Proceedings of the Royal Society of Edinburgh, **154**, 154–187, *2024*
DOI:10.1017/prm.2023.1

Subsymmetric exchanged braids and the Burau matrix

Alexander Stoimenow
School of Computing, Korean Advanced Institute of Science and Technology, Daejeon 34141, Korea (stoimeno@stoimenov.net)

(Received 7 June 2021; accepted 4 January 2023)

We develop a method based on the Burau matrix to detect conditions on the linking numbers of braid strands. Our main application is to iterated exchanged braids. Unless the braid permutation fixes both braid edge strands, we establish under some fairly generic conditions on the linking numbers a 'subsymmetry' property; in particular at most two such braids can be mutually conjugate. As an addition, we prove that the Burau kernel is contained in the commutator subgroup of the pure braid group. We discuss also some properties of the Burau image.

Keywords: Exchange move; braid group; conjugacy; Burau matrix; linking number

2020 *Mathematics Subject Classification:* 20F36 (primary); 57K10, 57K14, 15A24, 20C08 (secondary)

1. Motivation and summary

Alexander's and Markov's theorems (§2.1) exhibit a fundamental relation between braids and links in 3-space. One of the basic problems in understanding *braid representatives* of a given link L, i.e. those braids $b \in B_n$ with *closure* $\hat{b} = L$, is to describe the conjugacy classes of such braid representatives.

In this context the *exchange move* (§2.7) was extensively studied. Assume $b \in B_n$ is of the form $b = \alpha\beta$ with $\alpha \in B_{1,n-1}$ having isolated right strand and $\beta \in B_{2,n}$ having isolated left strand, which we formalize in §2.8 under the term *exchangeable structure (ES)*. Then there is a sequence of braids b_m, indexed by $m \in \mathbb{Z}$, with $b = b_0$, obtained by (iterated) exchange moves from b, satisfying

$$\hat{b}_m = \hat{b}. \tag{1.1}$$

The main question we are concerned in is the conjugacy of these b_m, which we write $\sim$.

There is a condition of *degeneracy* (3.6), identified in two equivalent forms in [**14**, **24**], under which all b_m are conjugate. We will exclude this trivial case. Then the work in [**24**] protruded, and extensive experimental evidence [**28**] cemented, the evidence of an unexpected *subsymmetric* pattern (definition 3.3):

$$\text{there is a } \mu \in \mathbb{Z} \text{ such that whenever } b_m \sim b_{m'} \text{ (for } m \neq m'), \text{then } m + m' = \mu.$$

This is a very strong restriction; it implies for instance that at most two b_m can be conjugate, and that $b_m \not\sim b_{m'}$ are pairwise non-conjugate for all $m > m' > 0$ or all $m < m' < 0$. Both many experimental examples in B_n for small n and more sporadic but systematically constructable ones for higher n show that the subsymmetry property cannot be generally strengthened, i.e. there exist *symmetric* ES. However, it is very much possible that subsymmetry is universal, as formulated in conjecture 3.7.

Let $\pi : B_n \to S_n$ be the *permutation homomorphism*. When $\pi(b)(1) \neq 1$ and $\pi(b)(n) \neq n$, then the conjecture holds by the work in [**24**] (theorem 3.4). Here we will be concerned with the case

$$\pi(b)(1) = 1 \text{ and } \pi(b)(n) \neq n \tag{1.2}$$

[and some equivalent forms of it, like (2.21)], which requires a very different (but still quite substantial) treatment. It involves degenerate ES that obviously have to be excluded somehow. We accomplish this by formulating a *linking number* condition (1.3). Linking numbers can be visually combinatorially defined (§2.6) and calculated. Under (1.2), for every $j > 1$, the linking number $lk_j = lk(1, j)$ is an integer, and $lk_n = 0$.

For degenerate cases all lk_j for $1 < j < n$ are equal, say, to some $lk \in \mathbb{Z}$ [see (4.1)]. In particular, when $C \neq \{1\}$ is a cycle of $\pi(b)$, then

$$\text{there is a } lk \in \mathbb{Z} \text{ so that for every } C \text{ we have } lk(1, C) = lk \cdot |C \setminus \{n\}|. \tag{1.3}$$

(Note that $lk(1, C)$ is the linking number of the corresponding components of the closure link $\hat{b}$.) This condition is generically violated [under (1.2)], and can be rapidly tested from a braid picture (easier than, e.g. degeneracy). We express (1.3) in §4 through specifying *defective cycles* [so that the absence of such a cycle is a slightly stronger form of (1.3)].

Theorem 4.2 states, among others, the following.

THEOREM 1.1. *If there is a defective cycle, then conjecture 3.7 holds.*

This is sufficient to extend, in §8, some knot-theory applications of non-degeneracy given in [**28**].

Our proof (see §2.1) consists in deriving a conjugacy invariant from the *Burau matrix* (§2.3). The Burau representation ψ_n plays a fundamental role in the study of braid groups, and has been extensively treated, e.g. [**4**, **17**]. By replacing conjugacy of Burau matrices by equality, we gain some characterization result (theorem 9.1), which is motivated here by its two following simple consequences.

COROLLARY 1.2. *(see corollary 9.4) The Burau kernel* $\ker(\psi_n)$ *is contained in the commutator subgroup* P_n^c *of the pure braid group* P_n*.*

COROLLARY 1.3. *The only scalar matrices that occur in the image of* ψ_n *are the image of* center (B_n) *[i.e. powers of (2.9)].*

2. Some basics about braids and links

2.1. General background

The *braid groups* B_n were introduced in the 1930s in the work of Artin [**2**]. We consider the n-strand braid group B_n on Artin's generators σ_i. Until §9, will assume $n \geqslant 4$.

Alexander [**1**] related braids to *links* in real 3-dimensional space (henceforth always assumed oriented), by means of a *closure* operation $\hat{\ }$. *Markov's theorem* relates braid representatives of a link by two moves, the *conjugacy* in the braid group, and *(de)stabilization*, which passes between $b \in B_n$ and $b\sigma_n^{\pm 1} \in B_{n+1}$ (see, e.g. [**21**]). The *exchange move* was apparently discovered by Markov in an earlier version of his theorem, but later showed a consequence of his other two moves. It was then, however, extensively studied by Birman and Menasco [**6**–**9**].

Let $\alpha \in B_n$ have isolated right strand (do not involve $\sigma_{n-1}^{\pm 1}$), β have isolated left strand (no $\sigma_1^{\pm 1}$), and $b = \alpha\beta$. Write $\delta^2_{[2,n-1]}$ for the (right) full-twist on strands 2 to $n-1$ (see §2.2). Then for $m \in \mathbb{Z}$ the braids

$$b_m = \alpha\delta^{2m}_{[2,n-1]}\beta\delta^{-2m}_{[2,n-1]} \tag{2.1}$$

are obtained by iterated exchange moves on b and have the same closure link (1.1).

The property (1.1) (which means that the closure link is useless as a conjugacy invariant), together with theorem 3.1 strongly motivate that

$$\{\, b_m \,:\, m \in \mathbb{Z} \,\} \tag{2.2}$$

are the most important infinite families of braids, on which the conjugacy problem is worth studying.

The question when the exchange move generates non-conjugate braids has been considered for some time. In [**24**] we proved that if $\pi(b)(1) \neq 1$ and $\pi(b)(n) \neq n$, then infinitely many b_m are non-conjugate (theorem 3.4), extending the case of a cycle $\pi(b)$ in [**23**]. This was later improved by Ito [**14**] (theorem 3.5), using some dilatation bound in the mapping class group.

The main goal here is to study the cases excluded in theorem 3.4, while obtaining stronger non-conjugacy properties of iterated exchanged braids b_m than those arising from geometric analysis. For the case (1.2), we will introduce a method using the Burau matrix, which essentially shows how it can account for the linking numbers in strands of a pure braid. It applies under very relaxed (and easy to test) assumptions, but requires effort to derive (theorems 4.1 and 4.2). The scenario $\pi(b)(1) = 1$ and $\pi(b)(n) = n$ is (even) more difficult, and will likely require the Lawrence–Krammer matrix (see §7).

Algorithmic decision of conjugacy $b_m \sim b_{m'}$ for particular m, m' is, of course, possible starting with Garside's [**11**], and later many others' work. This process runs efficiently on a computer [**12**], and is very useful for experimental tests as in [**28**], on a large – but finite – number of instances. It is well-known, though, to be too involved to be manually manageable, even on such explicit infinite families of braids as (2.1).

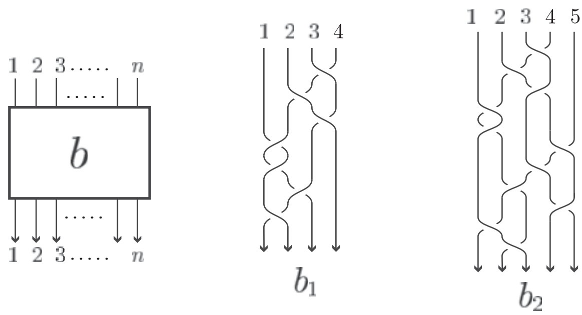


Figure 1. An n-braid.

The practical approach (behind all results summarized in §3) is rather to seek some, sufficiently successful, conjugacy invariant υ, for which $\upsilon(b_m)$ can be evaluated. The core qualitative contribution of this paper can be formulated in exploring (and exploiting) a new type of such invariant.

2.2. Braid groups and closures

For many standard terms and facts about braids, see [5].

Definition 2.1. *The braid group B_n on n strands can be defined by generators and relations as*

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} [\sigma_i, \sigma_j] = 1 & |i-j| > 1 \\ \sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i & |i-j| = 1 \end{array} \right\rangle. \tag{2.3}$$

The σ_i are called Artin standard generators. An element $b \in B_n$ is called an n-braid.

For example, in b_1 of figure 1, we have $n = 4$ and the word $b_1 = \sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^2\sigma_2\sigma_1^{-1}$.

Let

$$\delta_n = (\sigma_1 \cdot \dots \cdot \sigma_{n-1}) \cdot (\sigma_1 \cdot \dots \cdot \sigma_{n-2}) \cdot \dots \cdot (\sigma_1\sigma_2) \cdot \sigma_1 \tag{2.4}$$

be the (right-handed) half-twist on n strands. The *center* center(B_n) of B_n (elements that commute with all B_n) is infinite cyclic and generated by the full twist

$$\delta_n^2 = (\sigma_1 \cdot \dots \cdot \sigma_{n-1})^n.$$

Let similarly

$$\delta_{[i,j]}^2 = (\sigma_i \cdot \dots \cdot \sigma_{j-1})^{j-i+1}$$

be the restricted full twist on strands i to j. Let also for $1 \leqslant i < j \leqslant n$,

$$B_{i,j} := \langle\, \sigma_i, \dots, \sigma_{j-1} \,\rangle \tag{2.5}$$

be the subgroup of B_n of braids operating on strands $i, \dots, j$. Where ambiguity is avoided (as indicated by diagrams we will draw), we can identify $B_{i,j} \simeq B_{j-i+1}$.

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

Specifically, B_{n-1} as a subset of B_n will by default be considered to be $B_{1,n-1}$, e.g. in (2.7).

There is a *permutation homomorphism* of B_n,

$$\pi : B_n \to S_n\,, \quad \text{given by} \quad \pi(\sigma_i) = (i,\ i+1). \tag{2.6}$$

The permutation on the right is a transposition. More generally, we will write $(x_1\ x_2 \dots x_l)$ for the *cycle* $x_i \mapsto x_{i+1}$ for $i = 1, \dots, l-1$, and $x_l \mapsto x_1$. By abuse of notation, we will also sometimes identify $C = (x_1 \dots x_l)$ with its set $\{x_1, \dots, x_l\}$ of elements. In particular, we will use $|C|$ for the *length* of the cycle.

We call $\pi(b)$ the *braid permutation* of b. We call b a *pure braid* if $\pi(b) = Id$. We write $P_n = \ker \pi$ for the pure braid group.

Also, there is a homomorphism $e : B_n \to \mathbb{Z}$ sending all σ_i to 1. We will write $e = e(b)$ for the image, and call it *exponent sum* or *writhe* of b.

When we choose a (non-empty) subset C of $\{1, \dots, n\}$ whose elements form a subset of the cycles of $\pi(b)$, we can define a *subbraid* $b' = b_{[C]}$ of b by choosing only strings numbered in C. Then $\hat{b}_{[C]}$ is a *sublink* of $\hat{b}$. For example, in b_2 of figure 1, the two components $\hat{b}_2'$ and $\hat{b}_2''$ of $\hat{b}_2$ are given by the subbraids $b_2' = (b_2)_{[\{1,3,5\}]}$ comprising the strings starting at the top as number $1, 3, 5$, and $b_2'' = (b_2)_{[\{2,4\}]}$ of strings $2, 4$.

Markov's theorem (see, e.g. [**21**]) relates braid representatives [**29**] of the same link by two moves, the *conjugacy* in the braid group, and the pair of *stabilization*, which is the move to the right in

$$b \in B_{n-1} \longleftrightarrow b\sigma_{n-1}^{\pm 1} \in B_n, \tag{2.7}$$

together with its inverse (move to the left), called *destabilization*. As mentioned, Markov's moves have gained importance in knot theory, among others, as a tool for defining link invariants via braids.

We call a braid $b' \in B_n$ positively resp. negatively *stabilized* if $b'\sigma_{n-1}^{-1}$ resp. $b'\sigma_{n-1}$ lies in $B_{1,n-1}$. We say that $b \in B_n$ is *irreducible*, if b is not conjugate to a stabilized braid b'.

2.3. Burau representation

The (reduced) n-strand *Burau representation* ψ_n, of dimension $n-1$, which we simply call 'Burau', can be found for example in [**15**, §2]. It associates to a braid $\beta \in B_n$ a matrix $\psi_n(\beta)$ of size $(n-1) \times (n-1)$ and entries in $\mathbb{Z}[t^{\pm 1}]$.

Let us for square matrices M, N write for their block sum

$$M \oplus N = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array} \right].$$

Then ψ_n is defined by

$$\psi_n(\sigma_i)(t) = Id_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus Id_{n-i-2}, \tag{2.8}$$

with the first (resp. last) row and column of the 3×3 block removed for σ_1 (resp. σ_{n-1}).

The following formula for the Burau matrix of the center is well-known (see e.g. [**15**, **27**]):

$$\psi_n(\delta_n^2) = t^n \cdot Id_{n-1}. \tag{2.9}$$

We remark also that

$$\det \psi_n(\beta)(t) = (-t)^{e(\beta)} \tag{2.10}$$

and in particular $\det \psi_n(\beta)(1) = (-1)^{\pi(\beta)}$ is the sign of the permutation $\pi(\beta)$.

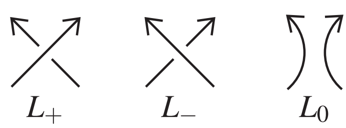
2.4. Links and link polynomials

Among the different braid representatives of a link L the one with the fewest strands is called a *minimal braid*. The number of strands of a minimal braid is called the *braid index* $b(L)$ of L (see e.g. [**10**, **20**, **22**, **26**]). It makes sense to consider throughout

$$n \geqslant \max(4, b(L)). \tag{2.11}$$

Obviously for a braid minimal implies irreducible, but the converse is not true [**18**] (although it is for $n \leqslant 3$ [**8**]).

Consider links with diagrams differing just near one crossing. We call the three diagram fragments in (2.12) from left to right a *positive* crossing, a *negative* crossing and a *smoothed out* crossing (in the skein sense).



(2.12)

Below Δ is the *Alexander polynomial*. It is an invariant with values in $\mathbb{Z}[t, t^{-1}]$, and can be defined by being 1 on the unknot and the relation

$$\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\, \Delta(L_0)\,.$$

The *Conway polynomial* is an oriented link invariant that takes values in $\mathbb{Z}[z]$. It is given by the value 1 on the unknot and the skein relation

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0). \tag{2.13}$$

We have

$$\nabla(L)(t^{1/2} - t^{-1/2}) = \Delta(L)(t),$$

so that ∇ and Δ are interconvertible (and equivalent as invariants).

If $\beta \in B_n$ has exponent sum e, then for the Alexander polynomial there is the formula in terms of the Burau matrix

$$\left(-\sqrt{t}\right)^{e-n+1} \Delta_{\hat{\beta}}(t) \frac{1-t^n}{1-t} = \det(Id_{n-1} - \psi_n(\beta)). \tag{2.14}$$

This is discussed, for instance, in [**15**].

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

2.5. Combed normal form

We present below some argument based on the *combed normal form* of a pure braid. (See [**5**] for some more detailed discussion.)

For $1 \leqslant i < j \leqslant n$, let

$$\kappa_{i,j} = \delta^2_{[i,j]}\delta^{-2}_{[i+1,j]} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma^2_{j-1}\sigma_{j-2}\cdots\sigma_i \tag{2.15}$$

('strand i goes around strands $i+1,\dots,j$'), with $\delta^2_{[i,i]} = Id$. Then every pure braid $\alpha \in P_n$ can be written as

$$\alpha = \prod_{i=1}^{n-1} \alpha_i, \quad \alpha_i = \prod_{j=1}^{k_i} \kappa^{\varepsilon_{i,j}}_{i,p_{i,j}}, \tag{2.16}$$

for $p_{i,j} > i$ and $\varepsilon_{i,j} = \pm 1$. This representation is also *unique*, except for obvious cancellations $\kappa^{\pm 1}_{i,k}\kappa^{\mp 1}_{i,k}$. More often one seems to use this form with

$$\kappa'_{i,j} = \kappa_{i,j}\kappa^{-1}_{i,j-1} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma^2_{j-1}\sigma^{-1}_{j-2}\cdots\sigma^{-1}_i \tag{2.17}$$

('strand i goes around strand j behind strands $i+1,\dots,j-1$'), setting $\kappa_{i,i} = Id$.

Both forms are equivalent, because a group is free in $e_1,\dots,e_l$ if and only if it is free in $e_1, e_1e_2, \dots, e_1\cdots e_l$. We will be more convenienced to use (2.15).

2.6. Linking numbers

For $b \in P_n$, one can define the *linking number* $lk_{ij} = lk_{i,j}(b)$ by numbering strands from left to right, and taking half the sum of the signs of all crossings (exponents of letters σ_k) involving strands i and j.

This definition can be extended to non-pure braids b, when for $\pi(b)(i) \neq i$ one has to fix that strands are numbered where (with their orientation) they enter the braid, and if $(\pi(b)(i) - \pi(b)(j))(i-j) < 0$, then $lk_{i,j}$ will only be a half-integer. For example, in b_2 of figure 1, we have $lk(2,3) = 1$ and $lk(3,5) = -1/2$.

For subbraids $b' = b_{[C']}$ and $b'' = b_{[C'']}$ of a fixed braid b one can define the *subbraid linking number* $lk(C', C'')$ by the linking number

$$lk(\hat{b}', \hat{b}'') = \sum_{i \in C', j \in C''} lk_{ij}$$

between sublinks of $\hat{b}$. In (1.3), let $lk(1, C) = lk(\{1\}, C)$. For example, in b_2 of figure 1, we have $lk(\{1,3,5\},\{2,4\}) = 0$.

In the presentation (2.16), one can see for $\alpha \in P_n$ that for $i < j$,

$$\sum_{k : p_{i,k} = j} \varepsilon_{i,k} = lk_{i,j} - lk_{i,j+1}, \tag{2.18}$$

with $lk_{i,n+1}$ set to 0. We will use this property several times below.

When G is a group. we will write G^c for the *commutator subgroup* of G, generated by elements $ghg^{-1}h^{-1}$. [We do not prefer to use the more standard notation $[G,G]$ to avoid confusion, since we will heavily deploy commutators starting from (5.11) in a ring-theoretic sense.] Then set $G^a = G/G^c$ to be the *abelianization*.

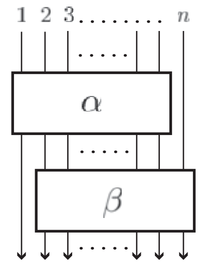


Figure 2. The exchangeable n-braid b.

It follows easily from the combed normal form that the abelianization P_n^a is isomorphic to $\mathbb{Z}^{n(n-1)/2}$, with the identification given by the vector of linking numbers $\alpha \mapsto (lk_{i,j}(\alpha))_{1\leqslant i<j\leqslant n}$. Also note that

$$e(\alpha) = 2\sum_{i<j} lk_{i,j}, \tag{2.19}$$

which is what specifies the braid commutator subgroup $B_n^c = \{\beta : e(\beta) = 0\}$, so that the inclusion $P_n^c \subset B_n^c \cap P_n$ is (very) proper.

Since we need this a few times, let us write

$$\Lambda_n := P_n^c \cdot \text{center}\,(B_n) \tag{2.20}$$

for the set of pure braids with equal linking numbers.

Also, for a few schematic displays, it is useful to introduce the *linking graph* $\Upsilon(b)$, which has a vertex labelled $|C|$ for each cycle C of $\pi(b)$ and edges between C, C' labelled by $lk(C, C')$.

2.7. Exchange move

We say that $b \in B_n$ *admits an exchange move* or is *exchangeable*, if b is as illustrated in figure 2, where $\alpha \in B_{1,n-1}$, $\beta \in B_{2,n}$ and $n \geqslant 4$. One transformation of figure 2 into the same form is to conjugate with $\delta_n\alpha^{-1}$. Then for instance the case

$$\pi(b)(1) \neq 1 \text{ and } \pi(b)(n) = n \tag{2.21}$$

is seen equivalent to (1.2) (and obsolete to further discuss).

An (iterated) *exchange move* [**6**] is the transformation between the braid b and the braids (2.1) shown in figure 3. Here m is some non-zero integer, and the boxes labelled $\pm m$ represent the full twists $\delta_{[2,n-1]}^{\pm 2m}$ respectively, acting on the middle $n-2$ strands. (Thus a positive number of full twists are understood to be right full twists, and $-m$ full twists mean m full left-handed twists.) We can set $b_0 = b$.

Of course, no non-trivial braid on 2 strands admits an exchange move, and all exchange moves on 3 strands are trivial, so that *we will naturally assume* $n \geqslant 4$ *throughout*.

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

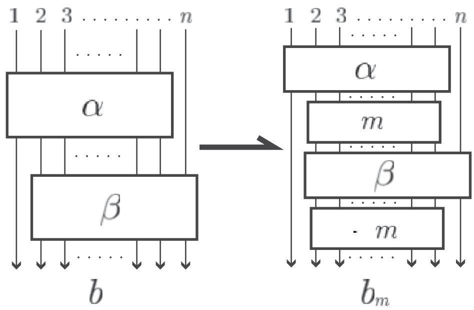


Figure 3. The braid b_m.

There is another, more common, way to describe the exchange move, namely by

$$\alpha\beta \longleftrightarrow \alpha\kappa^{-m}\beta\kappa^m, \text{ where } \quad \kappa = (\sigma_1 \cdot \dots \cdot \sigma_{n-2})(\sigma_{n-2} \cdot \dots \cdot \sigma_1). \tag{2.22}$$

Thus $\kappa = \kappa_{1,n-1}$ in (2.15). This description is equivalent to the previous one, because $\kappa \cdot \delta^2_{[2,n-1]} = \delta^2_{[1,n-1]}$, and this element commutes with α.

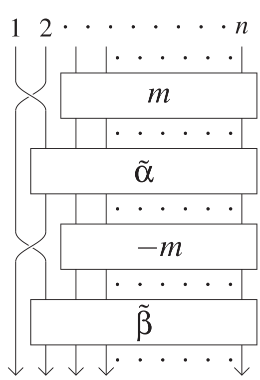
Up to conjugating and changing the sign of m, a further equivalent formulation of the move is

$$b_0 = \tilde{\alpha}\sigma_1\tilde{\beta}\sigma_1^{-1} \longleftrightarrow b_1 = \tilde{\alpha}\sigma_1^{-1}\tilde{\beta}\sigma_1, \tag{2.23}$$

with $\tilde{\alpha}, \tilde{\beta} \in B_{2,n}$, which can be generalized (up to conjugacy) by

$$b_m = \delta^{2m}_{[3,n]}\tilde{\alpha}\delta^{-2m}_{[3,n]}\sigma_1\tilde{\beta}\sigma_1^{-1}. \tag{2.24}$$

The below diagram displays this braid (again up to conjugacy).



The form (2.24) is a variant of (2.1), equivalent under conjugacy. It will be more convenient for our treatment of exchangeable braids from §5 on. To contain the (soon unfolding considerable) technicality of notation, we do not wish to introduce different symbols for the different forms; rather we will remind below (5.13) in §5 that we use (2.24). Similarly see the start of the proof of theorem 4.2 in §6.

Note that the exchange move in figure 3 is trivial when the leftmost strand of α (or the rightmost strand of β) are isolated, i.e.

$$\alpha \in B_{2,n-1}$$

[for $B_{2,n-1}$ from (2.5)]. We observed in [**24**] this failure to extend to braids b with

$$\alpha \in \langle\kappa\rangle \cdot B_{2,n-1}, \tag{2.25}$$

for κ in (2.22), since this element commutes with $B_{2,n-1}$.

Note that the exchange move preserves the linking graph: there is an obvious identification of cycles in $\pi(b)$ and $\pi(b_m)$ so that $\Upsilon(b) = \Upsilon(b_m)$.

2.8. Exchangeable structure

It should be kept in mind that the result b_m in (2.1) does depend on the decomposition

$$b = \alpha\beta \text{ with } \alpha \in B_{1,n-1} \text{ and } \beta \in B_{2,n}, \tag{2.26}$$

although some different pairs (α, β) give equal or conjugate b_m. To formalize this, let us say that the pair

$$(\alpha, \beta) \in B_{1,n-1} \times B_{2,n} \text{ with (2.26)},$$

regarded up to the equivalences for $\gamma \in B_{2,n-1}$

$$(\alpha\gamma, \beta) \cong (\alpha, \gamma\beta) \text{ and } (\gamma\alpha, \beta) \cong (\alpha, \beta\gamma), \tag{2.27}$$

forms an *exchangeable structure (ES)* of b, regarded up to conjugacy in $B_{2,n-1}$. An easy argument with the combed normal form in [**28**] shows that, if a $B_{2,n-1}$-conjugacy class admits an exchangeable structure, then it is unique. An ES for a link L is henceforth to be understood as one of a braid representative of L.

When we consider the family (2.2), we will then always understand that the exchangeable structure is kept fixed. We must point out that when we later talk about braids exchangeable 'up to conjugacy', we will mean conjugacy in the full B_n, though. This raises the question how to identify (all) exchangeable structures on braids in such a conjugacy class, if such exist. For instance, we know from [**28**] that a conjugacy class can have infinitely many different exchangeable structures.

One should also notice, that an ES has no canonical preferred choice of $m = 0$ in (2.2), i.e. the indexing by m of the family (2.2) is only unique up to transitions on $\mathbb{Z}$.

2.9. The axis addition link

The *axis (addition) link* L_b of $b \in B_n$ can be specified by the closure of the braid

$$b \cdot (\sigma_n \cdot \cdots \cdot \sigma_1) \cdot (\sigma_1 \cdot \cdots \cdot \sigma_n) \in B_{n+1}. \tag{2.28}$$

We call the closure of strand $n+1$ the *axis* of b.

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

If $b \sim b'$ are conjugate in B_n, then L_b and $L_{b'}$ are isotopic links. Hence, one can deploy link invariants of L_b as conjugacy invariants of b. In this connection, it is useful here to briefly return to the Burau matrix.

It can be inferred from formulas (2.14) and (2.9) above that the characteristic polynomial of the Burau matrix $\chi(\psi_n(b))$ for $b \in B_n$ holds, as an invariant of b, equivalent information to the map

$$\nabla(L_b^*) \,:\, \nu \mapsto \nabla(L_b^\nu).$$

Here ν is, say, a braid pattern in the solid torus (arbitrary cable degree allowed), and L_b^ν the satellite link of L_b in which the axis component is cabled by ν, but without cabling the component(s) of $\hat{b}$. Another way of expressing the invariant of $\nabla(L_b^*)$ is as the multi-variable Alexander polynomial of L_b, with all variables corresponding to components of $\hat{b}$ set equal, but different from the variable for the axis (so that a two-variable polynomial remains). See, e.g., also [**19**].

3. Non-conjugacy properties

In [**24**] we treated the question when infinitely many conjugacy classes of n-braid representatives of a given link L occur. Obviously it makes sense to consider only $n \geqslant b(L)$. Birman and Menasco [**6**] proved that an exchange move necessarily underlies the switch between many conjugacy classes of braid representatives of L.

Theorem 3.1 Birman–Menasco [**6**]. *The n-braid representatives of a given link decompose into a finite number of classes under the combination of exchange moves and conjugacy.*

We proved in [**24**] that it is also sufficient for generating infinitely many such classes, under a very mild restriction. This leads to the question which braids b_m, indexed by $m \in \mathbb{Z}$, of an ES are conjugate (in B_n).

There has now been a sequence of results in this direction, and to express ourselves succinctly, it is better to specify some qualities of infiniteness, most of which are rather self-motivating.

Definition 3.2. *We will write $\sim$ for conjugacy. We say an ES (α, β) is*

- *infinitely non-conjugate (INC) if there are infinitely many mutually non-conjugate b_m, i.e. the intersection*

$$\Sigma_{\mathcal{E}} = \{\, m \in \mathbb{Z} \,:\, b_m \in \mathcal{E} \} \tag{3.1}$$

  is non-empty for infinitely many conjugacy classes $\mathcal{E}$ in B_n,

- *finitely conjugate (FC) if for only finitely many m, the braids b_m are mutually conjugate, i.e. $\Sigma_{\mathcal{E}}$ is finite for any conjugacy class $\mathcal{E}$ in B_n. We will also write $FC(s)$ if there is an upper bound $s \geqslant |\Sigma_{\mathcal{E}}|$ independent of $\mathcal{E}$,*

- *totally non-conjugate (TNC) if $b_m \not\sim b_{m'}$ whenever $m \neq m'$, i.e. $FC(1)$,*

- *totally non-conjugate at infinity (TNCI) if, when allowing* $M > 0$ *to depend on* (α, β) *and choosing* M *large,*

$$b_m \not\sim b_{m'} \text{ are pairwise non-conjugate for all } m > m' > M \text{ and all } m' < m < -M. \tag{3.2}$$

These properties are interrelated with some other features we considered, which were partly (but not fully) formalized in [**24**, **28**], and whose role will become evident soon.

Definition 3.3. *We specify an ES to be*

- *symmetric (S) if there is a* $\mu \in \mathbb{Z}$ *so that* $b_m \sim b_{m'}$ *whenever* $m + m' = \mu$ *but* $b_m \not\sim b_{m'}$ *whenever* $m + m' \neq \mu$ *(and* $m \neq m'$*).*
- *subsymmetric (SS) if, whenever* $b_{m_1} \sim b_{m'_1}$ *and* $b_{m_2} \sim b_{m'_2}$ *for* $m_i \neq m'_i$*, we have*

$$m_1 + m'_1 = m_2 + m'_2. \tag{3.3}$$

- *quasi-subsymmetric (QSS) if there is a finite set* $\Sigma \subset \mathbb{Z}$ *so that (3.3) holds for* $m_i, m'_i \notin \Sigma$*.*

The implications are

$$\begin{tikzcd}[column sep=small] & & & & & \mathrm{TNC} \arrow[dl, Rightarrow] \\ \mathrm{INC} & \mathrm{FC} \arrow[l, Rightarrow] & \mathrm{TNCI} \arrow[l, Rightarrow] & \mathrm{QSS} \arrow[l, Rightarrow] & \mathrm{SS} \arrow[l, Rightarrow] & \\ & & & & & \mathrm{S} \arrow[ul, Rightarrow] \end{tikzcd}$$

The properties of definition 3.3 may not appear of obvious relevance at first. They emerged from the method of proof of theorem 3.4, but were experimentally found in [**28**] to be far more than a mere artefact of this technique. While TNC and S clearly appear too strong to be expected in general, SS turns out practically omnipresent (see conjecture 3.7).

Extensive experiments are made in [**28**]. As a brief extract of them, we mention that there exist SS, but not symmetric, ES with up to two pairs (m_i, m'_i) as in (3.3). We do not know if three distinct pairs (m_i, m'_i) always imply that the ES is symmetric.

It then also became clear how to construct symmetric ES (although, of course, this is far from the generic case).

Theorem 3.4 [**24**]. *Let a braid* $b \in B_n$ *be exchangeable as in figure 2 and the permutation* $\pi(b)$ *satisfy*

$$\pi(b)(1) \neq 1 \quad \text{and} \quad \pi(b)(n) \neq n. \tag{3.4}$$

Then the ES is SS. If $\pi(b)$ *is a cycle (i.e.* $\hat{b}$ *is a knot) and* n *is even, then the ES is TNC.*

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

The method consisted of evaluating coefficients of the Conway polynomial ∇ of the axis addition link L_{b_m} of b_m, or subbraids thereof. More precisely, there is always a conjugacy invariant υ so that

$$m \mapsto \upsilon(b_m) \tag{3.5}$$

is a (non-constant) at most quadratic polynomial in m. If the polynomial is quadratic, it shows SS, and if it is linear, TNC. We will apply a similar strategy later, just using as υ an invariant we derive from the Burau matrix.

Then Ito [**14**] much more recently obtained using the mapping class group a very similar version of our theorem, in which (3.4) is replaced by the most general assumption of *non-degeneracy*, namely that in figure 3

$$\delta^2_{[2,n-1]}\,\alpha \neq \alpha\,\delta^2_{[2,n-1]} \quad \text{and} \quad \delta^2_{[2,n-1]}\,\beta \neq \beta\,\delta^2_{[2,n-1]}. \tag{3.6}$$

THEOREM 3.5 [**14**]. *If the ES is non-degenerate, then it is INC.*

Non-degeneracy is obviously the weakest possible assumption, since for degenerate braids all exchange moves give conjugate braids. Also, despite not stated explicitly, FC follows from Ito's proof (as recalled for observation 7.1). However, there is no control on the size of (3.1): it can depend not only on n, but on (the ES of) b and on m (or, equivalently, $\mathcal{E}$). In contrast, QSS implies $FC(|\Sigma|+2)$ with a bound depending on b only (but not on m), and SS implies $FC(2)$. Both SS and QSS imply TNCI from (3.2). (Cf. also the remarks below observation 7.1.)

We found that Ito's conditions (3.6) coincide with our previously observed instances (2.25) of failure:

$$\alpha \text{ is of the form (2.25)} \iff \alpha \text{ fails (3.6)}. \tag{3.7}$$

(Only the reverse direction is non-trivial.) The form (2.25) also makes clear why degeneracy can be defined on the ES (rather than the braid itself).

After finding a proof of (3.7) using theorem 3.4 and the combed normal form, I was pointed by González-Meneses that (3.7) also follows from his work with Wiest on describing the centralizer in braid groups [**13**]. This rendered the alternative argument obsolete, but we may note that our proof of theorem 9.1 can be used to show a related form of the reverse direction in (3.7): when α fails (3.6) in $B_n/\ker(\psi_n)$, then α is of the form (2.25) in B_n/P_n^c.

We have in [**28**] the following version of theorem 3.4. This led to the property QSS, sharpening the conclusion of theorem 3.5 under still very general circumstances.

PROPOSITION 3.6 [**28**]. *Assume some $\mathbb{Q}$-Vassiliev braid conjugacy invariant υ distinguishes some b_{m_1} and b_{m_2} (for some $m_1 \neq m_2$). Then the ES is QSS.*

A $\mathbb{Q}$-Vassiliev conjugacy invariant is meant to be a conjugacy invariant of n-braids which is a $\mathbb{Q}$-valued Vassiliev invariant of braids [**3**]. (By standard arguments, $\mathbb{Q}$-valued is equivalent to $\mathbb{Z}$-valued.) Polynomial invariants of L_b, as well as $\nabla(L_b^*)$ or its equivalent $\chi(\psi_n(b))$ (see §2.9), can be understood as infinite collections of such υ.

Thus, while (3.6) remains the most general assumption, it is clear that, practically, the one of proposition 3.6 is very likely no restriction. One can conjecture this equivalence directly, but still, the construction of such invariant for large classes of exchange moves is not straightforward. The invariants of [**24**], operating under (3.4), and yielding theorem 3.4, can be argued to lie in this class.

We formulated in [**28**] the most optimistic (and simplest) expectation regarding the (non-)conjugacy of b_m, which combines Ito's (weakest) assumption and our (strongest) assertion, and which is supported not only by the above results but also by some (and not yet refuted by any) computational evidence.

CONJECTURE 3.7. Every non-degenerate ES is SS.

Our main results can be seen as adding further pieces towards this conjecture.

4. Main results

There is some insight in [**28**] that failures of the method behind theorem 3.4 are related to braids where strand 1 in α must have equal linking number with all strands $2, \ldots, n-1$:

$$lk_2 = \cdots = lk_{n-1}, \tag{4.1}$$

which can be written as

$$\alpha \in P^c_{n-1} \cdot \text{center}\,(B_{n-1}) \cdot B_{2,n-1}. \tag{4.2}$$

[Note that this turns into (2.25) when removing the first factor.] It was tempting to expect that under exclusion of this situation, and its analogue for β, one can always use the Conway polynomial to distinguish L_{b_m}.

We will show below that when replacing the Conway polynomial by the Burau matrix, there is a way to perform a manageable calculation to get out this linking number equality. While it is evident in (2.25), we will be led then to stronger non-conjugacy properties of b_m than FC, that one would have from theorem 3.5.

We have a complete result on linking number equality for two cycles in $\pi(b)$. [Note that, by the freedom of conjugating in $B_{2,n-1}$, the form (4.3), and likewise (4.5), is chosen only for convenience of notation and is no restriction of generality.]

THEOREM 4.1. *Let α, β as in figure 2 have $\pi(\alpha) = Id$ and*

$$\pi(\beta) = (2\ 3\ \ldots\ n-1\ n) \tag{4.3}$$

be a cycle. Let $lk_j = lk_{1,j}$ be the linking number between strands 1 and j in α for $j = 2, \ldots, n-1$. If the linking vector

$$(lk_2, \ldots, lk_{n-1}) \tag{4.4}$$

is not palindromic, then the ES is TNC. If not all lk_j are equal, then the ES is SS.

Now consider, w.l.o.g., that (still α is pure and) n lies in a non-trivial cycle

$$C_0 = (n_0\ n_0+1\ \ldots\ n-1\ n) \tag{4.5}$$

in $\pi(b) = \pi(\beta)$, but not of length $n-1$, so $2 < n_0 < n$. To convey the exact meaning of theorem 1.1, let us say that C_0 is *(linking-)defective* if the equality

$$lk_{n_0} = \cdots = lk_{n-1} \tag{4.6}$$

is not satisfied. (Keep in mind that always $lk_n = 0$.)

While Burau cannot yet, as least completely, detect linking number equality of individual strands j outside C_0, it can still detect it on cycles in $\pi(b)$. To make this precise, let

$$lk = lk_{n_0}, \tag{4.7}$$

and write

$$\{2, \ldots, n_0 - 1\} = C_1 \cup C_2 \cup \cdots \cup C_r$$

for the rest cycles of $\pi(b)$. We also need to consider cycle lengths

$$\lambda_i = |C_i|. \tag{4.8}$$

We include

$$\lambda_0 = |C_0| = n - n_0 + 1$$

in the notation, as it will be often needed. Set similarly

$$\lambda_{-1} = 1, \quad C_{-1} = \{1\}.$$

(see also example 6.1). Once lk is fixed in (4.7) from C_0, define for each cycle C_i, $1 \leqslant i \leqslant r$ (with the notation in §2.6) the *linking defect*

$$\tau(C_i) = lk \cdot \lambda_i - \sum_{j \in C_i} lk_j = lk \cdot \lambda_i - lk(\{1\}, C_i). \tag{4.9}$$

We may call the cycle C_i *(linking-)defective* if $\tau(C_i) \neq 0$.

It is obvious that the presence of defective cycles depends on the ES only, that it will generically occur, and that it can be examined directly from a braid picture (like figure 1) without any braid-group calculations. It is linear in both n and the braid word length. Thus it is slightly more economical than checking degeneracy, which (as the referee informed) is linear in n and quadratic in the braid word length.

Theorem 4.2. *(1) Let α, β as in figure 2 have $\pi(\alpha) = Id$ and the cycle decomposition*

$$\pi(\beta) = C_0 \cdot C_1 \cdot \cdots \cdot C_r \tag{4.10}$$

with (4.5) for $n > n_0 > 2$. If the linking vector

$$(lk_{n_0}, \ldots, lk_{n-1})$$

is not palindromic, then the ES is TNC. If not all lk_j, $n_0 \leqslant j \leqslant n-1$, are equal, then the ES is SS.

(2) Now assume (4.6) and set (4.7) in (4.9). If

$$\tau(C_{i_0}) \neq 0 \text{ for some } 1 \leqslant i_0 \leqslant r, \tag{4.11}$$

then the ES is SS.

5. Proof for two cycles

Proof of theorem 4.1. Fundamentally, throughout this proof, we will be concerned with the evaluation of $\psi_n^{(k)}(1)$, where the superscript means derivative taken w.r.t. t, entrywise, and the resulting matrix in $GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$ is evaluated at $t = 1$.

We prepare the following formulas for $\psi_n(\kappa_{1,l})$, $l = 2, \ldots, n-1$.

$$\psi_n(\kappa_{1,l}) = \begin{bmatrix} t^l & 0 & 0 & \cdots & 0 & 1-t \\ t^l - t^2 & t & 0 & \cdots & 0 & 1-t \\ t^l - t^3 & 0 & t & & & 1-t \\ \vdots & \vdots & & \ddots & & \vdots \\ t^l - t^{l-1} & 0 & \cdots & 0 & t & 1-t \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \oplus Id_{n-1-l}, \tag{5.1}$$

whence

$$\hat{A}_l = \psi_n'(\kappa_{1,l})(1) = \begin{bmatrix} l & 0 & 0 & \cdots & 0 & -1 \\ l-2 & 1 & 0 & & 0 & -1 \\ l-3 & 0 & 1 & & 0 & -1 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1 & 0 & \cdots & 0 & 1 & -1 \\ 0 & & \cdots & & 0 & 0 \end{bmatrix} \oplus 0_{n-1-l}.$$

One can check (5.1) as follows. There are explicit matrices X_n so that the embedding

$$\iota_1 : B_{n-1} \simeq B_{1,n-1} \subset B_n \tag{5.2}$$

gives

$$\psi_n \circ \iota_1 = X_n(\psi_{n-1} \oplus 1)X_n^{-1}. \tag{5.3}$$

The matrices are not difficult to find directly, and will also be explained in [**27**]. We just give the formula here. With $s = \sqrt{t}$,

$$\langle k \rangle = s^k - s^{-k} \quad \text{and} \quad a_k = \frac{s^{-k}\langle n-1-k \rangle}{\langle n-1 \rangle},$$

we have

$$X_n = \begin{bmatrix} 1 & & \cdots & 0 & a_{n-2} \\ 0 & 1 & \cdots & 0 & a_{n-1} \\ & & \ddots & & \vdots \\ & & & 1 & a_1 \\ & & & & 1 \end{bmatrix}. \tag{5.4}$$

(The splitting behaviour of $\psi_n \circ \iota_1$ is very well known [**15**].)

With (2.9), one can then calculate $\psi_n(\delta^2_{[1,n-1]})$ and $\psi_n(\delta^2_{[1,n-2]}) = \psi_{n-1}(\delta^2_{[1,n-2]}) \oplus 1$. The rest for obtaining (5.1) follows from (2.15).

We have the (non-commutative) matrix Leibniz rule (where prime refers to the derivative in t),

$$(AB)' = AB' + A'B (\neq AB' + BA'). \tag{5.5}$$

By differentiating $AA^{-1} = Id$ for $A = \psi_n(\kappa_{1,l})$, and using $A(1) = Id$, we see also

$$\psi'_n(\kappa^{-1}_{1,l})(1) = -\psi'_n(\kappa_{1,l})(1) = -\hat{A}_l. \tag{5.6}$$

It will be better in the following to move from ψ_n to the *non-reduced Burau representation* $\psi^\times_n$ of B_n acting on $\mathbb{C}^n$ by

$$\psi^\times_n(\sigma_i)(t) = Id_{i-1} \oplus \begin{bmatrix} 1-t & 1 \\ t & 0 \end{bmatrix} \oplus Id_{n-i-1}.$$

The form (2.8) comes from looking at the action of $\psi^\times_n$ on (the linear span of) the set of vectors $\mathbf{e}_i - \mathbf{e}_{i+1}$, with $\mathbf{e}_i$ the standard basis vector. There is an extra dimension coming from the fixvector $\sum \mathbf{e}_i$. Thus

$$\psi^\times_n = T(1 \oplus \psi_n)T^{-1} \quad T = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 1 & & \vdots \\ 1 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \cdots & -1 & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix}. \tag{5.7}$$

This map $(.)^\times$ augments matrix size, however, adding this extra dimension will simplify calculation enormously. For instance, the embedding ι_1 in (5.2) simplifies (5.3) to $\psi^\times_{n+1} \circ \iota_1 = \psi^\times_n \oplus 1$, while the one

$$\iota_2 : B_{n-1} \simeq B_{2,n} \subset B_n \tag{5.8}$$

gives $\psi^\times_{n+1} \circ \iota_2 = 1 \oplus \psi^\times_n$. Also

$$\Pi = \Pi(b) = \psi^\times_n(b)(1) \tag{5.9}$$

is just the permutation matrix of $\pi(b)$, with $\Pi_{i,\pi(b)(i)} = 1$ and $\Pi_{ij} = 0$ otherwise. Since the transition matrix T does not depend on t, it is also clear that $(\psi^\times_n)^{(k)} = (\psi^{(k)}_n)^\times$.

One can calculate straightforwardly that

$$A_l = \psi_n^\times(\kappa_{2,l+1})'(1) = 0 \oplus (\hat{A}_l)^\times \ominus 0, \tag{5.10}$$

where $(M \oplus N) \ominus N = M$ and (with a matrix having $l-1$ rows and columns with a '-1' entry)

$$(\hat{A}_l)^\times = \begin{bmatrix} l-1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \ddots & & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 1 \end{bmatrix} \oplus 0_{n-l}.$$

Letting the commutator be

$$[A, B] = AB - BA, \tag{5.11}$$

we also prepare for $l = 2, \ldots, n-2$,

$$[A_l, A_{n-1}] = 0 \oplus \left[\begin{array}{c|c|c} 0 & n-1-l & 1-l \\ \hline l-n+1 & & \\ \vdots & 0 & 1 \\ l-n+1 & & \\ \hline l-1 & & \\ \vdots & -1 & 0 \\ l-1 & & \end{array}\right] \tag{5.12}$$

with the blocks of (both horizontal and vertical) size $1, l-1, n-1-l$, respectively (and all entries within each block identical).

It will be easier to work with the form (2.24), which can be obtained from (2.1) by conjugating β by

$$\sigma_1\sigma_2 \cdots \sigma_{n-1}, \tag{5.13}$$

and isolating the two letters $\sigma_1^{\pm 1}$ [so that (4.3) remains the same], while changing α to $\iota_2(\alpha)$ from (5.8). *For the rest of the proof, we will use (2.24),* but continue writing α for $\tilde{\alpha}$ and β for $\tilde{\beta}$. Thus, for $\alpha, \beta \in B_{2,n}$, write

$$b(\alpha, \beta) = \alpha\sigma_1\beta\sigma_1^{-1}. \tag{5.14}$$

Then by (2.24),

$$b_0 = b(\alpha, \beta). \tag{5.15}$$

Furthermore, we keep n fixed and often simplify notation $\psi^\times = \psi_n^\times$.

Consider now the combed normal form for α. In (2.16), we can assume $\alpha_1 = Id$ because $\alpha \in B_{2,n}$. Also, since any right factor of α in $B_{3,n}$ can be moved into β [and, being pure, will not affect (4.3)], we can assume w.l.o.g. $\alpha_i = Id$ for $i > 2$.

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

We write thus

$$\alpha = \prod_{i=1}^{k} \kappa_{2,j_i}^{\varepsilon_i}. \tag{5.16}$$

Using that $\delta_{[2,n]}^2$ commutes with α, we will also rewrite (2.24) in the form that, to obtain b_1 from b_0, one adds a $\kappa_{2,n}^{-1}$ at the beginning and a $\kappa_{2,n}$ at the end of the product in (5.16). Thus, in conformance with (5.14), and as a generalization of (5.15),

$$b_1 = b(\kappa_{2,n}^{-1}\alpha\kappa_{2,n}, \beta), \quad b_m = b(\kappa_{2,n}^{-m}\alpha\kappa_{2,n}^{m}, \beta). \tag{5.17}$$

This is equivalent to (2.24), and thus also to (2.1), under conjugacy, and we will afford to maintain the same notation b_m.

Next, we will start evaluating

$$(\psi^\times)^{(k)}(b_1)(1) - (\psi^\times)^{(k)}(b_0)(1) \tag{5.18}$$

by using the product (5.16) and its modification $\kappa_{2,n}^{-1}\kappa_{2,j_1}^{\varepsilon_1}\cdots\kappa_{2,j_k}^{\varepsilon_k}\kappa_{2,n}$ in b_1 explained below (5.16).

The case $k = 0$ is completely trivial, thus let $k = 1$. Using (5.5) iteratedly on the form (5.16), one can see that the terms for $(\psi^\times)'(b_1)(1)$ are the same as for $(\psi^\times)'(b_0)(1)$, except those coming from taking the derivative in the factors $\kappa_{2,n}^{\pm 1}$ added. But since α is pure (and $\psi_n^\times(\kappa_{2,j}) = Id$) and because of (5.6), these two terms cancel as well. The result of (5.18) is thus 0 for $k = 1$.

But so prepared, we examine now $k = 2$. This requires somewhat more careful collection of terms, but the procedure is clear. Since we need them often later, let us fix the permutation matrices of the cycle and transposition

$$P = \Pi((2\ 3\ \cdots\ n)) \quad \text{and} \quad \Gamma = \Pi((1\ 2)). \tag{5.19}$$

Only products involving the terms $\kappa_{2,n}^{\pm 1}$ added to α will contribute. We use again $\psi_n^\times(\alpha)(1) = Id$, and it is helpful to note that

$$(\psi^\times)''(\kappa_{2,n}^{-1})(1) + 2(\psi^\times)'(\kappa_{2,n}^{-1})(1)(\psi^\times)'(\kappa_{2,n})(1) + (\psi^\times)''(\kappa_{2,n})(1) = 0,$$

which is obtained like (5.6) taken one derivative further.

The result is given thus. Let in (5.16) for $l = 2, \ldots, n-2$

$$\eta_l = \sum_{j : k_j = l+1} \varepsilon_j$$

be the exponent sum of $\kappa_{2,l+1}$, also written as

$$\eta_l = lk_{2,l+1} - lk_{2,l+2}, \tag{5.20}$$

from (2.18). Then let A_l be as in (5.10), and with (5.11) and (5.12),

$$\Omega = \sum_{l=2}^{n-2} \eta_l [A_l, A_{n-1}]. \tag{5.21}$$

Then we have with (5.19)

$$(\psi^{\times})''(b_1)(1) - (\psi^{\times})''(b_0)(1) = \Omega\Gamma P\Gamma. \tag{5.22}$$

The only conjugacy invariant we can really control from this is the trace, since it is linear:

$$\mathrm{tr}((\psi_n^{\times})^{(k)}(b_m)(1)) = (\mathrm{tr}\psi_n^{\times}(b_m))^{(k)}(1).$$

Assume the non-vanishing condition

$$\mathrm{tr}\left((\psi^{\times})''(b_1)(1) - (\psi^{\times})''(b_0)(1)\right) = \mathrm{tr}(\Omega\Gamma P\Gamma) \neq 0. \tag{5.23}$$

Now note that this expression does not change when we modify b_1 to b_{m+1} and b_0 to b_m, since the contribution of $\kappa_{2,n}^{\pm m}$ added in (5.16) does not change (5.21). Then (5.23) will imply that $\mathrm{tr}((\psi^{\times})''(b_m))(1)$ is a linear progression in m, so all $\mathrm{tr}(\psi^{\times}(b_m))$ will be distinct, all b_m will be non-conjugate, and the ES will be TNC. We will now regard the opposite of (5.23) via (5.21) as a linear condition on the η_l.

The goal is to collect enough similar linear conditions, until only the trivial solution $\eta_l = 0$ remains. Now obviously, $\mathrm{tr}((\psi^{\times})''(b_m)(1))$ will give a scalar, which is utterly insufficient. To help ourselves, we replace b_m by b_m^p and repeat the process. This means that for $p > 0$ we consider (3.5) for the series of conjugacy invariants

$$\upsilon(b) = \upsilon_p(b) = \mathrm{tr}((\psi_n^{\times})''(b^p)(1)). \tag{5.24}$$

It is a technical, but with the above explanation straightforward, thought, that

$$(\psi^{\times})''(b_1^p)(1) - (\psi^{\times})''(b_0^p)(1) = \sum_{q=0}^{p-1} \Gamma P^q \Gamma\Omega\Gamma P^{p-q}\Gamma. \tag{5.25}$$

The matrices summed are conjugated, thus the generalization of the logical negation of (5.23) becomes

$$p \cdot \mathrm{tr}(\Omega\Gamma P^p\Gamma) = 0, \tag{5.26}$$

wherein we remove the unnecessary first factor. Since $\pi(b) = \pi(\beta)$ has order $n-1$, it is also clear that only

$$1 \leqslant p < n-1 \tag{5.27}$$

makes sense. (The case $p = 0$ will give nothing.) This is again a linear condition

$$\sum_{l=2}^{n-2} q_{p,l}\eta_l = 0,$$

where

$$q_{p,l} = \mathrm{tr}([A_l, A_{n-1}] \cdot \Gamma P^p \Gamma).$$

Since $\Gamma P^p \Gamma$ is a permutation matrix from (5.19), and the commutator is given in (5.12), one can directly evaluate $q_{p,l}$. We obtain a homogeneous linear system with

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

$n-2$ equations for (5.27) and $n-3$ variables for $l=2,\ldots,n-2$. The matrix $(q_{p,l})$ of this system (with rows indexed by p and columns by l) looks

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & & \ddots & \iddots & & \vdots & \vdots \\ \vdots & \vdots & & \iddots & \ddots & & \vdots & \vdots \\ \vdots & & -1 & \cdots & \cdots & -1 & & \vdots \\ 0 & -1 & -1 & \cdots & \cdots & -1 & -1 & 0 \\ -1 & -1 & -1 & \cdots & \cdots & -1 & -1 & -1 \end{bmatrix}. \tag{5.28}$$

(When n is odd, there will be a zero row in the middle.) Its kernel elements $(\eta_2,\ldots,\eta_{n-2})^T$ satisfy

$$\eta_l = -\eta_{n-l}, \tag{5.29}$$

which with (5.20) (and shifting indices down by 1, to undo the change of form of the exchange move we performed at the beginning of the proof), gives the palindromicity of (4.4). In this situation, (5.24) run out of use.

This palindromicity problem was well-known in [**28**], with an example showing that then indeed two distinct b_m can be conjugate. Under such circumstance for any conjugacy invariant υ a linear progression (3.5) will be trivial. So we look for a quadratic one.

We will take $k=3$, in the context of (5.18), and replace in (3.5)

$$\upsilon(b) = \upsilon_p(b) = \mathrm{tr}((\psi_n^{\times})'''(b^p)(1)). \tag{5.30}$$

Our goal is, instead of (5.18), to determine

$$\upsilon_p(b_{m+2}) - 2\upsilon_p(b_{m+1}) + \upsilon_p(b_m) \tag{5.31}$$

first for $m=0$, and notice that this expression again does not depend on m (for fixed p), and to show that it is not zero (for some p), unless all $\eta_l = 0$. This will imply that $m \mapsto \upsilon_p(b_m)$ is a non-constant quadratic polynomial in m, and thus complete the proof.

Write

$$\overline{\alpha}_m = \kappa_{2,n}^{-m}\alpha\kappa_{2,n}^{m}\,, \quad \overline{\beta}_{m'} = \kappa_{2,n}^{m'}\beta\kappa_{2,n}^{-m'}, \tag{5.32}$$

and set using (5.14)

$$b(m,m') = b\left(\overline{\alpha}_m,\ \overline{\beta}_{m'}\right), \tag{5.33}$$

so that with (2.24)

$$b_{m+m'} \sim b(m,m').$$

Instead of (5.31), we will evaluate, equivalently up to sign,

$$(\upsilon_p(b(1,1)) - \upsilon_p(b(1,0))) - (\upsilon_p(b(0,1)) - \upsilon_p(b(0,0)))\,. \tag{5.34}$$

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

The calculation will also make soon clear that this expression (5.34) remains the same if we replace 1 by $m+1$ and 0 by m in the *first* argument of $b(.,.)$.

It will thus be enough to show that (5.34) $\neq 0$ for some p.

Again apply the Leibniz rule on the product form (5.16). The linear combination (5.34) was designed again so as to see that most terms cancel.

Since β is not pure, the above argument for $k=1$ does not fully apply to $\overline{\beta}_{m'}$; rather one is left to cancel terms of differentiated $\kappa_{2,n}^{\pm 1}$ in $\overline{\beta}_{m'}$ [occurring in the explicit form (5.32), not inside β] in a different way.

In the end, when writing $b(m,m')^p$ as p copies of the r.h.s. of (5.33) one after the other, one is left with terms, for $m,m'=0,1$,

$$\begin{aligned}&(\psi_n^\times)'''(b(1,1)^p)(1)-(\psi_n^\times)'''(b(1,0)^p)(1)-(\psi_n^\times)'''(b(0,1)^p)(1)+(\psi_n^\times)'''(b(0,0)^p)(1)\\&\quad=\sum_{i,j=1}^{p}\begin{pmatrix}\text{2 derivatives}\\ \text{in } i\text{-th } \overline{\alpha}_m\end{pmatrix}\cdot\begin{pmatrix}\text{one derivative}\\ \text{in } j\text{-th } \overline{\beta}_{m'}\end{pmatrix}.\end{aligned}\tag{5.35}$$

Hereby we understand $\overline{\alpha}_m$ and $\overline{\beta}_{m'}$ differentiated, using (5.5), in their product forms (5.32), using the representation (5.16) for α. The number of derivatives 'in' $\overline{\alpha}_m$ is the number of factors differentiated in the subword corresponding to one of the p copies of $\overline{\alpha}_m$.

With a bit of care in collecting terms one sees that, the way the linear combination on the left of (5.35) was designed, the terms with no derivative in $\overline{\beta}_{m'}$ and at most one derivative in $\overline{\alpha}_m$ cancel out, as well as those with two derivatives in two different copies of $\overline{\alpha}_m$.

The two derivatives in $\overline{\alpha}_m$ are known to give Ω from (5.21), while those of $\kappa_{2,n}^{-1}$ in the j-th $\overline{\beta}_1$ and of $\kappa_{2,n}$ in the $j+1$-st copy (up to cyclic permutations) cancel (in the trace). There remain one derivative of $\kappa_{2,n}^{-1}$ in a $\overline{\beta}_1$ and one of $\kappa_{2,n}$ for each Ω, those not separated by a copy of P from Ω (but by a copy of Γ). We set

$$\Xi_l=[\Gamma[A_l,A_{n-1}]\Gamma,A_{n-1}]=\left[\begin{array}{c|c|c|c}0&0&n-1-l&1-l\\ \hline 0&0&l+1-n&l-1\\ \hline n-l-1&l+1-n&&\\ \vdots&\vdots&0&0\\ n-l-1&l+1-n&&\\ \hline 1-l&l-1&&\\ \vdots&\vdots&0&0\\ 1-l&l-1&&\end{array}\right],\tag{5.36}$$

with the blocks of sizes $1,1,l-1,n-1-l$, respectively (in both rows and columns, and all entries within each block identical).

When the $\kappa_{2,n}^{-1}$ in the last copy of $\overline{\beta}_1$ is differentiated, we cycle it to the left, and get, up to this modification (which does not affect the trace),

$$\mathrm{tr}(5.35)=\mathrm{tr}\left(\sum_{d=0}^{p-1}\Gamma P^d[\Gamma\Omega\Gamma,A_{n-1}]P^{p-d}\Gamma\right).\tag{5.37}$$

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

As Ω does not depend on m, it is also already clear that (5.31) does not either. We will evaluate it for $m = 0$ via (5.34).

When taking traces, one can, as passing from (5.25) to (5.26), cycle powers of $\Gamma P \Gamma$ (the remnants, under setting $t = 1$, of undifferentiated copies of β) to one side. We obtain thus

$$(5.34) = p \cdot \mathrm{tr}\left(\sum_{l=2}^{n-2} \eta_l \Xi_l P^p\right). \tag{5.38}$$

For the purpose of testing whether this is zero or not, we can again discard the factor p.

Build a matrix M (of $n-2$ rows), whose $n-3$ columns,

$$\begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(n-2) \end{bmatrix},$$

for $l = 2, \ldots, n-2$ and $\phi = \phi_l$, are given by

$$\phi_l(p) = \mathrm{tr}(\Xi_l P^p) = \left\{ \begin{array}{lll} 2l-n & p \leqslant l-1 & \\ 2l-2 & p > l-1,\ p \leqslant n-1-l; & l-1 \leqslant n-1-l \\ 2l-n & p > n-l-1 & \\ \hline 2l-n & p \leqslant n-1-l & \\ 2l+2-2n & p > n-1-l,\ p \leqslant l-1; & l-1 > n-1-l \\ 2l-n & p > l-1 & \end{array} \right. . \tag{5.39}$$

The matrix M is the analogue of (5.28), but is now *anti*symmetric when rows are reflected. (The dotted diagonals do *not* meet at the same element, and when n is even, the middle *column* is zero.)

$$M = \begin{bmatrix} 4-n & 6-n & 8-n & \cdots & \cdots & & & n-4 \\ 2 & 6-n & 8-n & \cdots & \cdots & n-8 & n-6 & -2 \\ 2 & 4 & 8-n & \cdots & \cdots & n-8 & -4 & -2 \\ 2 & 4 & 6 & \cdots & \cdots & -6 & -4 & -2 \\ \vdots & \vdots & & \ddots & \iddots & & \vdots & \vdots \\ \vdots & \vdots & & \iddots & \ddots & & \vdots & \vdots \\ \vdots & & 6 & \cdots & \cdots & -6 & & \vdots \\ 2 & 4 & 8-n & \cdots & \cdots & n-8 & -4 & -2 \\ 2 & 6-n & 8-n & \cdots & \cdots & n-8 & n-6 & -2 \\ 4-n & 6-n & 8-n & \cdots & \cdots & n-8 & n-6 & n-4 \end{bmatrix}. \tag{5.40}$$

The resulting equation system from setting (5.38) to 0 becomes

$$M \cdot (\eta_2, \ldots, \eta_{n-2})^T = 0, \tag{5.41}$$

and it does enough to enforce

$$\eta_l = \eta_{n-l}.$$

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

With (5.29), this implies that all $\eta_l = 0$, and thus completes the proof of theorem 4.1. □

6. Proof for three or more cycles

For this we use subbraids. There are two ways, outlined in [**24**]. The first is given in [**24**, lemma 4.1]. A better alternative, though, is followed in [**24**, §8], and consists in summing over invariants of suitably chosen subbraids of b_m. We will use this approach here, which we clarify below.

Proof of theorem 4.2. We will adapt the proof of theorem 4.1.

Again, it will be easier to work with the form (2.24). Since we can w.l.o.g. conjugate α, β in (2.24) by $B_{3,n}$, we can assume (4.5) turning into

$$C_0 = (2\ 3\ \dots\ n'-1\ n'), \tag{6.1}$$

with $n'-2 = n-n_0 > 0$ and $\lambda_0 = n'-1$. A few (obvious) modifications from the formulation of the theorem have then to be made below.

We have now $lk_2 = 0$ [instead of $lk_n = 0$ with (2.1)]. Let us first argue that

$$lk_3 = \cdots = lk_{n'}. \tag{6.2}$$

For every n_0, we described in the proof of theorem 4.1 an invariant $v_{[n_*]}$ on B_{n_*} with $v_{[n_*]}(b_m)$ being a quadratic polynomial in m, whose quadratic term we proved to be non-trivial to establish SS.

For $b \in B_n$, define a conjugacy invariant

$$\xi(b) = \sum_{D_{1,2}} v_{[|D_1|+|D_2|]}(b_{[D_1 \cup D_2]}), \tag{6.3}$$

where the sum goes over unordered pairs of (distinct) cycles $D_{1,2}$ of $\pi(b)$ whose lengths are

$$\{|D_1|, |D_2|\} = \{1, n'-1\}.$$

(This is not immediately relevant, but let us specify already here that in selecting collections of cycles, their length condition is meant as a *multiset*, with repeated elements counted, i.e., it is an equality of tuples modulo permutations of their entries.)

This ξ is a well-defined conjugacy invariant, and in $\xi(b_m)$ the contribution of all pairs $D_{1,2}$ of cycles will vanish, unless $1 \in C_{-1} = D_1$ and $2 \in C_0 = D_2$. This identifies

$$\xi(b_m) = v_{[n']}((b_m)_{[D_1 \cup D_2]}),$$

and the rest follows from the proof of theorem 4.1. This establishes (6.2). (The non-palindromicity assertion follows similarly.) So we obtain part 1.

For the rest of the proof we deal with part 2 and assume (4.11).

Also assume (6.2), so that

$$\eta_l = 0 \text{ for } l = 2, \ldots, n' - 2. \tag{6.4}$$

We can try to repeat the calculation in the proof of theorem 4.1, with the only difference being that now P in (5.19) is the matrix of the new permutation

$$\pi(b) = C_0\, C_1 \cdots C_r.$$

If we try to evaluate (5.24), we can see that we get nothing new. However, for (5.34), we do.

Assuming from (6.4) that $l \geqslant n' - 1 = \lambda_0$, we see that the minor of the matrix in (5.36) consisting of rows and columns $2, \ldots, n'$ is comprised within the four central blocks in the form on the right of (5.36). It includes the upper left one among the four blocks, which is a single 0 but, since $n' \geqslant 3$, has at least one further row and column. This matrix does not change with P.

Then for any p not divisible by $n' - 1$, say, $p = 1$ (here we need $n' > 2$), we get instead of (5.39), with our new P,

$$\phi_l(1) = 2(l + 1 - n) \quad l = n' - 1, \ldots, n - 2. \tag{6.5}$$

What remains of (5.41) is the scalar equation

$$0 = 2 \sum_{l=n'-1}^{n-2} (l + 1 - n)\eta_l, \tag{6.6}$$

which with (5.20) can be rewritten as

$$(n - n')lk_{n'} = lk_{n'+1} + \cdots + lk_n. \tag{6.7}$$

(Note that the contributions from C_i, $i > 0$ in P give nothing for the trace. This observation will reappear for modification later.)

Also note that the symmetry of the right in $n' < j \leqslant n$ is necessary because of the freedom to conjugate with $B_{n'+1,n}$. If (6.7) fails, which is the same as saying

$$\sum_{i=1}^{r} \tau(C_i) \neq 0,$$

we have in (5.30) that $\upsilon(b_m)$ has a non-trivial quadratic term in m.

The expression on the right of (6.7) goes over all cycles C_i, $0 < i \leqslant r$. However, the idea now is to apply this argument to $(b_m)_{[D_1 \cup D_2 \cup D_3]}$ where $1 \in D_1 = C_{-1}$, $2 \in D_2 = C_0$ and $D_3 = C_{i_0}$ is a proper single cycle (different from $D_{1,2}$).

The statement of part 2 is then what one can do about it. A problem is that when one tries to create an invariant like (6.3), we need to specify some condition, preserved under conjugacy, to tell our favourite cycle C_{i_0} apart from others. However, this is not a problem if we can 'confuse' C_{i_0} only with 'similar' cycles that give the same contribution to the m-quadratic term of $m \mapsto \upsilon_{[n^*]}(b_m)$. Then this quadratic term only multiplies by some (non-zero) number.

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

To this vein, fix [using the notation (4.8)]

$$lk = \frac{lk(C_{-1}, C_0)}{\lambda_0 - 1},$$

and an i_0 in (4.11). Modify (6.3) to

$$\xi(b) = \sum_{D_{1,2,3}} \upsilon_{[|D_1|+|D_2|+|D_3|]}(b_{[D_1 \cup D_2 \cup D_3]}), \tag{6.8}$$

where the sum goes over *ordered* triples (D_1, D_2, D_3) of distinct cycles $D_{1,2,3}$ of $\pi(b)$, with the following conditions:

(a) $|D_1| = \lambda_0$, $|D_2| = 1$, $|D_3| = \lambda_{i_0}$

(b) $lk(D_1, D_2) = (|D_1| - 1) \cdot lk$ and
$lk(D_2, D_3) = lk(C_{-1}, C_{i_0})$.

This is again a (suitably chosen) conjugacy invariant of $b \in B_n$.

Note again that the contribution of all (D_1, D_2, D_3) in (6.8) to $\xi(b_m)$ is constant (in m) unless both C_{-1} and C_0 are among the D_i. Also, C_{-1} and C_0 are distinguished by $\lambda_{-1} = 1 < \lambda_0$. Because of (4.11), the m-quadratic contribution of

$$D_1 \mapsto C_0, \quad D_2 \mapsto C_{-1}, \quad D_3 \mapsto C_{i_0} \tag{6.9}$$

will be non-zero. Keep in mind that this contribution (as well as the choice of invariant $\upsilon_{[n^*]}$ contributed to) depends only on the ordered quadruple

$$(\lambda_0, \lambda_{i_0}, lk(C_{-1}, C_0), lk(C_{-1}, C_{i_0})). \tag{6.10}$$

To specify what other matchings are possible, let us say that $C_{i_0'}$ is *similar* to C_{i_0} for $1 \leqslant i_0' \leqslant r$, if $\lambda_{i_0'} = \lambda_{i_0}$ and $lk(C_{-1}, C_{i_0'}) = lk(C_{-1}, C_{i_0})$. The contribution of

$$D_1 \mapsto C_0, \quad D_2 \mapsto C_{-1}, \quad D_3 \mapsto \text{cycle } C_{i_0'} \text{ similar to } C_{i_0} \tag{6.11}$$

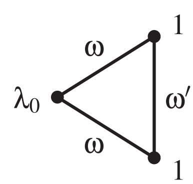
should then be clearly equal to the one of (6.9).

Only in few situations do extra matchings occur, and they create no harm either.

If $\lambda_{i_0} = 1$ and $\omega := lk(C_{i_0}, C_0) = lk(C_{-1}, C_0)$, then there is the additional possibility

$$D_1 \mapsto C_0, \quad D_2 \mapsto C_{i_0}, \quad D_3 \mapsto C_{-1},$$

but its contribution remains the same because (6.10) does not change compared to (6.9); it is the vertical symmetry of the subgraph



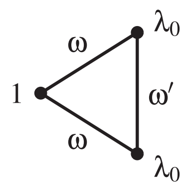
of $\Upsilon(b) = \Upsilon(b_m)$.

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

Similarly if $\lambda_{i_0} = \lambda_0$ and $\omega := lk(C_{-1}, C_{i_0}) = lk(C_{-1}, C_0)$, one can have

$$D_1 \mapsto C_{i_0}, \quad D_2 \mapsto C_{-1}, \quad D_3 \mapsto C_0.$$

This corresponds to the vertical reflection of



and does not change (6.10) either. How to proceed with cycles $C_{i_0'}$ similar to C_{i_0} instead of C_{i_0} should be clear. □

We include one example to illustrate the claim of the theorem.

Example 6.1. Use the convention (6.1) for the form (2.23). Let $n = 10$, $n' = 4$, and

$$\pi(b) = (1)(2\ 3\ 4)(5\ 6\ 7\ 8)(9\ 10),$$

so that $\lambda_0 = 3$, $\lambda_1 = 4$ and $\lambda_2 = 2$. Part 1 of the theorem then states that SS (and $FC(2)$) hold unless $lk := lk_2 = lk_3 = lk_4$ are equal, and part 2 amplifies this condition with $\tau(C_i) = 0$, which is

$$4lk = lk_5 + lk_6 + lk_7 + lk_8, \quad 2lk = lk_9 + lk_{10}. \tag{6.12}$$

Remark 6.2. Regarding part 2, although this was not needed above, one may also further separate different C_i in b_m by a condition like this. Initially fix for every $\lambda > 0$ a one-cycle conjugacy class $\mathcal{E}_\lambda$ or a two-cycle one $\mathcal{E}'_\lambda \subset B_\lambda$ (or a union of such). Then one can build a conjugacy invariant of $b \in B_n$ by requiring $D_1 = C_{i_0}$ in (6.3) to be so that $b_{[D_1]} \in \mathcal{E}_{|D_1|}$, or that there is a $j \neq i_0$ with $b_{[D_1 \cup C_j]} \in \mathcal{E}'_{|D_1|+|C_j|}$, etc. The use of conjugacy classes $\mathcal{E}_\lambda$, $\mathcal{E}'_\lambda$ can also restrict the matching process that occurs in the proof of [**24**, lemma 4.1] and so forth.

7. Relations, limitations and prospects

Probably, the present method can recover theorem 3.4 as well, by using for P in (5.19) a proper permutation matrix. We will not try, though, to reinvent another proof here; we focused on what is new.

It is worth pointing to a well-known connection between the eigenvalues of the Burau matrix and entropy; see e.g. [**16**]. If we regard $b \in B_n$ as a homeomorphism of the punctured disk, then for the *spectral radius* ρ, the *topological entropy* $h(b)$ satisfies

$$h(b) \geqslant \max\{\,\ln \rho(\psi_n(b))(t) \,:\, t \in \mathbb{C}\,,\ |t| = 1\,\}. \tag{7.1}$$

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

OBSERVATION 7.1. Every non-degenerate ES is an example where for $|m| \to \infty$, the inequality (7.1) on b_m becomes arbitrarily unsharp.

Proof. It follows from Ito's proof of theorem 3.5 that $h(b_m) \to \infty$ when $|m| \to \infty$. On the other hand, one can see using (2.9) and the embedding properties (5.3) that for fixed unit norm $t \in \mathbb{C}$, all entries of $\psi_n(b_m)(t)$ will remain bounded as $|m| \to \infty$. Thus so need to behave its eigenvalues as well. This bound is also universal (i.e. depends on b only) for all such t. $\square$

This leaves unclear, at least asymptotically, how to profit from the Burau matrix to improve dilatation bounds (as asked by Ito [**14**]) for iterated exchanged braids. The proof of our main results (as well as the one of theorem 3.4, via the relation in §2.9) thus also reveals to show the eigenvalues change within a confined domain.

REMARK 7.2. The method does definitely hit its limits when $\pi(b)(1) = 1$ and $\pi(b)(n) = n$. In that case, one would have to examine two vectors (η_l) for α and $(\eta'_{l'})$ for β. The invariants in (5.18) would have to be combined as in (5.34), but for $k = 4$. This will then give some condition on $(\eta), (\eta')$ under some bilinear form. Higher p will not bring more than the order of $\pi(b)$. If $\pi(b) = Id$, then all p would only give multiples of the same form. If taking $k > 4$, the result like in (5.34), or other linear combinations of the same sort, will depend on more than η_l, but on the order of letters in (5.16).

In general, ψ_n exhibits some rigidity under operations, which often poses problems if wished to overcome. In addition to our above experienced difficulties, for example, strand cabling will bring no new information in the proof of theorem 4.2 than strand omission.

REMARK 7.3. Similarly unhelpful is the status of the braid power p. We saw with (6.5) that the calculation (6.6) of the (now altered) trace (5.39) does no longer depend on which power p we evaluate $v(b_m^p)$ on, as long as $n' - 1 \nmid p$. Also, a power of cycles C_i for $i > 0$ in (4.10) of equal length will split into (possibly more) cycles of equal (even if smaller) length. Thus if one likes to restrict lk_j for strands $n' < j \leqslant n$ which lie in the same cycle C_i of $\pi(b)$, one sees that they cannot be distinguished in a power of b more than in b itself. In particular, the defectiveness condition (4.11) on $\tau(C_i)(b^p)$ from (4.9) will not refine (but can weaken) this same condition on $\tau(C_i)(b)$. Likewise, subbraids of braid powers are conjugate, so if one uses a restriction like in remark 6.2, and evaluates $\tau_{\mathcal{E}_\lambda}(C_i)$ for $b_{[C_i]} \in \mathcal{E}_\lambda$ only, then a condition on $\tau_{\mathcal{E}_\lambda}(b_m^p)$ will again be dominated by a similar condition on $\tau_{\tilde{\mathcal{E}}_\lambda}(b_m)$. It is not clear how to profitably adapt [**24**, lemma 4.1], either.

In these situations, where algebra (apparently yet) leaves the edge to geometry, it is then interesting to examine, for instance, what (more) the Lawrence–Krammer representation may reveal about non-conjugacy. But we will not stretch our account with this separate endeavour.

8. Application to links

In connection with the form (2.25), we studied in [**28**] a notion of regularity of links that allows one to exclude a degenerate ES for (a braid representative of) a link L. Theorem 4.1 allows one to extend some of these considerations while improving the quality of statement about non-conjugacy among the b_m.

We will not repeat all discussion; let a few remarks suffice. The hyperbolicity part of regularity and twisting cannot be used. The linking number conditions of regularity can be used unless L has a $U_{[2]}$ sublink with the (there) discussed properties. (We wrote $U_{[2]}$ for the two-component unlink.) The total linking number of a cycle in theorem 4.2 is sufficient (instead of individual strand linking numbers), since this is what transpires in the linking number of components of $\hat{b}_m$.

For demonstration, we revisit more explicitly only the two-component case. We assume L is a 2-component link and write ℓ for the linking number of its components.

Corollary 8.1. *If $n-2 \nmid \ell$, then every n-braid ES of L is SS.*

Proof. Since L has 2 components, $\pi(b)(1)=1$ and $\pi(b)(n)=n$ will imply $n=2$, which is irrelevant in (2.11). If (3.4) holds, apply theorem 3.4. Thus we are left in the situation of theorem 4.1. If the ES is not SS, then all lk_j are equal, and $\ell = \sum_{j=2}^{n-1} lk_j$ is divisible by $n-2$. □

Corollary 8.2. *If $0 < \ell < b(L)-2$, then every ES of L is SS.*

Proof. Because no $n \geqslant b(L)$ can satisfy $n-2 \mid \ell$. □

This applies, among others, also to arbitrary component links L with no $U_{[2]}$ sublink, for which $0 < lk(U, L\setminus U) < b(L)-2$ for every unknotted component U.

One then also obtains some of the consequences of regularity in [**28**] without using theorem 3.5. The below statement, for instance, could be derived in [**24**, corollary 6.2] only modulo 2.

Corollary 8.3. *If $b(L)-2 \nmid \ell$, then (for any $n \geqslant b(L)$) L has infinitely many conjugacy classes of n-braid representatives if and only if L has an n-braid representative admitting an exchange move.*

Proof. The case $n = b(L)$ follows from combining theorem 3.1 and corollary 8.1. Let $n > b(L)$. Let $T(p,q)$ for the (p,q)-torus link. Since $n-1 \geqslant 3$ from (2.11), which is more than the components of L, we have that $L \neq T(n-1, p(n-1))$ is not such a torus link. Then L has infinitely many conjugacy classes of (reducible, and hence exchangeable) n-braid representatives by [**25**]. □

9. On the Burau kernel and image

The problem on the kernel of the Burau representation has been long-standing [**4**, **17**]. Our proof of theorem 4.1 can be used to obtain a proof of the following [using the designation (2.20)]. The case $n=3$ should create no problem (although less interesting), but $n \leqslant 2$ will be continuously excluded. See also remark 9.6.

THEOREM 9.1. *Let $\beta \in B_n$ be so that $\psi_n(\beta)$ commutes with every matrix in $\psi_n(P_n)$, i.e.*

$$[\psi_n(\beta), \psi_n(\gamma)] = 0 \tag{9.1}$$

for all $\gamma \in P_n$. Then

$$\beta \in \Lambda_n = P_n^c \cdot \mathrm{center}\,(B_n). \tag{9.2}$$

Also $\psi_n(\beta)$ is a scalar matrix.

Proof. Let us first argue that $\beta \in P_n$. Assume $\pi(\beta)$ has a non-trivial cycle C. W.l.o.g. conjugate β so that $1 \in C$. Take some $\alpha \in B_{2,n+1}$ with $\pi(\beta\alpha)$ being a single cycle. Then consider

$$b_m = \delta_{[2,n]}^{2m} \beta \delta_{[2,n]}^{-2m} \alpha = \kappa_{1,n}^{-m} \beta \kappa_{1,n}^{m} \alpha.$$

They satisfy (3.4), with n replaced by $n+1$. On the other hand, since $\psi_n(\beta)$ commutes with $\psi_n(\kappa_{1,n})$, all $\psi_{n+1}(b_m)$ are equal, in particular by the remarks in §2.9, so are $\nabla(L_{b_m})$. Under (3.4) we proved theorem 3.4 in [**24**] (as explained below its formulation in §3) by using some coefficient of $\nabla(L_{b_m})$. (Since in our case, $\hat{b}_m$ is a knot, no subbraids of b_m were taken.) This gives a contradiction. So β is pure.

The assertion $\beta \in \Lambda_n$ now can be read as saying that all linking numbers in β are equal. This is most easily proved by assuming the opposite. Assume again w.l.o.g. by conjugacy (in B_n) that $lk_{1,j} \neq lk_{1,k}$ and choose the maximal such $j < k$. So by combed normal form,

$$\beta = \beta_0 \cdot \beta' \,, \quad \beta_0 = \prod_{i=1}^{k_1} \kappa_{1,j_i}^{\varepsilon_i},$$

with $\beta' \in P_{2,n} = B_{2,n} \cap P_n$, and the exponent sums

$$\eta_l = \sum_{\substack{i=1 \\ j_i = l}}^{k_1} \varepsilon_i,$$

satisfying $\eta_j \neq 0$. But by looking at the form (5.12), we see that $[A_l, A_n]$ are linearly independent matrices for $1 < l < n$, and

$$(\psi_n^{\times})''(\beta_0\kappa_{1,n})(1) - (\psi_n^{\times})''(\kappa_{1,n}\beta_0)(1) = \sum_{l=2}^{n-1} \eta_l [A_l, A_n] \neq 0. \tag{9.3}$$

(Since we assume equality of matrices, and not only conjugacy, we do no longer need to restrict ourselves to their traces.) Thus

$$\psi_n(\beta_0\kappa_{1,n}) \neq \psi_n(\kappa_{1,n}\beta_0),$$

and by noting that $\beta' \in P_{2,n}$ commutes with $\kappa_{1,n} \in P_n$,

$$\psi_n(\beta\kappa_{1,n}) = \psi_n(\beta_0\kappa_{1,n}\beta') \neq \psi_n(\kappa_{1,n}\beta_0\beta') = \psi_n(\kappa_{1,n}\beta).$$

So $\psi_n(\beta)$ does not commute with $\psi_n(\kappa_{1,n})$. This gives the contradiction to finish the proof of (9.2).

Since we proved in [**25**] that ψ_n is dense in a unitary group for proper t even on some subgroups of P_n, it also follows that ψ_n is irreducible on P_n, and that thus, for $\beta \in P_n$ as we already argued, $\psi_n(\beta)$ is a scalar matrix. □

REMARK 9.2. Also, if $\beta \in P_n$ (*not* B_n), one sees from (9.3) that one can replace ψ_n by $\psi_n/(t-1)^3 \in GL_{n-1}(\mathbb{Z}[t^{\pm 1}]/(t-1)^3)$ (where $t^{-1} = t^2 - 3t + 3$).

We say that an element g in a group G is a *root* of another element $h \in G$ if $g^d = h$ for some $d \neq 0$. A matrix is considered here to be *idempotent* if it is a root (in $G = GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$) of the identity matrix. (This is to be separated from the common conflicting terminology for a projector.)

COROLLARY 9.3. *Assume additionally $\beta \in P_n$. Then it is enough to demand that for some non-zero p, the power $\psi_n(\beta)^p$ commutes in (9.1),*

$$[\psi_n(\beta)^p, \psi_n(\gamma)] = 0 \tag{9.4}$$

(or then, equivalently, is scalar). That is, we have

$$\{\, \beta \in P_n \,:\, \psi_n(\beta) \textit{ is a root of a scalar matrix}\} \subset \Lambda_n. \tag{9.5}$$

Proof. If $\beta \in P_n$, then (9.5) follows from the remainder of the proof of theorem 9.1. One can generalize (9.3) to

$$(\psi_n^\times)''(\beta_0^p \kappa_{1,n})(1) - (\psi_n^\times)''(\kappa_{1,n}\beta_0^p)(1) = p \cdot \sum_{l=2}^{n-1} \eta_l [A_l, A_n] \neq 0.$$

[Note that thus we may even allow in (9.4), p to depend on γ, but this remains equivalent, as P_n is finitely generated.] □

COROLLARY 9.4. *Any $b \in P_n$ with idempotent $\psi_n(b)$ must lie in P_n^c. In particular, $\ker(\psi_n) \subset B_n$ is contained in P_n^c.*

Proof. If $\psi_n(b)$ is idempotent, then because of the determinant (2.10), we have $e(b) = 0$, and the part of b in the second factor of the decomposition on the right of (9.2) is trivial. This leaves $b \in P_n^c$.

For the second assertion, observe that $b \in \ker(\psi_n)$ readily implies $b \in P_n$ by setting $t = 1$ [and keeping (5.7) and (5.9) in mind]. □

It is known that P_n is residually nilpotent (and thus residually solvable). Since ψ_n is not faithful in general, one could ask how far down in the derived (or lower central) series the kernel lies. The above corollary shows that ψ_n will detect at least the first (coinciding) part of both filtrations.

The property (9.5) is obviously false for $\beta \in B_n \setminus P_n$, as central elements have non-pure roots, like δ_n or (5.13). However, something more interesting happens if we replace 'scalar' by 'identity'. We would like to know if all roots (in B_n) of $\ker(\psi_n)$ must lie in P_n^c, but it seems not obvious whether there are non-pure b whose $\psi_n(b)$ is idempotent.

https://doi.org/10.1017/prm.2023.1 Published online by Cambridge University Press

In fact, it is not even clear if there are non-identity idempotent Burau matrices, i.e. *does the Burau image have torsion* in $GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$? For example, $t^{-1} \cdot \psi_n(\sigma_1\sigma_2 \cdots \sigma_{n-1})$ has finite order. That this is not a Burau matrix is an application of corollary 1.3 and cannot be concluded by setting $t = 1$ or, for n odd, by the determinant.

At least we can say the following:

PROPOSITION 9.5. *Assume $b \in B_n$ so that $\psi_n(b)$ is idempotent. Then for each cycle C of $\pi(b)$, the subbraid exponent sum $e(b_{[C]}) = 0$. In particular, $\pi(b)$ has only odd(-length) cycles. Also*

$$lk(C, C') = 0 \tag{9.6}$$

for every other cycle $C' \neq C$.

Proof. We return to the explanation in remark 7.3, which now brings something.

Let C be a cycle of $\pi(b)$ with $\lambda = |C|$ and $e(b_{[C]}) \neq 0$. Then, because of (2.19), for $l = \lambda$, in b^l, we have

$$lk_{ij} \neq 0 \text{ for some } i, j \in C. \tag{9.7}$$

If (9.6) fails, then take $l = lcm\,(\lambda, \lambda')$ for $\lambda' = |C'|$, showing in b^l

$$lk_{ij} \neq 0 \text{ for some } i \in C, j \in C'. \tag{9.8}$$

Then there is some $p > 0$ so that $b^* = b^{pl}$ is pure, and (9.7) or (9.8) still holds in b^*. Now, if $\psi_n(b^*)$ were idempotent, then by corollary 9.4, $b^* \in P_n^c$, contradicting (9.7) or (9.8). Thus $\psi_n(b^*)$ is not idempotent, and neither is $\psi_n(b)$.

If $e(b_{[C]}) = 0$, then it is easy to see, essentially because even cycles require an odd number of transpositions, that λ must be odd. $\square$

If $\gcd(\lambda, \lambda') > 1$, then (9.8) may hold even with (9.6). An example is $b = \sigma_1\sigma_2^{-1}\sigma_4\sigma_5^{-1}\kappa'_{3,4}{\kappa'}_{3,5}^{-1}$ in B_6, with κ' from (2.17).

Returning to another thought, one can again replace in corollary 9.3, and its progeny, ψ_n by $\psi_n/(t-1)^3$. A similar remark applies to our last proved statement, with a short extra calculation.

Proof of corollary 1.3. Let $\psi = \psi_n(\beta)$ be a scalar matrix. From the determinant (2.10) one can restrict diagonal entries of ψ to the form $\pm t^l$. Then, setting $t = 1$, one also easily sees with (5.7) and (5.9) that a minus sign is ruled out (since Π is not a permutation matrix otherwise).

Next one sees from these same two formulas (or by using theorem 9.1) that a $\beta \in B_n$ with scalar $\psi_n(\beta)$ is pure. Now (9.2) implies that all linking numbers in β are equal, and thus $n(n-1) \mid e(\beta)$. Thus $\det \psi_n(\beta)$ is a power of $t^{n(n-1)}$, so that the diagonal entry t^l has l being a multiple of n. $\square$

REMARK 9.6. Note that now the conclusion (9.2) in theorem 9.1 improves to

$$\beta \in \ker(\psi_n) \cdot \text{center}\,(B_n). \tag{9.9}$$

But the theorem is needed to prove corollary 1.3 first, as well as corollary 9.4, to tell that (9.9) is indeed in improvement of (9.2).

Acknowledgements

This work was supported by the National Research Foundation of Korea (grant NRF-2017 R1E1A1 A0307 1032) and the International Research & Development Program of the Korean Ministry of Science and ICT (grant NRF-2016K1A3A7A03950702). The referee has made some helpful suggestions for improvements and minor corrections.

References

1 J. W. Alexander. A lemma on systems of knotted curves. *Proc. Natl. Acad. Sci. U.S.A.* **9** (1923), 93–95.

2 E. Artin. Theory of braids. *Ann. Math.* **48** (1947), 101–126.

3 D. Bar-Natan. *Vassiliev and quantum invariants of braids*. Proceedings of Symposia in Applied Mathematics, vol. 51 (Providence, RI: American Mathematical Society, 1996). https://www.math.toronto.edu/drorbn/papers/glN/glN.pdf.

4 S. Bigelow. The Burau representation is not faithful for $n = 5$. *Geom. Topol.* **3** (1999), 397–404.

5 J. S. Birman. *Braids, links and mapping class groups*, Ann. of Math. Studies, vol. 82 (Princeton: Princeton University Press, 1976).

6 J. S. Birman and W. W. Menasco. Studying links via closed braids VI. A nonfiniteness theorem. *Pac. J. Math.* **156** (1992), 265–285.

7 J. S. Birman and W. W. Menasco. Studying links via closed braids V. The unlink. *Trans. Am. Math. Soc.* **329** (1992), 585–606.

8 J. S. Birman and W. W. Menasco. Studying links via closed braids III. Classifying links which are closed 3-braids. *Pac. J. Math.* **161** (1993), 25–113.

9 J. S. Birman and W. W. Menasco. Stabilization in the braid groups I: MTWS. *Geom. Topol.* **10** (2006), 413–540.

10 J. Franks and R. F. Williams. Braids and the Jones–Conway polynomial. *Trans. Am. Math. Soc.* **303** (1987), 97–108.

11 F. Garside. The braid group and other groups. *Q. J. Math.* **20** (1969), 235–264.

12 J. González-Meneses. *Braiding: A C++ program for computations in braid groups*. https://personal.us.es/meneses/software.php.

13 J. González-Meneses and B. Wiest. On the structure of the centralizer of a braid. *Ann. Sci. Éc. Norm. Sup.* **37** (2004), 729–757.

14 T. Ito. *Non-degenerate exchange moves always produce infinitely many non-conjugate braids*, Nagoya Math. J. accepted.

15 V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. Math.* **126** (1987), 335–388.

16 B. Kolev. Entropie topologique et représentation de Burau. *C. R. Acad. Sci. Paris Ser. I Math.* **309** (1989), 835–838.

17 D. D. Long and M. Paton. The Burau representation is not faithful for $n \geqslant 6$. *Topology* **32** (1993), 439–447.

18 H. R. Morton. An irreducible 4-string braid with unknotted closure. *Math. Proc. Cambridge Philos. Soc.* **93** (1983), 259–261.

19 H. R. Morton. *Exchangeable braids, low-dimensional topology* (Chelwood Gate, 1982). London Math. Soc. Lecture Note Ser. vol. 95, pp. 86–105 (Cambridge: Cambridge University Press, 1985).

20 H. R. Morton. Seifert circles and knot polynomials. *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 107–109.

21 H. R. Morton. Threading knot diagrams. *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 247–260.

22 K. Murasugi. On the braid index of alternating links. *Trans. Am. Math. Soc.* **326** (1991), 237–260.

23 R. Shinjo and A. Stoimenow. Exchange moves and non-conjugate braid representatives of knots. *Nagoya Math. J.* **240** (2020), 298–321.

24 R. Shinjo and A. Stoimenow. Infinitely many non-conjugate braid representatives of links. Preprint http://www.stoimenov.net/stoimeno/homepage/papers.html.

25 A. Stoimenow. Non-conjugate braids with the same closure link from density of representations. *J. Math. Pures Appl.* **94** (2010), 470–496.

26 A. Stoimenow. *Properties of closed 3-braids and braid representations of links*. Springer Briefs in Mathematics (Switzerland: Springer, 2017). ISBN 978-3-319-68148-1.

27 A. Stoimenow. The Burau spectral test. To appear.

28 A. Stoimenow. Exchangeability and non-conjugacy of braid representatives. *Int. J. Comput. Geom. Appl.* **31** (2021), 39–73.

29 P. Vogel. Representation of links by braids: a new algorithm. *Comment. Math. Helv.* **65** (1990), 104–113.