the Ricci tensor. However, this leads to equations which are *singular* at the conformal boundary, so that the standard theory of partial differential equations (PDEs) cannot be applied. Remarkably, by introducing new variables, it is possible to obtain a system of equations for various *conformal fields* which is regular even at the conformal boundary and whose solutions imply, in turn, solutions to the Einstein field equations – this construction was first done in Friedrich (1981b). These equations are known as the *conformal Einstein field equations*.

This chapter provides derivations of two versions of the conformal field equations introduced by Friedrich: the so-called *standard conformal Einstein field equations* written in terms of the Levi-Civita connection of a conformally rescaled (unphysical) spacetime, and the *extended conformal field equations* which are given in terms of a Weyl connection. These two versions of the conformal equations can be expressed in tensorial, frame or spinorial form. The presentation in this chapter allows for the presence of general classes of matter models. It also provides a discussion of some basic properties of the equations, in particular, their conformal covariance and their relation to the Einstein field equations.

8.1 A singular equation for the conformal metric

Assume one has two spacetimes $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and $(\mathcal{M}, \boldsymbol{g})$ which are related to each other by means of a conformal transformation given by

$$g_{ab} = \Xi^2 \tilde{g}_{ab}. \tag{8.1}$$

Following the conventions of Section 7.1, $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is called the ***physical spacetime***, while $(\mathcal{M}, \boldsymbol{g})$ is known as the ***unphysical spacetime***. For simplicity, the discussion in this section is restricted to the case $\tilde{R}_{ab} = 0$.

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From the discussion in Chapter 5, the conformal rescaling (8.1) implies the transformation law

$$R_{ab} = \tilde{R}_{ab} - 2\Xi^{-1}\nabla_a\nabla_b\Xi - g_{ab}g^{cd}\left(\Xi^{-1}\nabla_c\nabla_d\Xi - 3\Xi^{-2}\nabla_c\Xi\nabla_d\Xi\right) \tag{8.2}$$

for the Ricci tensor. Combining this expression with the vacuum Einstein field equations one obtains the following *conformal vacuum Einstein field equation*:

$$R_{ab} - \frac{1}{2}Rg_{ab} = -2\Xi^{-1}\left(\nabla_a\nabla_b\Xi - \nabla_c\nabla^c\Xi g_{ab}\right) - 3\Xi^{-2}\nabla_c\Xi\nabla^c\Xi g_{ab}. \tag{8.3}$$

The latter equation can be interpreted as an Einstein field equation for the unphysical metric $\boldsymbol{g}$ with an *unphysical matter* with energy-momentum tensor T_{ab} given by

$$T_{ab} \equiv -2\Xi^{-1}\left(\nabla_a\nabla_b\Xi - \nabla_c\nabla^c\Xi g_{ab}\right) - 3\Xi^{-2}\nabla_c\Xi\nabla^c\Xi g_{ab}.$$

Equation (8.3) contains factors of Ξ^{-1} which become singular at $\Xi = 0$. Following the discussion of Chapter 7, such points correspond to the conformal boundary of the spacetime – a region of the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ for which one would like to be able to make analytic statements. This is not possible for Equation (8.3) as the standard theory of PDEs assumes equations which are *formally regular*. It is important to observe that multiplying Equation (8.3) by Ξ^2 does not improve the state of affairs as one has then an equation whose *principal part* (i.e. the terms containing the higher order derivatives) vanishes at $\Xi = 0$.

8.2 The metric regular conformal field equations

In what follows, it will be shown that by introducing new variables and reinterpreting old ones, it is possible to obtain a set of equations which is regular even at the conformal boundary. Under suitable conditions, a solution of this system implies a solution to the *physical* Einstein field equations.

The analysis of this section assumes a general matter content of the spacetime so that

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + \lambda\tilde{g}_{ab} = \tilde{T}_{ab} \tag{8.4}$$

and

$$\tilde{\nabla}^a\tilde{T}_{ab} = 0.$$

From the above it follows directly that

$$\tilde{R} = 4\lambda - \tilde{T}, \tag{8.5a}$$

$$\tilde{L}_{ab} \equiv \frac{1}{2}\tilde{R}_{ab} + \frac{1}{12}\tilde{R}\tilde{g}_{ab} = \frac{1}{2}\tilde{T}_{ab} + \frac{1}{6}(\lambda - \tilde{T})\tilde{g}_{ab}, \tag{8.5b}$$

where $\tilde{T} \equiv \tilde{g}^{ab}\tilde{T}_{ab}$ and $\tilde{L}_{ab}$ denotes the physical Schouten tensor.

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8.2.1 The regularisation of the transformation law for the Schouten tensor

The starting point of the construction is the *singular* transformation law for the Ricci tensor given by Equation (8.2). In practice, it is more convenient to work with the Schouten tensor than with the Ricci tensor. The analogue of Equation (8.2) for the Schouten tensor is given by

$$L_{ab} = \tilde{L}_{ab} - \Xi^{-1}\nabla_a\nabla_b\Xi + \frac{1}{2}\Xi^{-2}\nabla_c\Xi\nabla^c\Xi\, g_{ab}. \tag{8.6}$$

Formally, the most singular term in this equation is $\frac{1}{2}\Xi^{-2}\nabla_c\Xi\nabla^c\Xi$. From the transformation law

$$R = \Xi^{-2}\tilde{R} - 6\Xi^{-1}\nabla_c\nabla^c\Xi + 12\Xi^{-2}\nabla_c\Xi\nabla^c\Xi, \tag{8.7}$$

it follows that

$$\Xi^{-2}\nabla_c\Xi\nabla^c\Xi = \frac{1}{12}\left(R - \Xi^{-2}\tilde{R}\right) + \frac{1}{2}\Xi^{-1}\nabla_c\nabla^c\Xi. \tag{8.8}$$

The right-hand side of the last expression contains the singular term $-\frac{1}{12}\Xi^{-2}\tilde{R}$. Yet substituting Equation (8.8) into (8.6), some cancellations occur. Making use of Equation (8.5b) one obtains

$$L_{ab} = \frac{1}{2}\tilde{T}_{ab} + \frac{1}{6}(\lambda - \tilde{T})\tilde{g}_{ab} - \Xi^{-1}\nabla_a\nabla_b\Xi + \frac{1}{24}\left(R - \Xi^{-2}\tilde{R}\right)g_{ab} + \frac{1}{4}\Xi^{-1}\nabla^c\nabla_c\Xi g_{ab}.$$

Now, defining the ***Friedrich scalar***

$$s \equiv \frac{1}{4}\nabla^c\nabla_c\Xi + \frac{1}{24}R\Xi, \tag{8.9}$$

and writing $\Xi^{-2}\tilde{R}g_{ab} = \tilde{R}\tilde{g}_{ab}$, one obtains

$$\begin{aligned} L_{ab} &= \frac{1}{2}\tilde{T}_{ab} + \left(\frac{1}{6}\lambda - \frac{1}{6}\tilde{T} - \frac{1}{24}\tilde{R}\right)\tilde{g}_{ab} - \Xi^{-1}\nabla_a\nabla_b\Xi + \Xi^{-1}sg_{ab}, \\ &= \frac{1}{2}\tilde{T}_{ab} - \frac{1}{8}\tilde{T}\tilde{g}_{ab} - \Xi^{-1}\nabla_a\nabla_b\Xi + \Xi^{-1}sg_{ab}, \end{aligned} \tag{8.10}$$

where in the last expression, Equation (8.5a) has been used. The last expression brings about the question of the transformation law for the energy-momentum tensor $\tilde{T}_{ab}$ upon the conformal rescaling $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$. As $\tilde{T}_{ab}$ *is not a geometric object* derived from the metric $\tilde{\boldsymbol{g}}$ and concomitants thereof, one is free to choose the transformation law which best suits the analysis. As will be further elaborated in Chapter 9, a convenient choice is to define the ***unphysical energy-momentum tensor*** T_{ab} as

$$T_{ab} \equiv \Xi^{-2}\tilde{T}_{ab}.$$

It follows then that

$$\frac{1}{2}\tilde{T}_{ab} - \frac{1}{8}\tilde{T}\tilde{g}_{ab} = \Xi^2\left(\frac{1}{2}T_{ab} - \frac{1}{8}Tg_{ab}\right) = \frac{1}{2}\Xi^2 T_{\{ab\}}, \tag{8.11}$$

where $T \equiv g^{ab}T_{ab}$ so that $\tilde{T} = \Xi^4 T$ and $T_{\{ab\}}$ denotes the $\boldsymbol{g}$-trace-free part of T_{ab}. Substituting Equation (8.11) into Equation (8.10) one obtains

$$L_{ab} = \frac{1}{2}\Xi^2 T_{\{ab\}} - \Xi^{-1}\nabla_a\nabla_b\Xi + \Xi^{-1}sg_{ab}. \tag{8.12}$$

This last equation still contains formally singular terms. To get around this problem, one reads it not as determining the components of the conformal metric $\boldsymbol{g}$ contained in L_{ab}, but as conditions on the second covariant derivative of the conformal factor Ξ. Adopting this point of view, and multiplying Equation (8.12) by Ξ one obtains

$$\nabla_a\nabla_b\Xi = -\Xi L_{ab} + sg_{ab} + \frac{1}{2}\Xi^3 T_{\{ab\}}. \tag{8.13}$$

Equation (8.13) promotes the fields s and L_{ab} to the level of unknowns for which suitable equations need to be constructed. This will be done in the following sections.

8.2.2 The equation for s

In order to construct an equation for s, one applies ∇_c to Equation (8.13) and obtains

$$\begin{aligned}\nabla_c\nabla_a\nabla_b\Xi = {} & -\nabla_c\Xi L_{ab} - \Xi\nabla_c L_{ab} + \nabla_c s g_{ab} \\ & + \frac{3}{2}\Xi^2\nabla_c\Xi T_{\{ab\}} + \frac{1}{2}\Xi^3\nabla_c T_{\{ab\}}.\end{aligned} \tag{8.14}$$

By commuting covariant derivatives, the right-hand side of this equation can be rewritten as

$$\nabla_c\nabla_a\nabla_b\Xi = \nabla_a\nabla_c\nabla_b\Xi - R^d{}_{bca}\nabla_d\Xi.$$

Hence, contracting the indices $_b$ and $_c$ one finds that Equation (8.14) implies

$$\begin{aligned}\nabla_a(\nabla^c\nabla_c\Xi) + R_{ca}\nabla^c\Xi = {} & -L_{ca}\nabla^c\Xi - \Xi\nabla^c L_{ac} + \nabla_a s \\ & + \frac{3}{2}\Xi^2\nabla^c\Xi T_{\{ac\}} + \frac{1}{2}\Xi^3\nabla^c T_{\{ac\}}.\end{aligned} \tag{8.15}$$

Now, the definition of the field s, Equation (8.9), implies that

$$\nabla_a(\nabla^c\nabla_c\Xi) = 4\nabla_a s - \frac{1}{6}\Xi\nabla_a R - \frac{1}{6}R\nabla_a\Xi.$$

Using this expression in (8.15) and observing that

$$R_{ab} = 2L_{ab} + \frac{1}{6}Rg_{ab}, \tag{8.16}$$

one obtains

$$3\nabla_a s - \frac{1}{6}\Xi\nabla_a R = -3L_{ac}\nabla^c\Xi - \Xi\nabla^c L_{ac} + \frac{3}{2}\Xi^2\nabla^c\Xi T_{\{ac\}} + \frac{1}{2}\Xi^3\nabla^c T_{\{ac\}}.$$

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Now, let $G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab}$ denote the Einstein tensor of the metric $\boldsymbol{g}$. One has that $\nabla^a G_{ab} = 0$. This last equation can be rewritten in terms of the Schouten tensor as

$$\nabla^c L_{ca} - \frac{1}{6}\nabla_a R = 0. \tag{8.17}$$

Making use of this last expression one obtains

$$\nabla_a s = -L_{ac}\nabla^c\Xi + \frac{1}{2}\Xi^2\nabla^c\Xi T_{\{ac\}} + \frac{1}{6}\Xi^3\nabla^c T_{\{ca\}}. \tag{8.18}$$

This is a suitable equation for s.

8.2.3 The equations for the curvature

Equation (8.13) brings the unphysical Schouten tensor L_{ab} into play. Thus, one needs to obtain an equation which can be regarded as a differential condition on L_{ab}. The natural place to look for such an equation is the second Bianchi identity; see Section 2.4.3. In Section 5.2.2, it has been shown that the second Bianchi identity together with the decomposition of the Riemann tensor in terms of the Weyl and Schouten tensors lead to the expressions

$$\tilde{\nabla}_c\tilde{L}_{db} - \tilde{\nabla}_d\tilde{L}_{cb} = \tilde{\nabla}_a C^a{}_{bcd},$$
$$\nabla_c L_{db} - \nabla_d L_{cb} = \nabla_a C^a{}_{bcd};$$

compare Equations (5.11) and (5.13). As it stands, the second of the above equations is not a satisfactory differential condition for L_{ab} as it contains, in its right-hand side, the divergence of the Weyl tensor. One needs to find an expression for the latter in terms of undifferentiated fields. Observe that the right-hand side of this equation can be expressed in terms of the physical energy-momentum tensor $\tilde{T}_{ab}$ using formula (8.5b). This will not be done at this point. Instead, it is more convenient to expresses it in terms of the *physical Cotton* tensor $\tilde{Y}_{cdb} \equiv \tilde{\nabla}_c\tilde{L}_{db} - \tilde{\nabla}_d\tilde{L}_{cb}$, so that

$$\tilde{\nabla}_a C^a{}_{bcd} = \tilde{Y}_{cdb}. \tag{8.19}$$

Now, one would like to express the divergence $\tilde{\nabla}_a C^a{}_{bcd}$ in terms of an expression involving the covariant derivative $\boldsymbol{\nabla}$. For this, one makes use of the identity

$$\nabla_a(\Xi^{-1}C^a{}_{bcd}) = \Xi^{-1}\tilde{\nabla}_a C^a{}_{bcd};$$

see Equation (5.8). Making use of the latter in Equation (8.19) one obtains

$$\nabla_a(\Xi^{-1}C^a{}_{bcd}) = \Xi^{-1}\tilde{Y}_{cdb}.$$

This equation seems to lead to a dead end because of the Ξ^{-1} terms appearing on both sides, and which do not cancel out. However, defining the ***rescaled Weyl tensor***

$$d^a{}_{bcd} \equiv \Xi^{-1}C^a{}_{bcd}, \tag{8.20}$$

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and the ***rescaled Cotton tensor***

$$T_{cdb} \equiv \Xi^{-1}\tilde{Y}_{cdb}, \tag{8.21}$$

one obtains the formally *regular equation*

$$\nabla_a d^a{}_{bcd} = T_{cdb}. \tag{8.22}$$

This last equation suggests that the Weyl tensor $C^a{}_{bcd}$ be replaced by the rescaled Weyl tensor $d^a{}_{bcd}$ in the construction of a regular set of conformal field equations. In Chapter 10 it will be seen that the definitions of $d^a{}_{bcd}$ and T_{cdb} are justified in the sense that under suitable assumptions *the tensors $d^a{}_{bcd}$ and T_{cdb} are regular at the points where* $\Xi = 0$; see, in particular, Theorem 10.3.

One is now in the position of returning to the analysis of the equation for the Schouten tensor. Writing $C^a{}_{bcd}$ in terms of $d^a{}_{bcd}$ one obtains

$$\begin{aligned}\nabla_c L_{db} - \nabla_d L_{cb} &= \nabla_a(\Xi d^a{}_{bcd})\\ &= \nabla_a \Xi d^a{}_{bcd} + \Xi \nabla_a d^a{}_{bcd}.\end{aligned}$$

Finally, using Equation (8.22) in the last term yields

$$\nabla_c L_{db} - \nabla_d L_{cb} = \nabla_a \Xi d^a{}_{bcd} + \Xi T_{cdb}, \tag{8.23}$$

which, again, is formally regular if $\Xi = 0$.

8.2.4 The regularised transformation rule for the Ricci scalar

To relate solutions of the conformal field equations to solutions of the Einstein field equations, one also needs to consider a regularised version of the transformation rule for the Ricci scalar, Equation (8.7). Multiplying this transformation law by Ξ^2 and rearranging the various terms one obtains

$$\tilde{R} = \Xi^2 R + 6\Xi \nabla_c \nabla^c \Xi - 12 \nabla_c \Xi \nabla^c \Xi.$$

Finally, using Equations (8.5a) and (8.9) one concludes that

$$\lambda = 6\Xi s - 3\nabla_c \Xi \nabla^c \Xi + \frac{1}{4}\Xi^4 T. \tag{8.24}$$

To understand the role of this equation it is useful to compute the derivative of its right-hand side. One has that

$$\begin{aligned}\nabla_a\left(6\Xi s - 3\nabla_c\Xi\nabla^c\Xi + \frac{1}{4}\Xi^4 T\right)&\\ &= 6\nabla_a \Xi s + 6\Xi\nabla_a s - 6\nabla_a\nabla_c\Xi\nabla^c\Xi + \Xi^3\nabla_a\Xi T + \frac{1}{4}\Xi^4\nabla_a T\\ &= \Xi^4(\nabla^c T_{ca} + \Xi^{-1}\nabla_a \Xi T),\end{aligned}$$

where in the second equality Equations (8.13) and (8.18) have been used to remove, respectively, the terms $\nabla_a\nabla_c\Xi$ and $\nabla_a\Xi$.

As will be further discussed in Chapter 9, the tensors $\tilde{T}_{ab}$ and T_{ab} satisfy the relation

$$g^{bc}\nabla_b T_{ca} = \Xi^{-4}\tilde{g}^{bc}(\tilde{\nabla}_b\tilde{T}_{ca} - \Xi^{-1}\tilde{\nabla}_a\Xi\tilde{T}_{bc}).$$

Hence, if $\tilde{\nabla}^b\tilde{T}_{ba} = 0$ it follows that

$$\nabla^c T_{ca} + \Xi^{-1}\nabla_a\Xi T = 0. \tag{8.25}$$

This last relation implies that

$$\nabla_a\left(6\Xi s - 3\nabla_c\Xi\nabla^c\Xi + \frac{1}{4}\Xi^4 T\right) = 0.$$

One has the following result:

Lemma 8.1 (***propagation of the cosmological constant***) *If Equations* (8.13), (8.18) *and* (8.25) *are satisfied on $\mathcal{M}$ and, in addition, Equation (8.24) holds at a point $p \in \mathcal{M}$, then Equation (8.24) is also satisfied on $\mathcal{M}$.*

Thus, Equation (8.24) plays the role of a *constraint* which is preserved, upon evolution, by virtue of the other conformal field equations.

8.2.5 Properties of the metric conformal field equations

The discussion of the previous sections is summarised in the following list of equations:

$$\nabla_a\nabla_b\Xi = -\Xi L_{ab} + s g_{ab} + \frac{1}{2}\Xi^3 T_{\{ab\}}, \tag{8.26a}$$

$$\nabla_a s = -L_{ac}\nabla^c\Xi + \frac{1}{2}\Xi^2\nabla^c\Xi T_{\{ac\}} + \frac{1}{6}\Xi^3\nabla^c T_{\{ca\}}, \tag{8.26b}$$

$$\nabla_c L_{db} - \nabla_d L_{cb} = \nabla_a\Xi d^a{}_{bcd} + \Xi T_{cdb}, \tag{8.26c}$$

$$\nabla_a d^a{}_{bcd} = T_{cdb}, \tag{8.26d}$$

$$6\Xi s - 3\nabla_c\Xi\nabla^c\Xi + \frac{1}{4}\Xi^4 T = \lambda. \tag{8.26e}$$

These are known as the (regular) ***metric conformal Einstein field equations***. *Equations* (8.26a)–(8.26e) *should be read as differential conditions for the fields Ξ, s, L_{ab}, $d^a{}_{bcd}$.* As already mentioned, Equation (8.26e) plays the role of a constraint. At the points where $\Xi \neq 0$, these equations are complemented by the physical conservation equation $\tilde{\nabla}^a\tilde{T}_{ab} = 0$ expressed in terms of conformal quantities:

$$\nabla^c T_{ca} + \Xi^{-1}\nabla_a\Xi T = 0. \tag{8.27}$$

Observe that in contrast to Equations (8.26a)–(8.26e), *Equation (8.27) is not formally regular at $\Xi = 0$.* This equation will be analysed in more detail in Chapter 9.

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In what follows, for a ***solution to the metric conformal Einstein field equations*** it will be understood that a collection of fields

$$(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}, T_{ab})$$

satisfies Equations (8.26a)–(8.26e) and (8.27).

Remark. The discussion so far has not considered an equation for the components of the metric g_{ab}. To obtain the required condition assume that the Schouten tensor L_{ab} is determined through Equation (8.26c) and consider the relation (8.16) expressed in terms of some local coordinates (x^μ):

$$R_{\mu\nu} = 2L_{\mu\nu} + \frac{1}{6}Rg_{\mu\nu}.$$

Recalling that the components $R_{\mu\nu}$ can be expressed in terms of second-order derivatives of the components of the metric, one can read the previous expression as a differential condition for $g_{\mu\nu}$. To cast this equation in the form of some recognisable type of PDE one needs to make a particular choice of coordinates; see the discussions in Section 13.5.1 and in the Appendix of Chapter 13 on the *reduced Einstein field equations*.

The conformal vacuum Einstein field equations

An important particular case of Equations (8.26a)–(8.26e) occurs when $\tilde{T}_{ab} = 0$ on the whole of $\tilde{\mathcal{M}}$. Then one also has that $T_{abc} = 0$, and the conformal field equations reduce to:

$$\nabla_a \nabla_b \Xi = -\Xi L_{ab} + s g_{ab}, \tag{8.28a}$$

$$\nabla_a s = -L_{ac}\nabla^c \Xi, \tag{8.28b}$$

$$\nabla_c L_{db} - \nabla_d L_{cb} = \nabla_a \Xi d^a{}_{bcd}, \tag{8.28c}$$

$$\nabla_a d^a{}_{bcd} = 0, \tag{8.28d}$$

$$6\Xi s - 3\nabla_c \Xi \nabla^c \Xi = \lambda. \tag{8.28e}$$

Equations (8.28a)–(8.28e) are known as the ***conformal vacuum Einstein field equations***.

The metric conformal field equations and the Einstein field equations

Any solution to the Einstein field equations satisfies Equations (8.26a)–(8.26e) for *any* (smooth) choice of conformal factor Ξ. The converse of this observation is given in the following result.

Proposition 8.1 (***solutions of the conformal Einstein field equations as solutions to the Einstein field equations***) *Let*

$$(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}, T_{ab})$$

denote a solution to Equations (8.26a)–(8.26d) *and* (8.27) *such that* $\Xi \neq 0$ *on an open set* $\mathcal{U} \subset \mathcal{M}$*. If, in addition, Equation (8.26e) is satisfied at a point* $p \in \mathcal{U}$*, then the metric* $\tilde{g}_{ab} = \Xi^{-2} g_{ab}$ *is a solution to the Einstein field Equations (8.4) on* $\mathcal{U}$*.*

Proof It will be first shown that the Schouten tensor $\tilde{L}_{ab}$ of the metric $\tilde{g}_{ab} = \Xi^{-2} g_{ab}$ satisfies Equation (8.5b). Notice that the metric $\tilde{g}_{ab}$ is well defined on $\mathcal{U}$ as $\Xi \neq 0$. The transformation law for the Schouten tensor under conformal rescalings gives

$$\tilde{L}_{ab} = L_{ab} + \Xi^{-1} \nabla_a \nabla_b \Xi - \frac{1}{2} \Xi^{-2} \nabla_c \Xi \nabla^c \Xi g_{ab}.$$

Using Equations (8.26a) and (8.26b) the latter simplifies to

$$\tilde{L}_{ab} = \frac{1}{2} \tilde{T}_{ab} + \frac{1}{6} (\lambda - \tilde{T}) \tilde{g}_{ab},$$

as required. In order to conclude that the Einstein field equations hold, one also needs to compute the Ricci scalar of the metric $\tilde{g}_{ab}$. As a consequence of Lemma 8.1 one has that Equation (8.26e) holds on the whole of $\mathcal{U}$. From the latter, again using (8.26a) and (8.26b) and recalling that $\tilde{T} = \Xi^4 T$, it follows that $\tilde{R} = 4\lambda - \tilde{T}$. Combining the obtained expressions for $\tilde{L}_{ab}$ and $\tilde{R}$ one readily concludes that (8.4) is indeed satisfied. □

Conformal freedom and conformal gauge

Consider a solution $(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}, T_{ab})$ to the metric conformal field Equations (8.26a)–(8.26e) and (8.27). As a consequence of Proposition 8.1, one has that $\tilde{\boldsymbol{g}} = \Xi^{-2} \boldsymbol{g}$ and $\tilde{T}_{ab} = \Xi^2 T_{ab}$, give rise to a solution of the Einstein field equations as long as $\Xi \neq 0$. Consider now another conformal factor $\acute{\Xi}$. From $\acute{\Xi}$, together with the physical fields $\tilde{g}_{ab}$ and $\tilde{T}_{ab}$, one can construct, by direct computation using the definitions of Sections 8.2.1–8.2.3, a collection of conformal fields $(\acute{g}_{ab}, \acute{\Xi}, \acute{s}, \acute{L}_{ab}, \acute{d}^a{}_{bcd}, \acute{T}_{ab})$. In particular, one has that $\acute{g}_{ab} = \acute{\Xi}^2 \tilde{g}_{ab}$ and $\acute{T}_{ab} = \Xi^{-2} T_{ab}$. These fields constitute, in turn, a solution to the metric conformal field equations. That is, they satisfy

$$\begin{aligned}
&\acute{\nabla}_a \acute{\nabla}_b \acute{\Xi} = -\acute{\Xi} \acute{L}_{ab} + \acute{s} \acute{g}_{ab} + \frac{1}{2} \acute{\Xi}^3 \acute{T}_{\{ab\}}, \\
&\acute{\nabla}_a \acute{s} = -\acute{L}_{ac} \acute{\nabla}^c \acute{\Xi} + \frac{1}{2} \acute{\Xi}^2 \acute{\nabla}^c \acute{\Xi} \acute{T}_{\{ac\}} + \frac{1}{6} \acute{\Xi}^3 \acute{\nabla}^c \acute{T}_{\{ca\}}, \\
&\acute{\nabla}_c \acute{L}_{db} - \acute{\nabla}_d \acute{L}_{cb} = \acute{\nabla}_a \Xi \acute{d}^a{}_{bcd} + \acute{\Xi} \acute{T}_{cdb}, \\
&\acute{\nabla}_a \acute{d}^a{}_{bcd} = \acute{T}_{cdb}, \\
&6 \acute{\Xi} \acute{s} - 3 \acute{\nabla}_c \acute{\Xi} \acute{\nabla}^c \acute{\Xi} + \frac{1}{4} \acute{\Xi}^4 \acute{T} = \lambda, \\
&\acute{\nabla}^c \acute{T}_{ca} + \acute{\Xi}^{-1} \acute{\nabla}_a \acute{\Xi} \acute{T} = 0.
\end{aligned}$$

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The unphysical metrics $\boldsymbol{g}$ and $\acute{\boldsymbol{g}}$ are conformally related to each other: one has that $\acute{\boldsymbol{g}} = \kappa^2 \boldsymbol{g}$ with $\kappa \equiv \acute{\Xi}\Xi^{-1}$, $\Xi \neq 0$. Using the transformation formulae of Chapter 5, one can express the solution $(\acute{g}_{ab}, \acute{\Xi}, \acute{s}, \acute{L}_{ab}, \acute{d}^a{}_{bcd}, \acute{T}_{ab})$ in terms of $(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}, T_{ab})$ and κ. One has that

$$\acute{\Xi} = \kappa \Xi, \qquad \acute{g}_{ab} = \kappa^2 g_{ab}, \tag{8.29a}$$

$$\acute{s} = \kappa^{-1} s + \kappa^{-2} \nabla_c \kappa \nabla^c \Xi + \frac{1}{2}\kappa^{-3} \Xi \nabla_c \kappa \nabla^c \kappa, \tag{8.29b}$$

$$\acute{L}_{ab} = L_{ab} - \kappa^{-1} \nabla_a \nabla_b \kappa + \frac{1}{2}\kappa^{-2} \nabla_c \kappa \nabla^c \kappa \, g_{ab}, \tag{8.29c}$$

$$\acute{d}^a{}_{bcd} = \kappa d^a{}_{bcd}, \tag{8.29d}$$

$$\acute{T}_{ab} = \kappa^{-2} T_{ab}. \tag{8.29e}$$

The two sets of solutions to the metric conformal field equations are said to be ***conformally related***.

From the discussion of the previous paragraphs it follows that there exists an infinite number of solutions to the metric conformal field equations giving rise to the same solution of the Einstein field equations. This is a manifestation of the ***conformal invariance*** of the equations. This conformal invariance is tied to a ***conformal freedom (or gauge)*** which, in turn, manifests itself in the properties of the unphysical metric $\boldsymbol{g}$. This conformal freedom has to be fixed in some way if one is to apply the theory of PDEs to the metric conformal field equations.

The issue of the conformal gauge discussed in the previous paragraph is closely related to the Ricci scalar R of the unphysical metric $\boldsymbol{g}$. The scalar R does not explicitly appear in the conformal Equations (8.26a)–(8.26e) and (8.27). Hence, it is not determined by the equations. Of course, given a solution $(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}, T_{ab})$, one can readily compute R. In general, conformally related solutions to the metric conformal field equations will give rise to different Ricci scalars. In order to understand better the connection between the conformal gauge and the Ricci scalar, consider a metric $\acute{\boldsymbol{g}}$ conformally related to $\boldsymbol{g}$ via $\acute{\boldsymbol{g}} = \kappa^2 \boldsymbol{g}$. The transformation law for the Ricci scalar under conformal transformations implies that

$$6\nabla^a \nabla_a \kappa - R\kappa = -\acute{R}\kappa^3, \tag{8.30}$$

from where $\acute{R}$ can be determined. Alternatively, if $\acute{R}$ is an arbitrary scalar on $\mathcal{M}$, then Equation (8.30) can be read as a linear wave equation for κ. Given suitable initial data for this equation, it can be solved locally. The solution κ gives, in turn, the metric $\acute{\boldsymbol{g}} = \kappa^2 \boldsymbol{g}$. From this point of view, the scalar field $\acute{R}$ plays the role of a ***conformal gauge source function***. In particular, one could choose $\acute{R} = 0$. As will be seen in later chapters of this book, this choice, despite its simplicity, is not necessarily the best one.

8.3 Frame and spinorial formulation of the conformal field equations

8.3.1 The frame formulation

This section provides a discussion of a frame formulation of the conformal Einstein field equations. This version of the field equations is more flexible than the metric one.

General definitions, frame fields

In what follows, consider a set of frame fields $\{\boldsymbol{e_a}\}$, $\boldsymbol{a} = \boldsymbol{0}, \ldots, \boldsymbol{3}$ which is orthonormal with respect to the metric $\boldsymbol{g}$. Frames of this type will be said to be ***g-orthonormal***. One has that

$$\boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b}) = \eta_{\boldsymbol{ab}} = \mathrm{diag}(1, -1, -1, -1).$$

Following the conventions of Chapter 2, let $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \langle \boldsymbol{\omega}^{\boldsymbol{c}}, \nabla_{\boldsymbol{a}} \boldsymbol{e_b} \rangle$ denote the connection coefficients of the connection $\boldsymbol{\nabla}$. As a consequence of the metric compatibility of $\boldsymbol{\nabla}$ one has that

$$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} \eta_{\boldsymbol{dc}} + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} \eta_{\boldsymbol{bd}} = 0.$$

The components, $\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ of the torsion of $\boldsymbol{\nabla}$ are given by the relation

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \boldsymbol{e_c} = [\boldsymbol{e_a}, \boldsymbol{e_b}] - (\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}}) \boldsymbol{e_c}.$$

In the case of $\boldsymbol{\nabla}$ one naturally has that $\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0$.

The geometric and the algebraic curvature

The discussion of the conformal field equations in terms of a frame formalism requires the expression of the components $R^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ of the Riemann tensor $R^c{}_{dab}$ with respect to the frame $\{\boldsymbol{e_a}\}$; see Equation (2.31). Let $P^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ denote the right-hand side of equation (2.31), namely,

$$\begin{aligned} P^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv{}& \boldsymbol{e_a}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \boldsymbol{e_b}(\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) \\ &+ \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}) + \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}}. \end{aligned}$$

In what follows, $P^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ will be known as the ***geometric curvature***. To complete the discussion one also needs to consider the decomposition of the Riemann tensor in terms of the Weyl tensor $C^c{}_{dab}$ and the Schouten tensor L_{ab}; see Equation (2.21b). The frame version of the decomposition is given by

$$R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = C^{\boldsymbol{c}}{}_{\boldsymbol{dab}} + 2 S_{\boldsymbol{d}[\boldsymbol{a}}{}^{\boldsymbol{ce}} L_{\boldsymbol{b}]\boldsymbol{e}},$$

where, consistent with the general conventions of Chapter 2, $C^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ and $L_{\boldsymbol{ab}}$ denote, respectively, the components of the tensors $C^c{}_{dab}$ and L_{ab} with respect to $\{\boldsymbol{e_a}\}$. The components $\rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ of the ***algebraic curvature*** are given by

$$\rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \Xi d^{\boldsymbol{c}}{}_{\boldsymbol{dab}} + 2 S_{\boldsymbol{d}[\boldsymbol{a}}{}^{\boldsymbol{ce}} L_{\boldsymbol{b}]\boldsymbol{e}}.$$

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In the above definition it has been used that $C^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = \Xi d^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$. The geometric and algebraic curvature serve as useful shorthands of expressions which will be repeatedly used. Observe, in particular, that the equation $P^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = \rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ encodes the idea that the fields $C^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ and $L_{\boldsymbol{ab}}$ correspond to the components of the Weyl and Schouten tensor of the connection defined by $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$.

The frame zero quantities and the frame conformal field equations

The frame version of the conformal field Equations (8.26a)–(8.26e) and (8.27) are readily obtained by contraction with the frame $\{\boldsymbol{e_a}\}$ and the coframe $\{\boldsymbol{\omega^a}\}$. One obtains

$$
\begin{aligned}
&\nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi = -\Xi L_{\boldsymbol{ab}} + s\eta_{\boldsymbol{ab}} + \frac{1}{2}\Xi^3 T_{\{\boldsymbol{ab}\}},\\
&\nabla_{\boldsymbol{a}} s = -L_{\boldsymbol{ac}}\nabla^{\boldsymbol{c}}\Xi + \frac{1}{2}\Xi^2\nabla^{\boldsymbol{c}}\Xi T_{\{\boldsymbol{ac}\}} + \frac{1}{6}\Xi^3\nabla^{\boldsymbol{c}}T_{\{\boldsymbol{ca}\}},\\
&\nabla_{\boldsymbol{c}}L_{\boldsymbol{db}} - \nabla_{\boldsymbol{d}}L_{\boldsymbol{cb}} = \nabla_{\boldsymbol{a}}\Xi d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} + \Xi T_{\boldsymbol{cdb}},\\
&\nabla_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = T_{\boldsymbol{cdb}},\\
&6\Xi s - 3\nabla_{\boldsymbol{c}}\Xi\nabla^{\boldsymbol{c}}\Xi + \frac{1}{4}\Xi^4 T = \lambda,
\end{aligned}
$$

and

$$\nabla^{\boldsymbol{c}}T_{\boldsymbol{ca}} + \Xi^{-1}\nabla_{\boldsymbol{a}}\Xi T = 0,$$

where the directional derivative $\nabla_{\boldsymbol{a}}$ acts on components of tensorial fields according to the rules in (2.28). The above frame conformal field equations will be complemented by the *structure equations*

$$
\begin{aligned}
&\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c} = 0,\\
&P^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = \rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}},
\end{aligned}
$$

which express that for the connection $\boldsymbol{\nabla}$, its torsion must vanish and its geometric and algebraic curvature must coincide.

For convenience of the subsequent discussion one introduces a set of ***zero quantities***:

$$\Sigma_{\boldsymbol{ab}} \equiv \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c}, \tag{8.31a}$$

$$\Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv P^{\boldsymbol{c}}{}_{\boldsymbol{dab}} - \rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \tag{8.31b}$$

$$Z_{\boldsymbol{ab}} \equiv \nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi + \Xi L_{\boldsymbol{ab}} - s\eta_{\boldsymbol{ab}} - \frac{1}{2}\Xi^3 T_{\{\boldsymbol{ab}\}}, \tag{8.31c}$$

$$Z_{\boldsymbol{a}} \equiv \nabla_{\boldsymbol{a}} s + L_{\boldsymbol{ac}}\nabla^{\boldsymbol{c}}\Xi - \frac{1}{2}\Xi^2\nabla^{\boldsymbol{c}}\Xi T_{\{\boldsymbol{ac}\}} - \frac{1}{6}\Xi^3\nabla^{\boldsymbol{c}}T_{\{\boldsymbol{ca}\}}, \tag{8.31d}$$

$$\Delta_{\boldsymbol{cdb}} \equiv \nabla_{\boldsymbol{c}}L_{\boldsymbol{db}} - \nabla_{\boldsymbol{d}}L_{\boldsymbol{cb}} - \nabla_{\boldsymbol{a}}\Xi d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} - \Xi T_{\boldsymbol{cdb}}, \tag{8.31e}$$

$$\Lambda_{\boldsymbol{bcd}} \equiv \nabla_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} - T_{\boldsymbol{cdb}}, \tag{8.31f}$$

$$Z \equiv 6\Xi s - 3\nabla_{\boldsymbol{c}}\Xi\nabla^{\boldsymbol{c}}\Xi + \frac{1}{4}\Xi^4 T - \lambda, \tag{8.31g}$$

$$M_{\boldsymbol{a}} \equiv \nabla^{\boldsymbol{c}}T_{\boldsymbol{ca}} + \Xi^{-1}\nabla_{\boldsymbol{a}}\Xi T. \tag{8.31h}$$

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In terms of the above zero quantities, the ***frame version of the conformal field equations*** can be compactly written as

$$\Sigma_{\boldsymbol{ab}} = 0, \qquad \Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0, \qquad Z_{\boldsymbol{ab}} = 0, \qquad Z_{\boldsymbol{a}} = 0, \tag{8.32a}$$

$$\Delta_{\boldsymbol{cdb}} = 0, \qquad \Lambda_{\boldsymbol{bcd}} = 0, \qquad Z = 0, \qquad M_{\boldsymbol{a}} = 0. \tag{8.32b}$$

Accordingly, a ***solution to the frame conformal Einstein field equations*** is a collection $(\boldsymbol{e_a}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \Xi, s, L_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, T_{\boldsymbol{ab}})$ satisfying Equations (8.32a) and (8.32b). The equations associated to the zero quantities $\Sigma_{\boldsymbol{ab}}$ and $\Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ provide differential conditions for the components of the frame vectors $\{\boldsymbol{e_a}\}$ and for the connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$. The role of the equations associated to the zero quantities $Z_{\boldsymbol{ab}}$, $Z_{\boldsymbol{a}}$, $\Delta_{\boldsymbol{cdb}}$, $\Lambda_{\boldsymbol{bcd}}$, Z and $M_{\boldsymbol{a}}$ is similar to that of their metric counterparts in Section 8.2.

By considering a frame version of the conformal field equations, one introduces a further gauge freedom into the system. This gauge freedom corresponds to the Lorentz transformations preserving the $\boldsymbol{g}$-orthonormality of the frame vectors $\{\boldsymbol{e}_a\}$. In this case one speaks of a ***frame gauge freedom***. As in the case of the conformal freedom discussed in Section 8.2.5, this freedom needs to be fixed in order to be able to apply the methods of the theory of PDEs. These issues will be discussed further in Chapter 13.

The frame conformal field equations and the Einstein field equations

As in the case of the metric conformal field equations, a solution to the frame conformal field equations implies, under suitable conditions, a solution to the Einstein field equations; see Proposition 8.1. More precisely, one has:

Proposition 8.2 (***solutions to the frame conformal field equations as solutions to the Einstein field equations***) *Let*

$$(\boldsymbol{e_a}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \Xi, s, L_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, T_{\boldsymbol{ab}})$$

denote a solution to the frame conformal field Equations (8.32a) *and* (8.32b) *with* $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ *satisfying the metric compatibility condition*

$$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}}\eta_{\boldsymbol{dc}} + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}}\eta_{\boldsymbol{bd}} = 0,$$

and such that

$$\Xi \neq 0, \quad \det(\eta^{\boldsymbol{ab}}\boldsymbol{e_a} \otimes \boldsymbol{e_b}) \neq 0,$$

on an open set $\mathcal{U} \subset \mathcal{M}$. Then the metric $\tilde{\boldsymbol{g}} = \Xi^{-2}\eta_{\boldsymbol{ab}}\boldsymbol{\omega^a} \otimes \boldsymbol{\omega^b}$, where $\{\boldsymbol{\omega^a}\}$ is the dual frame to $\{\boldsymbol{e_a}\}$, is a solution to the Einstein field Equations (8.4) on $\mathcal{U}$.

Proof As a consequence of the metric compatibility assumption and $\Sigma_{\boldsymbol{ab}} = 0$, the coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ can be interpreted as the connection coefficients of a Levi-Civita connection with respect to the frame $\{\boldsymbol{e_a}\}$. By the uniqueness of the Levi-Civita connection, $\boldsymbol{g} = \eta_{\boldsymbol{ab}}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}$ is the metric associated to this connection. Notice that by assumption $\boldsymbol{g}$ is well defined on $\mathcal{U}$. Furthermore, because of $\Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0$ and exploiting the uniqueness of the decomposition of the Riemann tensor in terms of the Weyl and the Schouten tensors, it follows that $L_{\boldsymbol{ab}}$ are the components, with respect to the frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$, of the Schouten tensor of the metric $\boldsymbol{g}$. From here, following arguments analogous to those used in the proof of Proposition 8.1 one concludes that $\tilde{\boldsymbol{g}} = \Xi^{-2}\eta_{\boldsymbol{ab}}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}$ and $\tilde{T}_{ab} = \Xi^2\omega^{\boldsymbol{a}}{}_a\omega^{\boldsymbol{b}}{}_b T_{\boldsymbol{ab}}$ give a solution to the Einstein field Equations (8.4) on $\mathcal{U}$. □

8.3.2 Spinorial formulation of the conformal field equations

The frame conformal field equations lead, in a natural way, to a spinorial formulation. This formulation of the equations reveals in a more clear fashion the inherent algebraic structure of the equations and provides a systematic procedure for the construction of evolution equations. The formulation discussed in this section is not an abstract spinor formulation, but rather a *frame spinor formulation.*

General remarks concerning the spinorial formulation

Following the discussion in Section 3.1.13, the $\boldsymbol{g}$-orthonormal frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ gives rise to a frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ such that $\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{AA'}}, \boldsymbol{e}_{\boldsymbol{BB'}}) = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}$; that is, $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ is a null tetrad. In what follows, let

$$\Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}}, \quad P^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}}, \quad \rho^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}}, \quad T_{\boldsymbol{AA'BB'}},$$
$$L_{\boldsymbol{AA'BB'}}, \quad d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}}, \quad T_{\boldsymbol{AA'BB'CC'}},$$

denote, respectively, the spinorial counterparts of the fields

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \qquad P^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \qquad \rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \qquad T_{\boldsymbol{ab}}, \qquad L_{\boldsymbol{ab}}, \qquad d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \qquad T_{\boldsymbol{abc}}.$$

The spinorial counterpart of the geometric curvature, $P^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}}$, is expressed in terms of the spinorial connection coefficients $\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}}$. These, in turn, can be expressed in terms of the *reduced spin connection coefficients* $\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$; see formula (3.33). As the connection $\boldsymbol{\nabla}$ is metric, it follows that $\Gamma_{\boldsymbol{AA'BC}} = \Gamma_{\boldsymbol{AA'(BC)}}$; compare Section 3.2.2. By analogy to the split of the spinorial counterpart of the curvature tensor – Equation (3.35) – one can split the geometric curvature as

$$P^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} = P^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}\delta_{\boldsymbol{D'}}{}^{\boldsymbol{C'}} + \bar{P}^{\boldsymbol{C'}}{}_{\boldsymbol{D'AA'BB'}}\delta_{\boldsymbol{D}}{}^{\boldsymbol{C}}.$$

In what follows, the discussion will make use only of the ***reduced spinorial geometric curvature***

$$\begin{aligned} P^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} \equiv\; & \boldsymbol{e}_{\boldsymbol{AA'}}(\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}) - \boldsymbol{e}_{\boldsymbol{BB'}}(\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}) \\ & - \Gamma_{\boldsymbol{FB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{B}} - \Gamma_{\boldsymbol{BF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{B'}} + \Gamma_{\boldsymbol{FA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{A}} \\ & + \Gamma_{\boldsymbol{AF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{A'}} + \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{F}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{D}} - \Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{F}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{D}}. \end{aligned}$$

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The spinorial algebraic curvature has a similar split. Its information is encoded in the field

$$\rho^{C}{}_{DAA'BB'} \equiv -\Psi^{C}{}_{DAB}\epsilon_{A'B'} + L_{DB'AA'}\delta_{B}{}^{C} - L_{DA'BB'}\delta_{A}{}^{C},$$

where it is recalled that Ψ_{ABCD} is the Weyl spinor; see Equation (3.43). One then introduces the totally symmetric ***rescaled Weyl spinor*** ϕ_{ABCD} as

$$\phi_{ABCD} \equiv \Xi^{-1}\Psi_{ABCD}.$$

Consistent with Equation (3.43), ϕ_{ABCD} is related to the spinorial counterpart of $d^{a}{}_{bcd}$ via

$$d_{AA'BBCC'DD'} = -\phi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} - \bar{\phi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}. \tag{8.33}$$

Hence, the reduced spinorial algebraic curvature can be written as

$$\rho^{C}{}_{DAA'BB'} \equiv -\Xi\phi^{C}{}_{DAB}\epsilon_{A'B'} + L_{DB'AA'}\delta_{B}{}^{C} - L_{DA'BB'}\delta_{A}{}^{C}.$$

The spinorial counterpart of $T_{\{ab\}}$, the symmetric trace-free part of T_{ab}, is given by $T_{(AB)(A'B')}$; compare Equation (3.12). Finally, exploiting the antisymmetry $T_{cdb} = -T_{dcb}$ of the rescaled Cotton tensor, one has the split

$$T_{CC'DD'BB'} = T_{CDBB'}\epsilon_{C'D'} + \bar{T}_{C'D'BB'}\epsilon_{CD}, \tag{8.34}$$

where $T_{CDBB'} \equiv \frac{1}{2}T_{CQ'D}{}^{Q'}{}_{BB'}$. Observe that $T_{CDBB'} = T_{(CD)BB'}$.

The spinorial zero quantities

The spinorial counterparts of the frame conformal Einstein field equations are obtained by suitable contraction with the Infeld-van der Waerden symbols. Simpler expressions are obtained if one takes into account the remarks made in the previous subsection. It is convenient to introduce the following ***spinorial zero quantities***:

$$\Sigma_{AA'BB'} \equiv [e_{AA'}, e_{BB'}] - (\Gamma_{AA'}{}^{CC'}{}_{BB'} - \Gamma_{BB'}{}^{CC'}{}_{AA'})e_{CC'}, \tag{8.35a}$$

$$\Xi^{C}{}_{DAA'BB'} \equiv P^{C}{}_{DAA'BB'} - \rho^{C}{}_{DAA'BB'}, \tag{8.35b}$$

$$Z_{AA'BB'} \equiv \nabla_{AA'}\nabla_{BB'}\Xi + \Xi L_{AA'BB'} - s\epsilon_{AB}\epsilon_{A'B'} - \frac{1}{2}\Xi^{3}T_{(AB)(A'B')}, \tag{8.35c}$$

$$Z_{AA'} \equiv \nabla_{AA'}s + L_{AA'CC'}\nabla^{CC'}\Xi - \frac{1}{2}\Xi^{2}\nabla^{CC'}\Xi T_{(AC)(A'C')} - \frac{1}{6}\Xi^{3}\nabla^{CC'}T_{(AC)(A'C')}, \tag{8.35d}$$

$$\Delta_{CC'DD'BB'} \equiv \nabla_{CC'}L_{DD'BB'} - \nabla_{DD'}L_{CC'BB'} \tag{8.35e}$$

$$- \nabla_{AA'}\Xi d^{AA'}{}_{BB'CC'DD'} - \Xi T_{CC'DD'BB'}, \tag{8.35f}$$

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$$\Lambda_{\boldsymbol{BB'CC'DD'}} \equiv \nabla_{\boldsymbol{AA'}} d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}} - T_{\boldsymbol{CC'DD'BB'}}, \tag{8.35g}$$

$$Z \equiv 6\Xi s - 3\nabla_{\boldsymbol{CC'}}\nabla^{\boldsymbol{CC'}} + \frac{1}{4}\Xi^4 T - \lambda, \tag{8.35h}$$

$$M_{\boldsymbol{AA'}} \equiv \nabla^{\boldsymbol{CC'}} T_{\boldsymbol{CC'AA'}} + \Xi^{-1}\nabla_{\boldsymbol{AA'}}\Xi T. \tag{8.35i}$$

Hence, the spinorial conformal Einstein field equations are given in terms of the above zero quantities as

$$\Sigma_{\boldsymbol{AA'BB'}} = 0, \quad \Xi^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} = 0, \quad Z_{\boldsymbol{AA'BB'}} = 0, \quad Z_{\boldsymbol{AA'}} = 0, \tag{8.36a}$$

$$\Delta_{\boldsymbol{CC'DD'BB'}} = 0, \quad \Lambda_{\boldsymbol{BB'CC'DD'}} = 0, \quad Z = 0, \quad M_{\boldsymbol{AA'}} = 0. \tag{8.36b}$$

A reduced set of zero quantities can be obtained by explicitly making use of the antisymmetry of several of the spinorial zero quantities. In particular, it is noticed that as $\Delta_{\boldsymbol{CC'DD'BB'}} = -\Delta_{\boldsymbol{DD'CC'BB'}}$ and $\Lambda_{\boldsymbol{BB'CC'DD'}} = -\Lambda_{\boldsymbol{BB'DD'CC'}}$ one can write

$$\Delta_{\boldsymbol{CC'DD'BB'}} = \Delta_{\boldsymbol{CDBB'}}\epsilon_{\boldsymbol{C'D'}} + \bar{\Delta}_{\boldsymbol{C'D'BB'}}\epsilon_{\boldsymbol{CD}},$$
$$\Lambda_{\boldsymbol{BB'CC'DD'}} = \Lambda_{\boldsymbol{BB'CD}}\epsilon_{\boldsymbol{C'D'}} + \bar{\Lambda}_{\boldsymbol{B'BC'D'}}\epsilon_{\boldsymbol{CD}},$$

where

$$\Delta_{\boldsymbol{CDBB'}} \equiv \frac{1}{2}\Delta_{\boldsymbol{CQ'D}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{BB'}}, \qquad \Lambda_{\boldsymbol{BB'CD}} \equiv \frac{1}{2}\Lambda_{\boldsymbol{BB'CQ'D}}{}^{\boldsymbol{Q'}}.$$

A direct computation using the splits (8.33) and (8.34) yields

$$\Delta_{\boldsymbol{CDBB'}} = \nabla_{(\boldsymbol{C}}{}^{\boldsymbol{Q'}} L_{\boldsymbol{D})\boldsymbol{Q'BB'}} + \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\Xi\phi_{\boldsymbol{CDBQ}} + \Xi T_{\boldsymbol{CDBB'}}, \tag{8.37a}$$

$$\Lambda_{\boldsymbol{BB'CD}} = \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\phi_{\boldsymbol{BCDQ}} + T_{\boldsymbol{CDBB'}}. \tag{8.37b}$$

Thus, an equivalent spinorial formulation of the conformal field equations is given by

$$\Sigma_{\boldsymbol{AA'BB'}} = 0, \quad \Xi^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} = 0, \quad Z_{\boldsymbol{AA'BB'}} = 0, \quad Z_{\boldsymbol{AA'}} = 0 \tag{8.38a}$$

$$\Delta_{\boldsymbol{CDBB'}} = 0, \quad \Lambda_{\boldsymbol{BB'CD}} = 0, \quad Z = 0, \quad M_{\boldsymbol{AA'}} = 0. \tag{8.38b}$$

The antisymmetry of the zero quantities $\Sigma_{\boldsymbol{AA'BB'}}$ and $\Xi^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}$ can also be exploited to obtain reduced zero quantities $\Sigma_{\boldsymbol{AB}} = \Sigma_{(\boldsymbol{AB})}$ and $\Xi^{\boldsymbol{C}}{}_{\boldsymbol{DAB}} = \Xi^{\boldsymbol{C}}{}_{\boldsymbol{D}(\boldsymbol{AB})}$. This strategy will not be pursued further.

The spinorial conformal field equations and the Einstein field equations

As a consequence of the equivalence between spinorial and frame expressions discussed in Section 3.1.9, it follows that each of the two spinorial formulations of the conformal field Equations (8.36a) and (8.36b) or (8.38a) and (8.38b) is equivalent to the frame conformal field Equations (8.32a) and (8.32b). Thus, an analogue of Proposition 8.2 holds for the spinorial conformal field equations with the metric

$$\tilde{\boldsymbol{g}} = \Xi^{-2}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}\boldsymbol{\omega}^{\boldsymbol{AA'}} \otimes \boldsymbol{\omega}^{\boldsymbol{BB'}},$$

yielding the required solution to the Einstein field equations. In this last expression $\{\boldsymbol{\omega}^{\boldsymbol{AA'}}\}$ denotes the duals of the frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$.

8.3.3 Conformal freedom in the frame and spinorial conformal field equations

The transformation laws for the various conformal fields under a conformal gauge change follow from the tensorial version given in (8.29a)–(8.29e). As before, assume that one has two metrics $\boldsymbol{g}$ and $\acute{\boldsymbol{g}}$ such that $\boldsymbol{g} = \kappa^2 \acute{\boldsymbol{g}}$. Consider now a $\boldsymbol{g}$-orthonormal frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ with associated coframe $\{\boldsymbol{\omega}^{\boldsymbol{a}}\}$. From

$$\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}) = \kappa^2 \acute{\boldsymbol{g}}(\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}) = \eta_{\boldsymbol{ab}},$$

it follows that $\{\acute{\boldsymbol{e}}_{\boldsymbol{a}}\}$ and $\{\acute{\boldsymbol{\omega}}^{\boldsymbol{a}}\}$, with

$$\acute{\boldsymbol{e}}_{\boldsymbol{a}} \equiv \kappa \boldsymbol{e}_{\boldsymbol{a}}, \qquad \acute{\boldsymbol{\omega}}^{\boldsymbol{a}} \equiv \kappa^{-1} \boldsymbol{\omega}^{\boldsymbol{a}},$$

are a $\acute{\boldsymbol{g}}$-orthonormal frame and a $\acute{\boldsymbol{g}}$-orthonormal coframe, respectively. As a consequence, the tensorial transformation formulae (8.29a)–(8.29e) may pick up factors of κ depending on whether they are contracted with $\boldsymbol{e}_{\boldsymbol{a}}$ or $\acute{\boldsymbol{e}}_{\boldsymbol{a}}$. For, example

$$\begin{aligned}\acute{d}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} &\equiv \acute{\omega}^{\boldsymbol{a}}{}_a \acute{e}_{\boldsymbol{b}}{}^b \acute{e}_{\boldsymbol{c}}{}^c \acute{e}_{\boldsymbol{d}}{}^d \acute{d}^a{}_{bcd} \\ &= \kappa^3 \omega^{\boldsymbol{a}}{}_a e_{\boldsymbol{b}}{}^b e_{\boldsymbol{c}}{}^c e_{\boldsymbol{d}}{}^d d^a{}_{bcd} = \kappa^3 d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}.\end{aligned}$$

Similar considerations lead to

$$\begin{aligned}\acute{L}_{\boldsymbol{ab}} &= \kappa^2 L_{\boldsymbol{ab}} + \kappa^2 \nabla_{\boldsymbol{a}}(\kappa^{-1} \nabla_{\boldsymbol{b}} \kappa) - \frac{1}{2} S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}} \nabla_{\boldsymbol{c}} \kappa \nabla_{\boldsymbol{d}} \kappa, \\ \acute{T}_{\boldsymbol{ab}} &= \kappa^2 T_{\boldsymbol{ab}},\end{aligned}$$

where

$$\begin{aligned}S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}} &\equiv e_{\boldsymbol{a}}{}^a e_{\boldsymbol{b}}{}^b \omega^{\boldsymbol{c}}{}_c \omega^{\boldsymbol{d}}{}_d S_{ab}{}^{cd} \\ &\equiv \delta_{\boldsymbol{a}}{}^{\boldsymbol{c}} \delta_{\boldsymbol{b}}{}^{\boldsymbol{d}} + \delta_{\boldsymbol{a}}{}^{\boldsymbol{d}} \delta_{\boldsymbol{b}}{}^{\boldsymbol{c}} - \eta_{\boldsymbol{ab}} \eta^{\boldsymbol{cd}}.\end{aligned}$$

The spinorial counterparts of the conformal fields obey similar transformations. If $\{\epsilon_{\boldsymbol{A}}{}^A\}$ and $\{\acute{\epsilon}_{\boldsymbol{A}}{}^A\}$ denote the spin dyads associated, respectively, to the frame vectors $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ and $\{\acute{\boldsymbol{e}}_{\boldsymbol{a}}\}$, then

$$\acute{\epsilon}_{\boldsymbol{A}}{}^A = \kappa \epsilon_{\boldsymbol{A}}{}^A.$$

As a consequence one has, for example, that

$$\acute{\Psi}_{\boldsymbol{ABCD}} = \kappa^3 \Psi_{\boldsymbol{ABCD}}.$$

8.4 The extended conformal Einstein field equations

The conformal Einstein field equations discussed in the previous sections are expressed in terms of the Levi-Civita connection of the unphysical metric $\boldsymbol{g}$. This section provides a more general version of the equations by rewriting them in terms of a *Weyl connection*. The resulting system of equations is known as the ***extended conformal Einstein field equations***. The use of Weyl connections introduces a further freedom in the equations. This freedom can be exploited to incorporate *conformally privileged gauges*. The idea of reexpressing the vacuum conformal field equations in terms of a Weyl connection was first introduced in Friedrich (1995). Further discussions can be found in Friedrich (1998c, 2002, 2004). The extension of these ideas to the matter case has been given in Lübbe and Valiente Kroon (2012, 2013b).

In what follows, for ease of presentation, the discussion in this section is restricted to the vacuum case.

Basic setting

As in the previous sections of this chapter, let $\boldsymbol{g}$ denote an *unphysical* Lorentzian metric related to a *physical* metric $\tilde{\boldsymbol{g}}$ via $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$. The metric $\tilde{\boldsymbol{g}}$ is assumed to satisfy the vacuum Einstein field equations. Let $\boldsymbol{\nabla}$ and $\tilde{\boldsymbol{\nabla}}$ denote, respectively, the Levi-Civita connections of the metrics $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$.

In what follows, consider a Weyl connection $\hat{\boldsymbol{\nabla}}$ defined via

$$\hat{\boldsymbol{\nabla}} - \boldsymbol{\nabla} = \boldsymbol{S}(\boldsymbol{f}), \tag{8.39}$$

where $\boldsymbol{f}$ is a smooth covector. As

$$\boldsymbol{\nabla} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\Xi^{-1}\mathbf{d}\Xi),$$

it follows that $\hat{\boldsymbol{\nabla}} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\boldsymbol{f} + \Xi^{-1}\mathbf{d}\Xi)$. Hence, defining

$$\boldsymbol{\beta} \equiv \boldsymbol{f} + \Xi^{-1}\mathbf{d}\Xi,$$

one has that

$$\hat{\boldsymbol{\nabla}} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\boldsymbol{\beta}).$$

It is convenient to define

$$\boldsymbol{d} \equiv \Xi\boldsymbol{f} + \mathbf{d}\Xi, \tag{8.40}$$

so that $\boldsymbol{d} = \Xi\boldsymbol{\beta}$.

As the Weyl connection $\hat{\boldsymbol{\nabla}}$ is torsion free, it follows that its Riemann curvature tensor $\hat{R}^c{}_{dab}$ can be decomposed in terms of its Schouten tensor $\hat{L}_{ab}$ and the Weyl tensor of the conformal class of $C^c{}_{dab}$; see Equation (5.28a). Using the latter and recalling the definition of the rescaled Weyl tensor $d^c{}_{dab}$, Equation (8.20), one obtains the equation

$$\hat{R}^c{}_{dab} = 2S_{d[a}{}^{ce}\hat{L}_{b]e} + \Xi d^c{}_{dab}.$$

Consistent with the discussion in Section 5.3.2, Equations (5.29a)–(5.29c), the Schouten tensors of the connections $\tilde{\boldsymbol{\nabla}}$, $\boldsymbol{\nabla}$ and $\hat{\boldsymbol{\nabla}}$ are related to each other via

$$\hat{L}_{ab} - L_{ab} = \nabla_a f_b - \frac{1}{2} S_{ab}{}^{cd} f_c f_d,$$
$$\hat{L}_{ab} - \tilde{L}_{ab} = \hat{\nabla}_a \beta_b + \frac{1}{2} S_{ab}{}^{cd} \beta_c \beta_d,$$
$$L_{ab} - \tilde{L}_{ab} = \nabla_a (\Xi^{-1} \nabla_b \Xi) + \frac{1}{2} \Xi^{-2} S_{ab}{}^{cd} \nabla_c \Xi \nabla_d \Xi.$$

Taking into account the above expressions and recalling that $\hat{\boldsymbol{\nabla}} \boldsymbol{S} = 0$ one has that

$$\begin{aligned}
\hat{\nabla}_c \hat{L}_{db} - \hat{\nabla}_d \hat{L}_{cb} &= (\hat{\nabla}_c L_{db} - S_{cd}{}^{ef} f_e L_{fb} - S_{cb}{}^{ef} f_e L_{df}) \\
&\quad -(\hat{\nabla}_d L_{cb} - S_{dc}{}^{ef} f_e L_{fb} - S_{db}{}^{ef} f_e L_{cf}) \\
&= \hat{\nabla}_c \hat{L}_{db} - \hat{\nabla}_d \hat{L}_{cb} + (\hat{\nabla}_c \hat{\nabla}_d - \hat{\nabla}_d \hat{\nabla}_c) f_b \\
&\quad + S_{db}{}^{ef} f_e (\hat{\nabla}_c f_f - L_{cf}) - S_{cb}{}^{ef} f_e (\hat{\nabla}_d f_f - L_{df}).
\end{aligned}$$

A further computation using Equation (5.29a) and the definition of the tensor $\boldsymbol{S}$ yields

$$\begin{aligned}
S_{db}{}^{ef} f_e (\hat{\nabla}_c f_f - L_{cf}) - S_{cb}{}^{ef} f_e (\hat{\nabla}_d f_f - L_{df}) &= (S_{cb}{}^{ef} \hat{L}_{df} - S_{db}{}^{ef} \hat{L}_{cf}) f_e \\
&= 2 S_{b[c}{}^{ef} \hat{L}_{d]f} f_e.
\end{aligned}$$

Hence, recalling the split (5.28a) of the Riemann tensor one obtains

$$\hat{\nabla}_c \hat{L}_{db} - \hat{\nabla}_d \hat{L}_{cb} = \nabla_c L_{db} - \nabla_d L_{cb} + f_a C^a{}_{bcd}.$$

Thus, the Weyl connection version of the vacuum Cotton equation is given by

$$\hat{\nabla}_a \hat{L}_{bc} - \hat{\nabla}_b \hat{L}_{ac} = d_e d^e{}_{cab}.$$

Now, for the Bianchi Equation (8.22) one has that

$$\begin{aligned}
\hat{\nabla}_a d^a{}_{bcd} &= \nabla_a d^a{}_{bcd} - S_{ah}{}^{fa} f_f d^h{}_{bcd} + S_{ab}{}^{fh} f_f d^a{}_{hcd} \\
&\quad + S_{ac}{}^{fh} f_f d^a{}_{bhd} + S_{ad}{}^{fh} f_f d^a{}_{bch} \\
&= \nabla_a d^a{}_{bcd} - f_a d^a{}_{dcb} + f_a d^a{}_{cdb} \\
&= \nabla_a d^a{}_{bcd} - f_a d^a{}_{bcd},
\end{aligned}$$

where in the last line it has been used that $d^a{}_{bcd}$ satisfies the first Bianchi identity $d^a{}_{bcd} + d^a{}_{cdb} + d^a{}_{dbc} = 0$. Hence, Equation (8.22) expressed in terms of the Weyl connection $\hat{\boldsymbol{\nabla}}$ takes the form:

$$\hat{\nabla}_a d^a{}_{bcd} = f_a d^a{}_{bcd}.$$

As a summary of this section one has the two equations:

$$\hat{\nabla}_a \hat{L}_{bc} - \hat{\nabla}_b \hat{L}_{ac} = d_d d^d{}_{cab}, \tag{8.41a}$$
$$\hat{\nabla}^d d_{dcab} = f_d d^d{}_{cab}. \tag{8.41b}$$

These two equations will be regarded as the core of the *extended conformal Einstein field equations*. They provide differential conditions on the Schouten tensor of the Weyl connection and the rescaled Weyl tensor.

8.4.1 The frame version of the extended conformal field equations

Equations (8.41a) and (8.41b) need to be supplemented with equations which provide information about the metric $\boldsymbol{g}$ associated to the conformal factor Ξ and which also allow to determine the covector $\boldsymbol{f}$ giving rise to the Weyl connection $\hat{\boldsymbol{\nabla}}$. The most convenient way of doing this is to make use of a frame formalism.

As in Section 8.3.1 let $\{\boldsymbol{e}_{\boldsymbol{a}}\}$, $\boldsymbol{a} = \boldsymbol{0}, \ldots, \boldsymbol{3}$ denote a frame field which is $\boldsymbol{g}$-orthonormal so that $\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}) = \eta_{\boldsymbol{ab}}$. As $\boldsymbol{\nabla}$ is the Levi-Civita connection of $\boldsymbol{g}$, its connection coefficients, $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \langle \boldsymbol{\omega}^{\boldsymbol{c}}, \nabla_{\boldsymbol{a}} \boldsymbol{e}_{\boldsymbol{b}} \rangle$, satisfy the metric compatibility condition of Equation (2.29).

Let now $\hat{\boldsymbol{\nabla}}$ denote the Weyl connection constructed from the Levi-Civita connection $\boldsymbol{\nabla}$ and the covector $\boldsymbol{f}$ using Equation (8.39). If $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \langle \boldsymbol{\omega}^{\boldsymbol{c}}, \hat{\nabla}_{\boldsymbol{a}} \boldsymbol{e}_{\boldsymbol{b}} \rangle$ denotes the connection coefficients of $\hat{\boldsymbol{\nabla}}$ with respect to the frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$, one has that

$$\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} + S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}} f_{\boldsymbol{d}}, \tag{8.42a}$$

$$= \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} + \delta_{\boldsymbol{a}}{}^{\boldsymbol{c}} f_{\boldsymbol{b}} + \delta_{\boldsymbol{b}}{}^{\boldsymbol{c}} f_{\boldsymbol{a}} - \eta_{\boldsymbol{ab}} \eta^{\boldsymbol{cd}} f_{\boldsymbol{d}}. \tag{8.42b}$$

In particular, one has that

$$f_{\boldsymbol{a}} = \frac{1}{4} \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{b}}, \tag{8.43}$$

as $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{b}} = 0$ in the case of a metric connection.

Let $\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ denote the torsion of the connection $\hat{\boldsymbol{\nabla}}$. Using the transformation formula for the torsion under change of connections, Equation (2.15), together with Equation (8.42a), one obtains

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = -2 S_{[\boldsymbol{ab}]}{}^{\boldsymbol{cd}} f_{\boldsymbol{d}} = 0.$$

Thus,

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0,$$

as $\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0$. As in Section 8.3.1 it is convenient to distinguish between the *geometric curvature* $\hat{P}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ – that is, the expression for the components of the Riemann tensor of the connection $\hat{\boldsymbol{\nabla}}$ in terms of the connection coefficients $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ – and the *algebraic curvature* $\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ – that is, the expression of the Riemann tensor in terms of the Schouten and Weyl tensors. These are given by

$$\begin{aligned}
\hat{P}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} &\equiv \boldsymbol{e}_{\boldsymbol{a}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \boldsymbol{e}_{\boldsymbol{b}}(\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) \\
&\quad + \hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}) + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}}, \\
\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} &\equiv \Xi d^{\boldsymbol{c}}{}_{\boldsymbol{dab}} + 2 S_{\boldsymbol{d}[\boldsymbol{a}}{}^{\boldsymbol{ce}} \hat{L}_{\boldsymbol{b}]\boldsymbol{e}}.
\end{aligned}$$

In analogy to the discussion of Section 8.3.1, it is convenient to introduce a set of *geometric zero quantities* associated to the various equations. In the present case let:

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$$\hat{\Sigma}_{\boldsymbol{ab}} \equiv [\boldsymbol{e_a}, \boldsymbol{e_b}] - (\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})\boldsymbol{e_c}, \tag{8.44a}$$

$$\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \hat{P}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} - \rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \tag{8.44b}$$

$$\hat{\Delta}_{\boldsymbol{cdb}} \equiv \hat{\nabla}_{\boldsymbol{c}}\hat{L}_{\boldsymbol{db}} - \hat{\nabla}_{\boldsymbol{d}}\hat{L}_{\boldsymbol{cb}} - d_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \tag{8.44c}$$

$$\hat{\Lambda}_{\boldsymbol{bcd}} \equiv \hat{\nabla}_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} - f_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}. \tag{8.44d}$$

Now, taking into account Equation (8.43) one has that

$$\begin{aligned}
\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{cab}} &= \boldsymbol{e_a}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{c}}) - \boldsymbol{e_b}(\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{c}}) + \hat{\Gamma}_{\boldsymbol{d}}{}^{\boldsymbol{c}}{}_{\boldsymbol{c}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{a}} - \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}}) - 2S_{\boldsymbol{c}[\boldsymbol{a}}{}^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{b}]\boldsymbol{e}},\\
&= 4\left(\boldsymbol{e_a}(f_{\boldsymbol{b}}) - \boldsymbol{e_b}(f_{\boldsymbol{a}}) - f_{\boldsymbol{d}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{a}} - \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}}) - \hat{L}_{\boldsymbol{ba}} + \hat{L}_{\boldsymbol{ab}}\right),\\
&= 4\left(\hat{\nabla}_{\boldsymbol{a}}f_{\boldsymbol{b}} - \hat{\nabla}_{\boldsymbol{b}}f_{\boldsymbol{a}} - \hat{L}_{\boldsymbol{ba}} + \hat{L}_{\boldsymbol{ab}}\right).
\end{aligned}$$

In view of the latter, it is convenient to define

$$\hat{\Xi}_{\boldsymbol{ab}} \equiv \frac{1}{4}\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{cab}} = \hat{\nabla}_{\boldsymbol{a}}f_{\boldsymbol{b}} - \hat{\nabla}_{\boldsymbol{b}}f_{\boldsymbol{a}} - \hat{L}_{\boldsymbol{ab}} + \hat{L}_{\boldsymbol{ba}}. \tag{8.45}$$

In terms of the zero quantities discussed in the previous paragraphs, one defines the *extended conformal vacuum Einstein field equations* as the conditions

$$\hat{\Sigma}_{\boldsymbol{ab}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0, \qquad \hat{\Delta}_{\boldsymbol{cdb}} = 0, \qquad \hat{\Lambda}_{\boldsymbol{bcd}} = 0. \tag{8.46}$$

These equations yield differential conditions, respectively, for the components of the frame $\{\boldsymbol{e_a}\}$, the spin coefficients $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ (including the components $f_{\boldsymbol{a}}$ of the covector $\boldsymbol{f}$), the components of the Schouten tensor $\hat{L}_{\boldsymbol{ab}}$ and the components of the rescaled Weyl tensor $d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$. In contrast to the standard conformal field Equations (8.32a) and (8.32b), there are no equations which can be regarded as differential conditions on the conformal factor Ξ and the components $d_{\boldsymbol{a}}$ of the covector $\boldsymbol{d}$. As will be seen in Chapter 13, these objects will be fixed through gauge conditions.

In order to relate the extended conformal field Equations (8.46) to the Einstein field equations, one introduces further zero quantities:

$$\delta_{\boldsymbol{a}} \equiv d_{\boldsymbol{a}} - \Xi f_{\boldsymbol{a}} - \hat{\nabla}_{\boldsymbol{a}}\Xi, \tag{8.47a}$$

$$\gamma_{\boldsymbol{ab}} \equiv \hat{L}_{\boldsymbol{ab}} - \hat{\nabla}_{\boldsymbol{a}}\beta_{\boldsymbol{b}} - \frac{1}{2}S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}}\beta_{\boldsymbol{c}}\beta_{\boldsymbol{d}} + \frac{1}{6}\lambda\Xi^{-2}\eta_{\boldsymbol{ab}}, \tag{8.47b}$$

$$\varsigma_{\boldsymbol{ab}} \equiv \hat{L}_{[\boldsymbol{ab}]} - \hat{\nabla}_{[\boldsymbol{a}}f_{\boldsymbol{b}]}. \tag{8.47c}$$

The associated equations

$$\delta_{\boldsymbol{a}} = 0, \qquad \gamma_{\boldsymbol{ab}} = 0, \qquad \varsigma_{\boldsymbol{ab}} = 0, \tag{8.48}$$

will be treated as constraints. The first equation expresses the relation between the covectors $\boldsymbol{d}$, $\boldsymbol{f}$ and the conformal factor Ξ. The second equation encodes the relation between the components of the Schouten tensor of the Weyl connection $\hat{L}_{\boldsymbol{ab}}$ and the physical Schouten tensor via the Einstein field equations – this constraint is the analogue of the standard conformal equation $Z_{\boldsymbol{ab}} = 0$. The role

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of the equations in (8.48) is similar to that of the equation $Z = 0$ of the standard conformal field equations.

In the particular case when $f_{\boldsymbol{a}} = 0$ it follows from (8.40) that $d_{\boldsymbol{a}} = \nabla_{\boldsymbol{a}}\Xi$. Hence, one has that $\hat{\boldsymbol{\nabla}} = \boldsymbol{\nabla}$. Under these circumstances the extended conformal field Equations (8.46) reduce to

$$\Sigma_{\boldsymbol{ab}} = 0, \qquad \Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0, \qquad \Delta_{\boldsymbol{cdb}} = 0, \qquad \Lambda_{\boldsymbol{bcd}} = 0,$$

where the zero quantities $\Sigma_{\boldsymbol{ab}}$, $\Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$, $\Delta_{\boldsymbol{cdb}}$ and $\Lambda_{\boldsymbol{bcd}}$ are as defined in Section 8.3.1.

The conformal covariance of the equations

As in the case of the standard conformal field equations, the extended conformal field equations discussed in the previous section are conformally covariant. To make this statement more precise, consider a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and two metrics $\boldsymbol{g}$ and $\acute{\boldsymbol{g}}$ conformally related to $\tilde{\boldsymbol{g}}$ via

$$\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}, \qquad \acute{\boldsymbol{g}} = \acute{\Xi}^2\tilde{\boldsymbol{g}},$$

so that $\boldsymbol{g} = \kappa^2\acute{\boldsymbol{g}}$ with $\kappa \equiv \Xi\acute{\Xi}^{-1}$. Let $\boldsymbol{\nabla}$, $\acute{\boldsymbol{\nabla}}$ denote, respectively, the Levi-Civita connections of the metrics $\boldsymbol{g}$ and $\acute{\boldsymbol{g}}$. One has that

$$\boldsymbol{\nabla} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\Xi^{-1}\mathbf{d}\Xi), \qquad \acute{\boldsymbol{\nabla}} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\acute{\Xi}^{-1}\mathbf{d}\acute{\Xi}), \tag{8.49}$$

and, furthermore,

$$\boldsymbol{\nabla} - \acute{\boldsymbol{\nabla}} = \boldsymbol{S}(\kappa^{-1}\mathbf{d}\kappa).$$

In addition, consider the covectors $\boldsymbol{f}$ and $\acute{\boldsymbol{f}}$ and define by means of these two the Weyl connections $\hat{\boldsymbol{\nabla}}$ and $\check{\boldsymbol{\nabla}}$ via

$$\hat{\boldsymbol{\nabla}} - \boldsymbol{\nabla} = \boldsymbol{S}(\boldsymbol{f}), \qquad \check{\boldsymbol{\nabla}} - \acute{\boldsymbol{\nabla}} = \boldsymbol{S}(\acute{\boldsymbol{f}}). \tag{8.50}$$

Combining Equations (8.49) and (8.50) one finds that the relation between the physical connection $\tilde{\boldsymbol{\nabla}}$ and the Weyl connections $\hat{\boldsymbol{\nabla}}$ and $\check{\boldsymbol{\nabla}}$ is given by

$$\hat{\boldsymbol{\nabla}} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\boldsymbol{\beta}), \qquad \check{\boldsymbol{\nabla}} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\acute{\boldsymbol{\beta}}),$$

where

$$\boldsymbol{\beta} \equiv \boldsymbol{f} + \Xi^{-1}\mathbf{d}\Xi, \qquad \acute{\boldsymbol{\beta}} \equiv \acute{\boldsymbol{f}} + \acute{\Xi}^{-1}\mathbf{d}\acute{\Xi}.$$

Combining these expressions one finds that

$$\begin{aligned}\hat{\boldsymbol{\nabla}} - \check{\boldsymbol{\nabla}} &= \boldsymbol{S}(\boldsymbol{\beta} - \acute{\boldsymbol{\beta}}) \\ &= \boldsymbol{S}(\boldsymbol{k} + \kappa^{-1}\mathbf{d}\kappa),\end{aligned}$$

with $\boldsymbol{k} \equiv \boldsymbol{f} - \acute{\boldsymbol{f}}$. Hence, letting $\boldsymbol{d} = \Xi\boldsymbol{\beta}$ and $\acute{\boldsymbol{d}} = \acute{\Xi}\acute{\boldsymbol{\beta}}$, one concludes that

$$\boldsymbol{d} = \kappa^{-1}\acute{\boldsymbol{d}} + \kappa\acute{\Xi}\boldsymbol{k} + \acute{\Xi}\mathbf{d}\kappa. \tag{8.51}$$

Assume now that the fields $(\boldsymbol{e_a}, \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \hat{L}_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \Xi, d_{\boldsymbol{a}})$ constitute a solution to the extended conformal field Equations (8.46). Then, proceeding in analogy to the discussion in Section 8.3.3, one finds that the fields $(\acute{\boldsymbol{e}}_{\boldsymbol{a}}, \check{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \check{L}_{\boldsymbol{ab}}, \acute{d}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \acute{\Xi}, \acute{d}_{\boldsymbol{a}})$ with

$$
\begin{aligned}
\acute{\boldsymbol{e}}_{\boldsymbol{a}} &= \kappa \boldsymbol{e}_{\boldsymbol{a}} \\
\check{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} &= \kappa \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} + \delta_{\boldsymbol{c}}{}^{\boldsymbol{b}} \hat{\nabla}_{\boldsymbol{a}} \kappa - \kappa S_{\boldsymbol{ac}}{}^{\boldsymbol{bd}} (k_{\boldsymbol{d}} + \kappa^{-1} \hat{\nabla}_{\boldsymbol{d}} \kappa), \\
\check{L}_{\boldsymbol{ab}} &= \kappa^2 \hat{L}_{\boldsymbol{ab}} - \kappa^2 \hat{\nabla}_{\boldsymbol{a}} (k_{\boldsymbol{b}} + \kappa^{-1} \hat{\nabla}_{\boldsymbol{b}} \kappa) \\
&\qquad - \frac{1}{2} \kappa^2 S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}} (k_{\boldsymbol{c}} + \kappa^{-1} \hat{\nabla}_{\boldsymbol{c}} \kappa)(k_{\boldsymbol{d}} + \kappa^{-1} \hat{\nabla}_{\boldsymbol{d}} \kappa), \\
\acute{d}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} &= \kappa^3 d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \\
\acute{\Xi} &= \kappa^{-1} \Xi, \\
\acute{d}_{\boldsymbol{a}} &= \kappa d_{\boldsymbol{a}} - \Xi \hat{\nabla}_{\boldsymbol{a}} \kappa - \kappa \Xi k_{\boldsymbol{a}},
\end{aligned}
$$

are also a solution of the extended conformal equations. Observe that the $\hat{\boldsymbol{\nabla}}$-quantities are components with respect to the frame $\{\boldsymbol{e_a}\}$ which is $\boldsymbol{g}$-orthonormal, while the $\check{\boldsymbol{\nabla}}$-quantities are components on the $\{\acute{\boldsymbol{e}}_{\boldsymbol{a}}\}$ frame which is $\acute{\boldsymbol{g}}$-orthonormal.

8.4.2 The spinorial version of the extended conformal field equations

The frame formulation of the extended conformal field equations discussed in the previous subsection leads directly to its spinorial counterpart. The strategy is analogous to the one adopted in Section 8.3.2.

The spinorial counterpart of the $\boldsymbol{g}$-orthogonal frame $\{\boldsymbol{e_a}\}$ is given by the null tetrad $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ satisfying $\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{AA'}}, \boldsymbol{e}_{\boldsymbol{BB'}}) = \epsilon_{\boldsymbol{AB}} \epsilon_{\boldsymbol{A'B'}}$. Furthermore, let $\hat{\nabla}_{\boldsymbol{AA'}} \equiv \boldsymbol{e}_{\boldsymbol{AA'}}{}^a \hat{\nabla}_a$. Similarly, let

$$
\hat{\Sigma}_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}}, \qquad \hat{P}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}}, \qquad \hat{\rho}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}}, \\
\hat{L}_{\boldsymbol{AA'BB'}}, \qquad d_{\boldsymbol{AA'}}, \qquad f_{\boldsymbol{AA'}},
$$

denote, respectively, the spinorial counterparts of the fields

$$
\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \qquad P^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \qquad \rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \qquad \hat{L}_{\boldsymbol{ab}}, \qquad d_{\boldsymbol{a}}, \qquad f_{\boldsymbol{a}}.
$$

The spinorial counterpart of the geometric curvature, $\hat{P}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}}$, is given in terms of the spinorial connection coefficients $\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}}$. These, in turn, can be expressed in terms of the *reduced* spin connection coefficients $\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$ by

$$
\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} = \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} \delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}} + \bar{\hat{\Gamma}}_{\boldsymbol{A'A}}{}^{\boldsymbol{B'}}{}_{\boldsymbol{C'}} \delta_{\boldsymbol{C}}{}^{\boldsymbol{B}}, \tag{8.52}
$$

consistent with formula (3.33). The reduced Weyl spin connection coefficients are related to the unphysical spin connection coefficients via

$$
\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} = \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} + \delta_{\boldsymbol{A}}{}^{\boldsymbol{B}} f_{\boldsymbol{CA'}}, \qquad \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{Q}} = f_{\boldsymbol{AA'}};
$$

see Equation (5.32).

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The geometric and algebraic Weyl curvature admit, respectively, the splits

$$\hat{P}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} = \hat{P}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}\delta_{\boldsymbol{D'}}{}^{\boldsymbol{C'}} + \bar{P}^{\boldsymbol{C'}}{}_{\boldsymbol{D'AA'BB'}}\delta_{\boldsymbol{D}}{}^{\boldsymbol{C}},$$
$$\hat{\rho}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} = \hat{\rho}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}\delta_{\boldsymbol{D'}}{}^{\boldsymbol{C'}} + \bar{\hat{\rho}}^{\boldsymbol{C'}}{}_{\boldsymbol{D'AA'BB'}}\delta_{\boldsymbol{D}}{}^{\boldsymbol{C}}.$$

The formula giving the reduced geometric curvature $\hat{P}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}$ in terms of the reduced spin connection coefficients is identical to that for a Levi-Civita connection. Namely, one has that

$$\begin{aligned}\hat{P}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} \equiv\; & \boldsymbol{e}_{\boldsymbol{AA'}}(\hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}) - \boldsymbol{e}_{\boldsymbol{BB'}}(\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}) \\ & - \hat{\Gamma}_{\boldsymbol{FB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{B}} - \hat{\Gamma}_{\boldsymbol{BF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\hat{\Gamma}}_{\boldsymbol{AA'}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{B'}} + \hat{\Gamma}_{\boldsymbol{FA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{A}} \\ & + \hat{\Gamma}_{\boldsymbol{AF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\hat{\Gamma}}_{\boldsymbol{BB'}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{A'}} + \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{E}}\hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{E}}{}_{\boldsymbol{D}} - \hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{E}}\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{E}}{}_{\boldsymbol{D}}.\end{aligned}$$

In particular, it can be verified that

$$\begin{aligned}\hat{P}^{\boldsymbol{Q}}{}_{\boldsymbol{QAA'BB'}} &= \boldsymbol{e}_{\boldsymbol{AA'}}(f_{\boldsymbol{BB'}}) - \boldsymbol{e}_{\boldsymbol{BB'}}(f_{\boldsymbol{AA'}}) + \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}}f_{\boldsymbol{QB'}} + \bar{\hat{\Gamma}}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{B'}}f_{\boldsymbol{BQ'}} \\ &\quad - \hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{A}}f_{\boldsymbol{QA'}} - \bar{\hat{\Gamma}}_{\boldsymbol{BB'}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{A'}}f_{\boldsymbol{AQ'}} \\ &= \hat{\nabla}_{\boldsymbol{AA'}}f_{\boldsymbol{BB'}} - \hat{\nabla}_{\boldsymbol{BB'}}f_{\boldsymbol{AA'}}.\end{aligned}$$

Hence, one can write

$$\hat{P}_{\boldsymbol{ABCC'DD'}} = \hat{P}_{(\boldsymbol{AB})\boldsymbol{CC'DD'}} + \frac{1}{2}\epsilon_{\boldsymbol{AB}}(\hat{\nabla}_{\boldsymbol{CC'}}f_{\boldsymbol{DD'}} - \hat{\nabla}_{\boldsymbol{DD'}}f_{\boldsymbol{CC'}}).$$

The reduced algebraic curvature spinor satisfies a similar expression. Namely, one has that

$$\hat{\rho}_{\boldsymbol{ABCC'DD'}} = \hat{\rho}_{(\boldsymbol{AB})\boldsymbol{CC'DD'}} - \frac{1}{2}\epsilon_{\boldsymbol{AB}}(\hat{L}_{\boldsymbol{CC'DD'}} - \hat{L}_{\boldsymbol{DD'CC'}}),$$

with

$$\hat{\rho}_{(\boldsymbol{AB})\boldsymbol{CC'DD'}} = -\Xi\phi_{\boldsymbol{ABCD}}\epsilon_{\boldsymbol{C'D'}} + \hat{L}_{\boldsymbol{BC'DD'}}\epsilon_{\boldsymbol{AC}} - \hat{L}_{\boldsymbol{BD'CC'}}\epsilon_{\boldsymbol{AD}};$$

compare Equation (5.33).

The objects discussed in the previous paragraphs can be used, in turn, to define the zero quantities:

$$\hat{\Sigma}_{\boldsymbol{AA'BB'}} \equiv [\boldsymbol{e}_{\boldsymbol{AA'}}, \boldsymbol{e}_{\boldsymbol{BB'}}] - (\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} - \hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{AA'}})\boldsymbol{e}_{\boldsymbol{CC'}}, \tag{8.53a}$$
$$\hat{\Xi}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} \equiv \hat{P}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} - \hat{\rho}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}, \tag{8.53b}$$
$$\hat{\Delta}_{\boldsymbol{CC'DD'BB'}} \equiv \hat{\nabla}_{\boldsymbol{CC'}}\hat{L}_{\boldsymbol{DD'BB'}} - \hat{\nabla}_{\boldsymbol{DD'}}\hat{L}_{\boldsymbol{CC'BB'}} \tag{8.53c}$$
$$\qquad - d_{\boldsymbol{AA'}}d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}}, \tag{8.53d}$$
$$\hat{\Lambda}_{\boldsymbol{BB'CC'DD'}} \equiv \hat{\nabla}_{\boldsymbol{AA'}}d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}}$$
$$\qquad - f_{\boldsymbol{AA'}}d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}} \tag{8.53e}$$

One can exploit the symmetries of some of the above zero quantities to obtain reduced zero quantities. In particular, one can write

$$\hat{\Lambda}_{\boldsymbol{BB'CC'DD'}} = \hat{\Lambda}_{\boldsymbol{BB'CD}}\epsilon_{\boldsymbol{C'D'}} + \bar{\hat{\Lambda}}_{\boldsymbol{B'BC'D'}}\epsilon_{\boldsymbol{CD}},$$

with

$$\hat{\Lambda}_{\boldsymbol{BB'CD}} \equiv \frac{1}{2}\hat{\Lambda}_{\boldsymbol{BB'CQ'D}}{}^{\boldsymbol{Q'}}.$$

A similar idea can be applied to $\hat{\Sigma}_{\boldsymbol{AA'BB'}}$, $\hat{\Xi}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}$ and $\hat{\Delta}_{\boldsymbol{CC'DD'BB'}}$. This idea will not be pursued here.

The spinorial version of the extended conformal Einstein field equations is expressed in terms of the above zero quantities as:

$$\hat{\Sigma}_{\boldsymbol{AA'BB'}} = 0, \quad \hat{\Xi}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} = 0, \quad \hat{\Delta}_{\boldsymbol{CC'DD'BB'}} = 0, \tag{8.54a}$$
$$\hat{\Lambda}_{\boldsymbol{BB'CD}} = 0. \tag{8.54b}$$

Finally, let $\delta_{\boldsymbol{AA'}}$, $\varsigma_{\boldsymbol{AA'BB'}}$ and $\gamma_{\boldsymbol{AA'BB'}}$ denote the spinorial counterparts of the zero quantities $\delta_{\boldsymbol{a}}$ and $\gamma_{\boldsymbol{ab}}$. One then requires that

$$\delta_{\boldsymbol{AA'}} = 0, \qquad \varsigma_{\boldsymbol{AA'BB'}} = 0, \qquad \gamma_{\boldsymbol{AA'BB'}} = 0. \tag{8.55}$$

8.4.3 The extended conformal Einstein field equations and the Einstein field equations

As in the case of the other versions of the conformal field equations discussed in this chapter, it is important to analyse the precise relation between the extended conformal field equations and the (physical) Einstein field equations. One has the following:

Proposition 8.3 (***solutions to the extended conformal field equations as solutions to the Einstein field equations***) *Let*

$$(\boldsymbol{e_a}, \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \hat{L}_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}})$$

denote a solution to the extended conformal field Equations (8.46) for some choice of the conformal gauge fields $(\Xi, d_{\boldsymbol{a}})$ *satisfying the supplementary Equations (8.48). Furthermore, suppose*

$$\Xi \neq 0, \quad \det(\eta^{\boldsymbol{ab}}\boldsymbol{e_a}\otimes\boldsymbol{e_b}) \neq 0,$$

on an open subset $\mathcal{U}\subset\mathcal{M}$. *Then the metric* $\tilde{\boldsymbol{g}} = \Xi^{-2}\eta_{\boldsymbol{ab}}\boldsymbol{\omega}^{\boldsymbol{a}}\otimes\boldsymbol{\omega}^{\boldsymbol{b}}$, *where* $\{\boldsymbol{\omega}^{\boldsymbol{a}}\}$ *is the dual frame to* $\{\boldsymbol{e_a}\}$, *is a solution to the Einstein field equations (8.4) on* $\mathcal{U}$.

Proof As a consequence of the conformal equation $\hat{\Sigma}_{\boldsymbol{ab}} = 0$, the fields $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ can be interpreted as the connection coefficients, with respect to the frame field $\{\boldsymbol{e_a}\}$, of a torsion-free connection $\hat{\boldsymbol{\nabla}}$. In order to show that $\hat{\boldsymbol{\nabla}}$ is a Weyl connection, one needs to compute $\hat{\nabla}_{\boldsymbol{a}}\eta_{\boldsymbol{bc}}$. This is best done using spinors. As $\boldsymbol{e}_{\boldsymbol{AA'}}(\epsilon_{\boldsymbol{BC}}) = 0$, one has that

$$\begin{aligned}\hat{\nabla}_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{BC}} &= -\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}}\epsilon_{\boldsymbol{QC}} - \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{C}}\epsilon_{\boldsymbol{BQ}} = -\hat{\Gamma}_{\boldsymbol{AA'CB}} + \hat{\Gamma}_{\boldsymbol{AA'BC}}\\ &= -\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{Q}}\epsilon_{\boldsymbol{BC}} = -f_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{BC}}.\end{aligned}$$

Recalling that $\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}$ is the spinorial counterpart of $\eta_{\boldsymbol{ab}}$ and observing the split (8.52) one concludes that $\hat{\nabla}_{\boldsymbol{a}}\eta_{\boldsymbol{bc}} = -2f_{\boldsymbol{a}}\eta_{\boldsymbol{bc}}$; that is, $\hat{\boldsymbol{\nabla}}$ is a Weyl connection. Now, as $\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0$, the fields $\hat{L}_{\boldsymbol{ab}}$ and $\Xi d^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ obtained as a solution to the extended conformal field equations correspond to, respectively, the Schouten tensor and the Weyl tensor of the connection $\hat{\boldsymbol{\nabla}}$, as a consequence of the uniqueness of the decomposition in terms of irreducible components.

Given the Weyl connection $\hat{\boldsymbol{\nabla}}$, one can define a new connection $\boldsymbol{\nabla}$ via $\boldsymbol{\nabla} \equiv \hat{\boldsymbol{\nabla}} - \boldsymbol{S}(\boldsymbol{f})$. By construction, this connection is metric. The Schouten tensor of $\boldsymbol{\nabla}$ is then given by

$$L_{\boldsymbol{ab}} = \hat{L}_{\boldsymbol{ab}} - \nabla_{\boldsymbol{a}}f_{\boldsymbol{b}} + \frac{1}{2}S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}}f_{\boldsymbol{c}}f_{\boldsymbol{d}}.$$

As $\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0$, it follows that $\hat{\Xi}_{\boldsymbol{ab}}$ as defined by Equation (8.45) also vanishes. As $\hat{\boldsymbol{\nabla}}$ is torsion free, so is $\boldsymbol{\nabla}$. Hence, one concludes that $\boldsymbol{\nabla}$ must be the Levi-Civita connection of the metric $\boldsymbol{g} \equiv \eta_{\boldsymbol{ab}}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}$. The latter expression is a well-defined Lorentzian metric on $\mathcal{U}$ as its determinant is, by hypothesis, non-vanishing.

Finally, one defines a *physical connection* $\tilde{\boldsymbol{\nabla}}$ via $\tilde{\boldsymbol{\nabla}} \equiv \boldsymbol{\nabla} - \boldsymbol{S}(\Xi^{-1}\mathbf{d}\Xi)$. As $\delta_{\boldsymbol{a}} = 0$ it follows that $d_{\boldsymbol{a}} = f_{\boldsymbol{a}} + \hat{\nabla}_{\boldsymbol{a}}\Xi$ so that $\tilde{\boldsymbol{\nabla}}$ is the Levi-Civita connection of the metric $\tilde{\boldsymbol{g}} \equiv \Xi^{-2}\eta_{\boldsymbol{ab}}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}$. The latter is well defined as long as $\Xi \neq 0$. The Schouten tensor of $\tilde{\boldsymbol{\nabla}}$ is given by

$$\tilde{L}_{\boldsymbol{ab}} = L_{\boldsymbol{ab}} - \nabla_{\boldsymbol{a}}(\Xi^{-1}\nabla_{\boldsymbol{b}}\Xi) - \Xi^{-2}S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}}\nabla_{\boldsymbol{c}}\Xi\nabla_{\boldsymbol{d}}\Xi.$$

As a consequence of $\delta_{\boldsymbol{a}} = 0$ and $\gamma_{\boldsymbol{ab}} = 0$ one concludes that

$$\tilde{L}_{\boldsymbol{ab}} = \frac{1}{6}\lambda\Xi^{-2}\eta_{\boldsymbol{ab}}.$$

Thus, $\tilde{\boldsymbol{g}}$ is a solution to the vacuum Einstein field equations on $\mathcal{U}$. □

Remark. Given the equivalence between the frame and spinorial versions of the extended conformal field equations, the latter result also provides the connection between the spinorial extended conformal field equations and the Einstein field equations.

8.5 Further reading

The standard conformal Einstein field equations were first introduced in Friedrich (1981a,b, 1982). General aspects of the Cauchy problem of the conformal equations were first discussed in Friedrich (1983); see also Friedrich (1984). A systematic discussion of gauge issues and hyperbolic reductions of the equations has been given in Friedrich (1985). A discussion of the conformal field

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equations with trace-free matter was first given in Friedrich (1991). The extended conformal field equations were introduced in Friedrich (1995). Reviews discussing various aspects of the conformal field equations can be found in Friedrich (2002, 2004). The extended conformal field equations with (trace-free) matter were first discussed in Lübbe and Valiente Kroon (2012).

The conformal field equations can be related to other geometrical objects (*twistors*); see Frauendiener and Sparling (2000). The extended conformal field equations can be set in a more geometrical framework involving the language of differential forms. A discussion of this has been given in Friedrich (1995). Certain applications of the conformal equations require the use of a *lift* of the equations to a suitable fibre bundle. A discussion of this type of procedure can be found in Friedrich (1986b, 1998c).

A different approach to the construction of regular conformal field equations based on the *Fefferman-Graham obstructions* – see, for example, Graham and Hirachi (2005) – has been elaborated in Anderson (2005a) and Anderson and Chruściel (2005). This approach gives rise to suitable field equations for an arbitrary number of odd space dimensions and has been used to prove (global and semi-global) existence and stability results of higher dimensional asymptotically simple spacetimes.

In the context of conformal geometry, given a metric $\boldsymbol{g}$, it is natural to ask whether there exists a further metric in the conformal class $[\boldsymbol{g}]$ which is an Einstein space. This problem has been addressed in, for example, Baston and Mason (1987), Kozameh et al. (1985) and Mason (1986).

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