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ON ISOMORPHISMS OF LOCALLY CONVEX SPACES WITH SIMILAR BIORTHOGONAL SYSTEMS

BY

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Introduction. The relationship between bases and isomorphisms (i.e. linear homeomorphisms) between complete metrizable linear spaces has been studied with great interest by Arsove and Edwards (see [1] and [2]). We prove (Theorem 1) that in the case of B-complete barrelled spaces, similar generalized bases imply existence of an isomorphism. This result was also proved by Dyer and Johnson [4], so we do not give a proof. We show (Theorem 6) that if one assumes that the bases are Schauder and similar, then Theorem 1 holds for countably barrelled spaces. We use Theorem 1 to advantage (Theorems 2–5) to show that one can improve some results due to Davis [3].

Let E be a real Hausdorff topological vector space (TVS), and (x_i) $i \in I$ a family of elements in E. Let E' denote the topological dual of E. A family (x_i, f_i) $i \in I$, $x_i \in E, f_i \in E'$ is said to be biorthogonal if $f_i(x_j) = \delta_{ij}$ (Kronecker delta). A *generalized basis* (GB) in E is a biorthogonal system (x_i, f_i) $i \in I$, $f_i \in E'$, such that $f_i(x) = 0$ for all i implies $x=0$. If a biorthogonal system is such that the closed linear span of (x_i) $i \in I$ is E, i.e. $[x_i]=E$, then the system is called a dual generalized basis (DGB). A biorthogonal system which is a DGB and a GB is called an M-basis (M for Markuschevitch). Let Φ denote the coefficient map associated with (x_i, f_i) $i \in I$, i.e. Φ is given by $\Phi(x) = (f_i(x))$, $i \in I$. Two systems (x_i, f_i) in E and (y_i, g_i) in F are *similar* if $\Phi(E) = \Psi(F)$, Ψ the coefficient map associated with (y_i, g_i) $i \in I$. The underlying indexing set will be I unless otherwise indicated.

Isomorphism theorems

THEOREM 1. *Let E, F be B-complete barrelled spaces with topologies u and v respectively. Let (x_i, f_i) be a* GB *in E. If T is an isomorphism of E onto F and $Tx_i = y_i$, $i \in I$, then (y_i, g_i) is a* GB *in F similar to (x_i, f_i). Here g_i is given by $g_i = f_i \circ T^{-1}$, $i \in I$. Conversely if (y_i, g_i) is a* GB *in F similar to (x_i, f_i), then there exists an isomorphism T of E onto F such that $Tx_i = y_i$ for all $i \in I$.*

Proof. Same as e.g. in [4].

REMARK. The above theorem holds for the class of (C)-barrelled spaces studied by McIntosh [8], since it has been shown that they are precisely the class of

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B-complete barrelled spaces. Jones and Retherford have observed [7] that this theorem does not hold in complete barrelled spaces.

As applications of Theorem 1 we have the following:

THEOREM 2. *Let E, F be B-complete barrelled spaces with (x_i, f_i) and (y_i, g_i) as M-bases in E, F respectively. Let $I' \subseteq I$ such that I–I' is finite. Consider $[x_i]=E_1$ and $[y_i]=F_1$, $i \in I'$ with the induced topologies. If T is an isomorphism of E_1 onto F_1, such that $Tx_i=y_i$ $i \in I'$, then T can be extended to an isomorphism E onto F such that $Tx_i=y_i$ for $i \in I$.*

Proof. If follows from the definition of M-bases that (x_i, f_i) and (y_i, g_i), $i \in I'$, are M-bases in E_1 and F_1 respectively. Since T is an isomorphism and $Tx_i=y_i$ for $i \in I'$, one gets similarity of the bases. Since the cardinality of I–I' is finite, $[x_\alpha]\alpha \in I$–I' is isomorphic to $[y_\alpha]\alpha \in I$–I' and x_α goes to y_α. By hypothesis E_1 is isomorphic to F_1 and x_α goes to y_α. Clearly $E=E_1 \oplus [x_\alpha]\alpha \in I$–$I'$ and $F=F_1 \oplus [y_\alpha]\alpha \in I$–$I'$. Hence (x_i, f_i) is similar to (y_i, g_i), $i \in I$. Now Theorem 1 gives the desired isomorphism.

THEOREM 3. *Let E, F be B-complete barrelled spaces and (x_i, f_i), (y_i, g_i), $i \in I$, similar biorthogonal systems for E, F respectively. Then there exists an isomorphism T of $E/\mathrm{Ker}\ \Phi$ onto $F/\mathrm{Ker}\ \Psi$ such that $T(\phi(x_i))=\phi'(y_i)$ for all $i \in I$, where ϕ, ϕ' are the quotient or canonical maps and Φ, Ψ the respective coefficient maps.*

Proof. It follows from ([4, Lemma 2]) that $(\phi(x_i), \tilde{f}_i)$ and $(\phi'(y_i), \tilde{g}_i)$, $i \in I$ are GB's in $E/\mathrm{Ker}\ \Phi$ and $F/\mathrm{Ker}\ \psi$ respectively where $\tilde{f}_i$, $\tilde{g}_i$ are defined by $\tilde{f}_i(\phi(x))=f_i(x)$ and $\tilde{g}_i(\phi'(y))=g_i(y)$. Similarity of these bases follows since (x_i, f_i) is similar to (y_i, g_i). Now we topologize $\Phi(E)$ and $\psi(F)$ with the topologies induced from $E/\mathrm{Ker}\ \Phi$ and $F/\mathrm{Ker}\ \Psi$ by $\tilde{\Phi}$, $\tilde{\Psi}$ respectively. Hence $\Phi=\tilde{\Phi} \circ \phi$, $\Psi=\tilde{\Psi} \circ \phi'$ are continuous, so $\mathrm{Ker}\ \Phi$ and $\mathrm{Ker}\ \psi$ are closed subspaces of E and F respectively. Since the quotient of B-complete barrelled subspaces with a closed subspace is also B-complete and barrelled, the desired isomorphism between $E/\mathrm{Ker}\ \Phi$ and $F/\mathrm{Ker}\ \psi$ follows by Theorem 1.

DEFINITION Biorthogonal systems (x_i, f_i) in E and (y_i, g_i) in F are **-similar* if $\hat{\Phi}(E'^{\beta})=\hat{\psi}(F'^{\beta})$, where $\hat{\Phi}$, $\hat{\psi}$ are the coefficient maps associated with (f_i, Jx_i) and (g_i, Jy_i) respectively, J being the appropriate canonical embedding of E into E'' or of F into F''.

REMARK. It is well known (see [3, Lemma 3]) that if (x_i, f_i) is a DGB for a TVS E, then (f_i, Jx_i) is a GB for E'^{β} (strong dual), where J is the canonical embedding of E into $E'^{\beta\prime\beta}$.

LEMMA 1. *Let E, F be* TVS's *where E'^{β}, F'^{β} are B-complete barrelled spaces. If (x_i, f_i) and (y_i, g_i) are *-similar* DGB's *in E and F respectively, then there exists an isomorphism S from E'^{β} onto F'^{β} such that $S(f_i)=g_i$, $i \in I$.*

Proof. Since (x_i, f_i) and (y_i, g_i) are *-similar *DGB*'s, it follows from the previous remark that (f_i, Jx_i) and (g_i, Jy_i) are GB's in E'^{β} and F'^{β} respectively. The definition of *-similarity immediately gives similarity of these bases. Since the strong duals are B-complete and barrelled, Theorem 1 gives the desired isomorphism.

COROLLARY. *Let E, F be Banach spaces. Let (x_i, f_i), (y_i, g_i) be *-similar DGB's in E, F. Then there exists an isomorphism S from E'^{β} onto F'^{β}.*

DEFINITION. A locally convex space E is quasi-barrelled if every strongly bounded subset of the dual E' is equicontinuous.

THEOREM 4. *Let E, F be quasi-barrelled spaces with *-similar DGB's (x_i, f_i) and (y_i, g_i). Assume E'^{β}, F'^{β} are B-complete barrelled spaces. Then there exists an isomorphism T of E onto F such that $Tx_i=y_i$, $i \in I$.*

Proof. Let J_E, J_F be the canonical embeddings of E and F into E'' and F'' respectively. That J_E and J_F are isomorphisms into follows by ([5, p. 229 Proposition 5]) since E and F are quasibarrelled. By Lemma 1 there exists an isomorphism $S: E'^{\beta} \to F'^{\beta}$ such that $S(f_i)=g_i$, $i \in I$. S is given by $\hat{\psi}^{-1} \circ \hat{\Phi}$, where $\hat{\Phi}$ and $\hat{\psi}$ are the coefficient maps associated with $(J_E x_i)$ $i \in I$ and $(J_F y_i)$ $i \in I$ respectively. Consider the conjugate map $(S^{-1})^*$. Then for $g \in F'$ and $i \in I$,

$$\begin{aligned}[(S^{-1})^*(J_E x_i) - J_F y_i](g) &= [(\hat{\Phi}^{-1} \circ \hat{\psi})^*(J_E x_i) - J_F y_i]\,(g) \\ &= (J_E x_i)(\hat{\Phi}^{-1} \circ \hat{\psi})(g) - (J_F y_i)(g) = 0\end{aligned}$$

because of *-similarity of (x_i, f_i) and (y_i, g_i). Thus $(S^{-1})^*(J_E x_i) = J_F y_i$, $i \in I$. Hence, $(S^{-1})^* \mid [J_E x_i] = [J_F y_i]$. Define T by $T = J_F^{-1} \circ (S^{-1})^* \circ J_E$. Clearly T is an isomorphism. Also $Tx_i = (J_F^{-1} \circ (S^{-1})^* \circ J_E)(x_i) = (J_F^{-1} \circ J_F) y_i = y_i$ for all $i \in I$. Hence T is the desired isomorphism.

REMARK. Theorem 4 was known previously only for Banach spaces. In locally convex spaces quasi-barrelledness insures that the canonical maps are isomorphism into.

The following is a particular case of Theorem 4.

COROLLARY. *Let E, F be reflexive spaces with *-similar* DGB's *(x_i, f_i) and (y_i, g_i) respectively. Assume E'^{β} and F'^{β} are B-complete spaces. Then there exists an isomorphism T of E onto F such that $Tx_i=y_i$, $i \in I$.*

THEOREM 5. *Let E, F be B-complete barrelled spaces where E'^{β} and F'^{β} are also B-complete and barrelled. If (x_i, f_i) is a* GB *in E *-similar and similar to a* DGB *(y_i, g_i) in F, then there exists an isomorphism T of E onto F such that $Tx_i = y_i$ for all $i \in I$ and both are M-bases.*

Proof. Theorem 3 and similarity of the systems gives an isomorphism S of $E/\text{Ker}\,\Phi$ and $F/\text{Ker}\,\psi$ such that $S(\Phi(x_i)) = \phi'(y_i)$, $i \in I$. It follows from ([3, Lemma 2(b)]) that $(\phi'(y_i))$ $i \in I$ is an M-basis for $F/\text{Ker}\,\psi$. Now $S^{-1}(\phi'(y_i)) = S^{-1}(S\phi(x_i)) = \phi(x_i)$ for all $i \in I$. Hence, $(\phi(x_i))$ $i \in I$ is an M-basis for $E/\text{Ker}\,\Phi$. But (x_i, f_i) $i \in I$ being a GB gives $\text{Ker}\,\Phi = \{0\}$, hence $E/\text{Ker}\,\Phi$ is isomorphic to E. Hence, (x_i, f_i) $i \in I$ is an M-basis for E. By *-similarity of (x_i, f_i) and (y_i, g_i) and Theorem 4, we get the isomorphism $T: E \to F$ such that $Tx_i = y_i$ for $i \in I$.

COROLLARY. *Theorem 5 holds true for the following pairs of locally convex spaces under the same hypotheses on the bases.*

(a) *E, F reflexive Fréchet (in particular reflexive Banach) spaces.*
(b) *E, F Montel Fréchet spaces.*

REMARK. Countably barrelled spaces have been introduced and studied by Husain [6], and he has established a Banach-Steinhaus theorem for these spaces. This result allows one to prove Theorem 1 for Schauder bases under weaker assumptions.

THEOREM 6. *Let E and F be countably barrelled spaces and (x_n, f_n), (y_n, g_n) $n \in N$ Schauder bases in E and F respectively. Then (x_n, f_n) is similar to (y_n, g_n) if and only if there exists an isomorphism T of E onto F such that $Tx_n = y_n$, $n \in N$.*

Proof. If such a T exists then similarity follows directly. For the converse, one defines a sequence of continuous linear maps as follows: For $x \in E$ $x = \sum_1^\infty f_n(x)x_n$. Define $T_m(x) = \sum_1^m f_n(x)y_n$, $m \in N$, and $T(x) = \sum_1^\infty f_n(x)y_n$. Clearly T is one-one and onto and T_m converges pointwise to T. By applying the Banach-Steinhaus theorem for countably barrelled spaces ([6, Theorem 4, Corollary 7]), one gets T continuous. Similarly T^{-1} is continuous, and T is the desired isomorphism.

REMARK. Theorem 4 of [6] can also be used to show that every weak Schauder basis in a countably barrelled TVS is a Schauder basis in the initial topology, which extends a result known for barrelled spaces (see [2, Theorem 11]).

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