Can. J. Math., Vol XLI, No. 4, 1989, pp. 676–701

ON THE SEMI-TENSOR PRODUCT OF THE DYER-LASHOF AND STEENROD ALGEBRAS

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0. Introduction. This paper arises out of joint work with F. R. Cohen and F. P. Peterson [**5**, **2**, **3**] on the joint structure of infinite loop spaces QX. The homology of such a space is operated on by both the Dyer-Lashof algebra, R, and the opposite of the Steenrod algebra A_*. We describe a convenient summary of these actions; let M be the algebra which is $R \otimes A_*$ as a vector space and where multiplication $Q^I \otimes P_*^J \cdot Q^{I'} \otimes P_*^{J'}$ is given by applying the Nishida relations in the middle and then the appropriate Adem relations on the ends. Then M is a Hopf algebra which acts on the homology of infinite loop spaces.

The paper is organized as follows. In Section one, for the convenience of the reader, we recall without proofs some of the results and notations of [**5**]. In Section 2, we define M and prove that it is a Hopf algebra over A_*. In addition we define the canonical subalgebras $M[k] = R[k] \otimes A_*$ and show that their duals $M[k]^*$ are completed polynomial algebras. Section 3 computes the action of the Steenrod algebra on $M[k]^*$. Finally, in Section 4, we apply our results to show that $Q\mathbf{R}P^{2^s}$ is a mod 2 H-atomic.

1. Notations, conventions and useful facts. In order to make this paper as self-contained as possible, we reproduce here many of the results (without proofs) of Section 1 from [**5**], which in turn relied heavily on J. P. May's article in [**4**]. We follow May's convention and write the paper as it would be for odd primes and put the minor modifications necessary when $p = 2$ in square brackets [ ]. When convenient P_*^r will be used to denote Sq_*^r when the prime is 2.

We will often use the lower notation for elements in the Dyer-Lashof algebra, R. That is, if $p > 2$,

$$Q_{s(p-1)}y = Q^{(s+|y|)/2}y$$

provided $s + |y|$ is even ($|y|$ means "degree of y") and if $p = 2$,

$$Q_s y = Q^{s+|y|}y \quad \text{for every } s.$$

This differs by a unit in $\mathbf{F}_p$ from the notation in [**4**]. Q_0 is the p-th power operation. We write simply I for the sequences $(i_1(p-1), \ldots, i_k(p-1)), (\epsilon_1, \ldots, \epsilon_k)$ where $\epsilon_l = 0$ or 1: then Q_I denotes the composition of operations

$$\beta^{\epsilon_1} Q_{i_1(p-1)} \beta^{\epsilon_2} Q_{i_2(p-1)} \cdots \beta^{\epsilon_k} Q_{i_k(p-1)}.$$

Received September 2, 1988. The authors gratefully acknowledge the support of NSERC.

https://doi.org/10.4153/CJM-1989-031-x Published online by Cambridge University Press

[If $p = 2$, all the ϵ_l are zero.] In this notation Q_I is admissible if $0 \leqq i_{l-1} \leqq i_l - \epsilon_l$, $2 \leqq l \leqq k$, but note that then the notation is somewhat redundant since all the ϵ_l except the first are determined by the i_l's. Namely $\epsilon_l \equiv i_l - i_{l-1} (\bmod 2)$. We say that the *length* of I is k. We have

$$|Q_I| = (p-1)\sum_{l=1}^{k} p^{l-1} i_l - \sum_{l=1}^{k} p^{l-1}\epsilon_l \qquad \left[|Q_I| = \sum_{l=1}^{k} s^{l-1} i_l \right]$$

and the excess of Q_I, denoted $e(Q_I)$, is $i_1 - \epsilon_1$.

In lower notation, the *Adem* relations have the following form: (first kind) if $i > j$,

$$\begin{aligned} &Q_{i(p-1)}Q_{j(p-1)} \\ &\quad = \sum_l (-1)^{\alpha(l)} \binom{(2l-j)\left(\frac{p-1}{2}\right) - 1}{i/2 - l - 1} Q_{(i+pj-2pl)(p-1)}Q_{2l(p-1)}, \end{aligned}$$

where

$$\alpha(l) = \left(\frac{i + j(p-1)}{2}\right) + l,$$

$$\left[\text{if } i > j, \quad Q_iQ_j = \sum_l \binom{l-j-1}{i-l-1} Q_{i+2j-2l}Q_l \right],$$

and (second kind), if $p > 2$ and $i > j$,

$$\begin{aligned} &Q_{i(p-1)}\beta Q_{j(p-1)} \\ &\quad = \sum_l (-1)^{\gamma(l)} \binom{(2l-j)\left(\frac{p-1}{2}\right)}{(i-1)/2 - l} \beta Q_{(i-1+pj-2pl)(p-1)}Q_{2l(p-1)} \\ &\quad + \sum_l (-1)^{\gamma(l)} \binom{(2l-j)\left(\frac{p-1}{2}\right) - 1}{(i-1)/2 - l} Q_{(i+pj-2pl)(p-1)}\beta Q_{2l(p-1)}, \end{aligned}$$

where

$$\gamma(l) = \frac{(i - 1 + j(p-1))}{2} + l.$$

We note that the Dyer-Lashof algebra, R, admits the structure of a coalgebra under the coproduct given on generators by

$$Q_i \to \sum Q_{i-j} \otimes Q_j.$$

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Furthermore, the set of elements determined by the sequences of length k, $R[k]$, is a sub-coalgebra.

For a space X with $H_*(X;\mathbf{F}_p)$ of finite type, the hom-duals of

$$P^r : H^q(X;\mathbf{F}_p) \longrightarrow H^{q+2r(p-1)}(X;\mathbf{F}_p)$$

$$[Sq^r : H^q(X;\mathbf{F}_2) \longrightarrow H^{q+r}(X;\mathbf{F}_2)]$$

are denoted

$$P^r_* : H_q(X;\mathbf{F}_p) \longrightarrow H_{q-2r(p-1)}(X;\mathbf{F}_p)$$

$$[Sq^r_* : H_q(X;\mathbf{F}_2) \longrightarrow H_{q-r}(X;\mathbf{F}_2)]$$

They turn $H_*(X;\mathbf{F}_p)$ into an A_*-module, where A_* denotes the opposite of the Steenrod algebra.

The Nishida relations in lower notation are (first kind)

$$P^r_* Q_{i(p-1)}x = \sum_j (-1)^{r+j} \binom{(i-2r+|x|)\left(\frac{p-1}{2}\right)}{r-pj} Q_{(i-2r+2pj)(p-1)} P^J_* x,$$

$$\left[Sq^r_* Q_i x = \sum_j \binom{i-r+|x|}{r-2j} Q_{i-r+2j} Sq^j_* x\right]$$

and if $p > 2$, (second kind)

$$\begin{aligned} &P^r_* \beta Q_{i(p-1)}x \\ &\quad = \sum_j (-1)^{r+j} \binom{(i-2r+|x|)\left(\frac{p-1}{2}\right)-1}{r-pj} \beta Q_{(i-2r+2pj)(p-1)} P^j_* \beta x \\ &\quad + \sum_j (-1)^{r+j} \binom{(i-2r+|x|)\left(\frac{p-1}{2}\right)-1}{r-pj-1} Q_{(i-2r+2pj)(p-1)} P^j_* \beta x \end{aligned}$$

The R and $R[k]$ are A_*-coalgebras. In order to describe their duals R^* and $R[k]^*$ let

$$a_{jk} = Q_{(\underbrace{0,\ldots,0}_{k-j},\underbrace{2(p-1),\ldots,2(p-1)}_{j})}{}^*$$

$$\left[a_{jk} = Q_{(\underbrace{0,\ldots,0}_{k-j},\underbrace{1,\ldots,1}_{j})}{}^*\right], \quad j \leqq k \text{ and if } p > 2$$

$$\tau_{jk} = Q_{(\underbrace{p-1,\ldots,p-1}_{k-j},\underbrace{2(p-1),\ldots,2(p-1)}_{j}),\Delta_{k-j+1}}{}^*, \quad j \leqq k$$

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$$\sigma_{ijk} = Q_{(\underbrace{0,\dots,0}_{k-j},\underbrace{p-1,\dots,p-1}_{j-i},\underbrace{2(p-1),\dots,2(p-1)}_{i}),\Delta_{k-j+1}+\Delta_{k-i+1}}{}^{*},$$

$$i < j \leqq k,$$

in the dual basis to the basis of admissible sequences for $R[k]$.

Thus

$$|a_{jk}| = 2(p^k - p^{k-j}) \quad [|a_{jk}| = 2^k - 2^{k-j}],$$

$$|\tau_{jk}| = 2(p^k - p^{k-j}) - 1 \quad \text{and}$$

$$|\sigma_{ijk}| = 2(p^k - p^{k-i} - p^{k-j}).$$

By convention $\sigma_{ijk} = 0$ if $i = j$. When dealing with a fixed k we drop the last subscript.

THEOREM 1.1. *As an A-algebra $R[k]^*$ is isomorphic to the free commutative graded algebra on $\{a_j, \tau_j, \sigma_{ij}\}$ $[\{a_j\}]$ modulo the relations*

(i) $\quad \tau_i\tau_j = a_k\sigma_{ij}$

(ii) $\quad \sigma_{ij}\tau_s = \tau_i\tau_j\tau_s/a_k$

(iii) $\quad \sigma_{ij}\sigma_{st} = \tau_i\tau_j\tau_s\tau_t/a_k{}^2$

with

$$P^{p^t} a_j = \begin{cases} -\delta_j^{k-t-1} a_{j+1} & , t < k-1 \\ a_1 a_j & , t = k-1 \\ 0 & , t > k-1 \end{cases}$$

and if $p > 2$,

$$P^{p^t} \tau_j = \begin{cases} -\delta_j^{k-t-1} \tau_{j+1} & , t < k-1 \\ a_1\tau_j + a_j\tau_1 & , t = k-1 \\ 0 & , t > k-1 \end{cases}$$

$$P^{p^t} \sigma_{ij} = \begin{cases} -\delta_i^{k-t-1}\sigma_{i+1,j} - \delta_j^{k-t-1}\sigma_{i,j+1} & , t < k-1 \\ a_1\sigma_{ij} + a_i\sigma_{ij} - a_j\sigma_{1i} & , t = k-1 \\ 0 & , t > k-1 \end{cases}$$

$\beta\tau_k = a_k$, $\beta\sigma_{ik} = -\tau_i$, $\beta\tau_j = \beta\sigma_{ij} = 0 \quad (j < k)$, $\beta a_j = 0$.

Proof. See [**4**, §I.3] for $p \geqq 2$ or [**7**, §3].

As an algebra $R[k]^* \cong P[k] \otimes M[k]$, where $P[k] = \mathbf{F}_p[a_1, \dots, a_k]$ is the subalgebra of $R[k]^*$ generated by $\{a_j\}$ and $M[k]$ is the subalgebra generated by $\{\tau_j, \sigma_{ij}\}$. Observe that $P[k]$ is closed under the Steenrod algebra and so forms an A-subalgebra. Let $P = \oplus P[k]$.

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Let P^{Δ_s}, $s \geqq 1$ and T^s, $s \geqq 0$ denote the elements of A inductively defined by Milnor [**6**] as

$$P^{\Delta_1} = P^1, \quad P^{\Delta_{s+1}} = [P^{p^s}, P^{\Delta_s}] \quad \text{and}$$

$$T^0 = \beta, \quad T^{s+1} = [P^{p^s}, T^s].$$

Using the inductive definition above and the fact that P^{Δ_s} and T^s act as derivations it is easy to verify that the actions of P^{Δ_s} and T^s for $s \leqq k$ are given by the following.

THEOREM 1.2. *For $s \leqq k$, we have*

$$P^{\Delta_s} a_j = -\delta_j^{k-1} a_k, \quad \textit{if } s < k, \textit{ and } P^{\Delta_k} a_j = a_j a_k$$

and if $p > 2$,

$$T^s a_j = 0,$$

$$P^{\Delta_s} \tau_j = -\delta_j^{k-s} \tau_k, \quad \textit{if } s < k, \textit{ and } P^{\Delta_k} \tau_j = a_k \tau_j + a_j \tau_k,$$

$$T^s \tau_j = -\delta_j^{k-s} a_k, \quad \textit{if } s < k, \textit{ and } T^k \tau_j = a_j a_k,$$

$$P^{\Delta_s} \sigma_{ij} = -\delta_i^{k-s} \sigma_{jk} - \delta_j^{k-s} \sigma_{ik}, \quad \textit{if } s < k, \textit{ and}$$

$$P^{\Delta_k} \sigma_{ij} = a_j \sigma_{ik} + a_k \sigma_{ij} - a_i \sigma_{jk},$$

$$T^s \sigma_{ij} = \delta_j^{k-s} \tau_i - \delta_i^{k-s} \tau_j, \quad \textit{if } s < k, \textit{ and } T^k \sigma_{ij} = a_i \tau_j - a_j \tau_i.$$

In Theorem 3.1, we compute the action of the higher-order Milnor elements on $R[k]^*$.

COROLLARY 1.3. *Let $a = a_1^{r_1}, \dots, a_k^{r_k}$ be a monomial in $P[k] = \mathbf{F}_p[a_1, \dots, a_k]$ and let $r = \Sigma r_i$. Then*

(i) $$P^{\Delta_s} a = (-1)^{r_{k-s}} r_{k-s} a \left(\frac{a_k}{a_{k-s}} \right), \quad 1 \leqq s \leqq k-1,$$

(ii) $$P^{\Delta_k} a = r a a_k.$$

COROLLARY 1.4. *Suppose $a \in P[k]$. The following are equivalent:*

(1) *a is a p-th power:*

(2) $$a \in \bigcap_{j=1}^{k} \operatorname{Ker} P^{\Delta_j};$$

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(3) $$a \in \bigcap_{j=1}^{\infty} \operatorname{Ker} P^{\Delta_j}.$$

Proof. See [**5**, Corollary 1.4].

THEOREM 1.5.

$$P[k] = \bigcap_{j=0}^{k-1} \operatorname{Ker} T^s.$$

Proof. See [**5**, Theorem 1.5].

2. The Hopf algebra M and its dual M^*. In this section, we define M and prove that it is a Hopf algebra over A_*. In addition, we define the canonical sub-coalgebras $M[k] = R[k] \otimes A_*$ and show that their duals are completed polynomial algebras.

Let A_* denote the opposite of the mod p Steenrod algebra and let R denote the mod p Dyer-Lashof algebra. We grade A_* by declaring $|P_*^J| = -|J|$. Here

$$J = (j_i, \dots, j_l), (\epsilon_i, \dots, \epsilon_l) \quad \text{and}$$

$$P^J = \beta^{\epsilon_1} P_*^{j_1} \beta^{\epsilon_2} P_*^{j_2} \dots \beta^{\epsilon_l} P_*^{j_l}.$$

We denote the respective multiplication maps (Adem relations) by

$$\phi_{A_*} : A_* \otimes A_* \longrightarrow A_* \quad \text{and} \quad \phi_R : R \otimes R \longrightarrow R.$$

We denote by $\eta : A_* \otimes R \longrightarrow R \otimes A_*$ the map which is inductively defined by the Nishida relations

$$\eta(P_*^r \otimes Q^i) = \sum_j (-1)^{r+j} \binom{(i+p^s-r)(p-1)}{r-pj} Q^{i-r+j} \otimes P_*^j, \text{ large } s,$$

$$\left[\eta(Sq_*^r \otimes Q^i) = \sum_j \binom{i+2^s-r}{r-2j} Q^{i-r+j} \otimes Sq_*^j, \text{ large } s\right],$$

$$\eta(P_*^r \otimes \beta Q^i) = \sum_j (-1)^{r+j} \binom{(i+p^s-1)(p-1)-1}{r-pj} \beta Q^{i-r+j} \otimes P_*^j$$
$$+ \sum_j (-1)^{r+j} \binom{(i+p^s-r)(p-1)-1}{r-pj-1} Q^{i-r+j} \otimes P_*^j \beta, \text{ large } s.$$

We show that η is well-defined in the proof of Theorem 2.1.

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We define M to be the algebra which is $R \otimes A_*$ as a vector space but which has multiplication given by the composite.

$$M \otimes M = R \otimes A_* \otimes R \otimes A_* \xrightarrow{1_R \otimes \eta \otimes 1_{A_*}} R \otimes R \otimes A_* \otimes A_* \xrightarrow{\phi_R \otimes \phi_{A_*}} R \otimes A_*$$
$$= M.$$

Intuitively, one thinks of this multiplication of $Q^I \otimes P_*^J \cdot Q^K \otimes P_*^L$ as being given by Nishida relations in the middle and then Adem relations on either end, as appropriate. We grade M by interpreting $Q^I \otimes P_*^J$ as acting on an element of $H_0(QX; \mathbf{F}_P)$ i.e.,

$$|Q^I \otimes P_*^J| = |I| - |J|.$$

We set $M[k] = R[k] \otimes A_*$.

Now consider the map $\zeta_r : R \longrightarrow R$ defined by

$$\zeta_r(Q_I) = Q_J$$

where if $I = (i_1, \ldots, i_k)$, $(\epsilon_1, \ldots, \epsilon_k)$, then

$$J = I - 2(p-1)(p^r, \ldots, p^r), (\epsilon_1, \ldots, \epsilon_k),$$

$$[J = I - (2^r, \ldots, 2^r)].$$

By convention, $Q_1 = 0$ if $1 < 0$.

LEMMA 2.1. *The map ζ_r is well-defined.*

Proof. Suppose $i > j$ so that

$$Q_{i(p-1)} Q_{j(p-1)} = \sum_l (-1)^{\alpha(l)} \binom{(2l-j)\left(\frac{p-1}{2}\right) - 1}{i/2 - l - 1} Q_{(i+pj-2pl)(p-1)} Q_{2l(p-1)},$$

where

$$\alpha(l) = \frac{i + j(p-1)}{2} + l.$$

Then

$$\zeta_r\left(Q_{i(p-1)} Q_{j(p-1)}\right) = Q_{(i-2p^r)(p-1)} Q_{(j-2p^r)(p-1)}.$$

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But $i-2p^r > j-2p^r$ so

$$Q_{(i-2p^r)(p-1)}Q_{(j-2p^r)(p-1)} = \sum_m (-1)^{\alpha(m)} \binom{(2m-j+2p^r)\left(\frac{p-1}{2}\right)-1}{\left((i-2p^r)/2\right)-m-1} Q_{(i-2p^r+p(j-2p^r)-2pm)(p-1)}Q_{2m(p-1)},$$

where

$$\alpha(m) = \frac{(i-2p^r+(j-2p^r)(p-1))}{2} + m.$$

On the other hand, taking ζ_r of the right hand side of the equation above we get

$$\sum_l (-1)^{\alpha(l)} \binom{(2l-j)\left(\frac{p-1}{2}\right)-1}{\left(i/2\right)-l-1} Q_{(i-2p^r+pj-2pl)(p-1)}Q_{(2l-2p^r)(p-1)}.$$

If we make the change of summation index $l = m+p^r$, we discover

$$(i-2p^r+pj-2pl) = i-2p^r+p(j-2p^r)-2pm, \quad 2l-2p^r = 2m,$$

$$\binom{(2l-j)\left(\frac{p-1}{2}\right)-1}{\left(i/2\right)-l-1} = \binom{(2m-j+2p^r)\left(\frac{p-1}{2}\right)-1}{\left(i-2p^r)/2\right)-m-1} \quad \text{and}$$

$$\alpha(l) \equiv \alpha(m) (\operatorname{mod} 2),$$

as required.

A similar calculation shows that ζ_r preserves Adem relations of the second kind.

THEOREM 2.2. *M is a Hopf algebra over A_*.*

Proof. Let $\tilde{M} = M/K$ where K is the subspace of M with basis

$$\{Q^I \otimes P_*^J \mid e(Q^I) < e(P_*^J)\}.$$

K is the subspace of operations which are identically zero when evaluated on any $H_*(QX;\mathbf{F}_p)$. Let

$$P = \prod_n \prod_{x\in H_*(K(\mathbf{F}_p,n);\mathbf{F}_p)} H_*(QK(\mathbf{F}_p,n);\mathbf{F}_p)$$

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and define $\Phi : \tilde{M} \longrightarrow P$ by

$$(\Phi(Q^I \otimes P^J_*))_{n,x} = Q^I P^J_*(x).$$

Then Φ is an injection and $\operatorname{Im}\Phi$ is a subset of P which is closed under the action of A_*. So there is an induced A_*-module structure on $\tilde{M}$, given explicitly by the Nishida relations used inductively.

Let $M(r)$ be the subspace of M with basis

$$\{Q_I \otimes P^J_* | e(P^J_*) \leqq p^r\} \quad [\{Q_I \otimes Sq^J_* | e(Sq^J_*) \leqq 2^r\}]$$

and let $A_* = F/S$ (where F is the free associative algebra on the symbols $P^i, \beta P^j$ and S is the opposite of the Adem relations (including $\beta^2 = 0$)). Now set $K(r) = K \cap M(r)$ and note that $K(r) \subset \ker(\zeta_r)$. Set

$$I(r) = \operatorname{Im}(\zeta_r | M(r)).$$

We have an epimorphism

$$M(r)/K(r) \twoheadrightarrow M(r)/(\ker(\zeta_r) \cap M(r)) \subset I(r)$$

whose kernel is $\ker(\zeta_r) \cap M(r)/K(r)$. Consider the following diagram

$$0 \longrightarrow S \otimes (M(r)/K(r)) + F \otimes ((\ker(\zeta_r) \cap M(r))/K(r))$$

$$\begin{tikzcd}
{} \arrow[r] & F \otimes M(r)/K(r) \arrow[r] \arrow[d] & A_* \otimes I(r) \arrow[r] \arrow[ddd] & 0 \\
& A_* \otimes M(r)/K(r) \arrow[d] & & \\
& A_* \otimes M/\ker(\zeta_r) \arrow[d, "\text{action = Nishida relations}"'] & & \\
& M/\ker(\zeta_r) \arrow[r, phantom, "\cong"] & \operatorname{Image}(\zeta_r) \arrow[r] & M
\end{tikzcd}$$

The action map makes sense because the action map takes $A_* \otimes \ker(\zeta_r)$ to $\ker(\zeta_r)$ since Nishida relations make the first entry in Q_I smaller and if Adem relations are needed, these will make the first entry smaller still. See also [**4**, Theorem 2.3 (iv), p. 17]. The left vertical composite is trivial on

$$S \otimes M(r)/K(r) + F \otimes (\ker(\zeta_r) \cap M(r))/K(r)$$

and also induces

$$A_* \otimes I(r) \longrightarrow \operatorname{Im}(\zeta_r).$$

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These maps are compatible, so taking the direct limit, they induce a map $A_* \otimes M \longrightarrow M$ making M into a A_*-module. The action is explicitly given by the map η defined above.

It follows that M becomes a Hopf algebra under the multiplication defined above.

In order to describe $M[k]^*$, let

$$a_j = (Q_{(\underbrace{0,\dots,0}_{k-j},\underbrace{2(p-1),\dots,2(p-1)}_{j})} \otimes 1)^*$$

$$[a_{jk} = (Q_{(\underbrace{0,\dots,0}_{k-j},\underbrace{1,\dots,1}_{j})} \otimes 1)^*] \quad j \leqq k,$$

$$e_s = (Q_{0,\dots,0)} \otimes P_*^{\Delta_s})^*, \quad s > 0,$$

and if $p > 2$

$$\tau_j = (Q_{(\underbrace{p-1,\dots,p-1}_{k-j},\underbrace{2(p-1),\dots,(p-1)}_{j})\Delta_{k-j+1}} \otimes 1)^* \quad j \leqq k,$$

$$\sigma_{ij} = (Q_{(\underbrace{0,\dots,0}_{k-j},\underbrace{(p-1),\dots,(p-1)}_{j-i},\underbrace{2(p-1),\dots,2(p-1)}_{i})(\Delta_{k-j-1}+\Delta_{k-i+1})} \otimes 1)^*, \quad i < j \leqq k$$

$$d_{sk} = (Q_{(0,\dots,0)} \otimes T_*^s)^*, \quad s \geqq 0,$$

in the dual basis to the basis of $\{\text{admissible elements}\} \otimes \{\text{Milnor basis}\}$, $Q_I \otimes P_*^J$ for $M[k]$. Thus

$$|a_{jk}| = 2(p^k = p^{k-j}) \quad [|a_{jk}| = 2^k - 2^{k-j}],$$

$$|\tau_{jk}| = 2(p^k = p^{k-j}) - 1, \quad |\sigma_{ijk}| = 2(p^k - p^{k-1} - p^{k-j}),$$

$$|e_{sk}| = -2(p^s - 1), \quad |e_{sk}| = -(2^s - 1)] \quad \text{and}$$

$$|d_{sk}| = -(2p^s - 1).$$

By convention, $\sigma_{ijk} = 0$ if $i = j$. When dealing with a fixed k we drop the last subscript.

THEOREM 2.3. *$M[k]^* \cong \hat{\mathbf{F}}_p[a_j, \tau_j, \sigma_{ij}, e_s, d_s]$ $i \leqq j < k$, $s \geqq 0$, subject to the relations*

(i) $\tau_i\tau_j = a_k\sigma_{ij}$

(ii) $\sigma_{ij}\tau_l = \tau_i\tau_j\tau_l/a_k$

(iii) $\sigma_{ij}\sigma_{lm} = \tau_i\tau_j\tau_l\tau_m/a_k^2$

(iv) $d_s^2 = 0$

$[M[k]^* \cong \hat{\mathbf{F}}_2[a_j, e_s]]$ where $\hat{}$ means completion with respect to the augmentation ideal. In other words, within any degree infinite sums of terms of that degree are allowed.

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Proof. We clearly obtain a map from

$$P = \mathbf{F}_p[a_j, \tau_j, \sigma_{ij}, e_s, d_s],$$

modulo the relations above to $M[k]^*$. Each basis element of $M[k]$ pairs non-trivially with only finitely many monomials in P so it is clear now to extend the pairing to get a map $\hat{P} \longrightarrow M[k]^*$. It is easy to check that this map is an isomorphism.

3. The action of the Steenrod algebra on $M[k]^*$**.** In this section, we compute the action of the Steenrod algebra on $M[k]^*$. We are interested not only in the operations P^{p^s}, and β but also in the Milnor operations P^{Δ_s} and T^s.

We recall Theorems 1.1 and 1.2 which give the action of P^{p^t}, β, P^{Δ_s} and T^s for $s \leqq k$ on $R[k]^*$. Our first result computes the action of the higher order Milnor elements on $R[k]^*$.

THEOREM 3.1. a) $P^{\Delta_{k+s}} a_i = cQ_I^*$, *for* $c \neq 0$, $s > 0$ *where*

$$I = (p-1)(\underbrace{2, \dots, 2}_{k-i}, \underbrace{4, \dots, 4, 2(p^s + \dots + p + 2)}_{i})$$

$$[Sq^{\Delta_{k+s}} a_i = Q_I^*, \quad I = (\underbrace{1, \dots, 1}_{k-i}, \underbrace{2, \dots, 2, 2^{s+1}}_{i})];$$

Note that

$$Q_I^* = a_1^{p^s + \dots + p} a_i a_k + \text{ others} \quad [Q_I^* = a_1^{2^{s+1}-2} a_i a_k + \text{ others}].$$

b) $T^s a_i = 0$, $s \geqq 0$.
c) $P^{\Delta_{k+s}} \tau_i = cQ_I^* + dQ_J^*$, *for* $c, d \neq 0$, $s > 0$ *where*

$$I = (p-1)(\underbrace{2, \dots, 2}_{k-i}, \underbrace{4, \dots, 4, 2(p^s + \dots + p + 2)}_{i}), \Delta_1$$

$$J = (p-1)(\underbrace{3, \dots, 3}_{k-i}, \underbrace{4, \dots, 4, 2(p^s + \dots + p + 2)}_{i}), \Delta_{k-i+1}$$

Note that

$$Q_I^* = a_1^{p^s + \dots + p} a_i \tau_k + \text{ others} \quad Q_J^* = a_1^{p^s + \dots + p} a_k \tau_i + \text{ others}.$$

d) $T_s \tau_i = 0$ *for* $s > k$.
e) $P^{\Delta k+s} \sigma_{ij} = sQ_I^* + dQ_J^* + eQ_K^*$ *for* $c, d, e \neq 0$, $s > 0$ *where*

$$I = (p-1)(\underbrace{1, \dots, 1}_{k-j}, \underbrace{3, \dots, 3}_{j-i}, \underbrace{4, \dots, 4, 2(p^s + \dots + p + 2)}_{i}),$$

$$\Delta_1 + \Delta_{k-i+1},$$

$$J = (p-1)(\underbrace{2,\ldots,2}_{k-j},\underbrace{3,\ldots,3}_{j-i},\underbrace{4,\ldots,4,2(p^s+\cdots+p+2)}_{i}),$$
$$\Delta_{k-j+1} + \Delta_{k-i+1}$$

and

$$K = (p-1)(\underbrace{1,\ldots,1}_{k-j},\underbrace{2,\ldots,2}_{j-i},\underbrace{4,\ldots,4,2(p^s+\cdots+p+2)}_{i}),$$
$$\Delta_1 + \Delta_{k-j+1}.$$

Note that

$$Q_I^* = a_1^{p^s+\cdots+p} a_j \sigma_{ik} + \text{ others, } Q_J^* = a_1^{p^s+\cdots+p} a_k \sigma_{jk} + \text{ others,}$$

$$Q_K^* = a_1^{p^s+\cdots+p} a_i \sigma_{jk} + \text{ others.}$$

f) $T^s \sigma_{ij} = 0$ *for* $s > k$.

Proof. Part (a) for $p = 2$ is Theorem 3 of [**1**]. The proofs of (b), (d) and (f) follow immediately from Theorem 1.2. We will give the proof of (c); the proofs of (a) and (e) are similar and left to the reader.

Write the I and J of (c) as $I(s)$ and $J(s)$, for $s > 0$ and write

$$I(0) = (p-1)(\underbrace{2,\ldots,2}_{k-i},\underbrace{4,\ldots,4}_{i}), \Delta_1,$$

$$J(0) = (p-1)(\underbrace{3,\ldots,3}_{k-i},\underbrace{4,\ldots,4}_{i}), \Delta_{k-i+1};$$

then Theorem 1.2 says that

$$P^{\Delta_k}\tau_i = Q_{I(0)}^* + Q_{J(0)}^*.$$

Our proof proceeds via induction on $s \geqq 1$; the step required to start the induction is the same as the general step. We leave the minor changes required when $i = 1$ or k to the reader.

So suppose

$$P^{\Delta_{k+s-1}}\tau_i = cQ_{I(s-1)}^* + dQ_{J(s-1)}^*$$

for $c, d \neq 0$, $1 < i < k$ and write $k + s = r$. Now

$$P^{\Delta_r}\tau_i = P^{p^r} P^{\Delta_{r-1}}\tau_i + P^{\Delta_{r-1}} P^{p^r}\tau_i = cP^{p^r} Q_{I(s-1)}^* + dP^{p^r} Q_{J(s-1)}^*$$

so we show that

$$P^{p^r} Q_{J(s-1)}^* = Q_{J(s)}^*$$

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the proof that

$$P^{p^r} Q^*_{I(s-1)} = Q^*_{I(s)}$$

is similar and is left to the reader.

A simple duality argument shows that

$$P^{p^r} Q^*_{J(s-1)} = Q^*_{J(s)};$$

if and only if $Q_{J(s)}$ is the only admissible element of $R[k]$ mapped to $Q_{J(s-1)}$ under $P^{p^r}_*$. So let

$$J = (p-1)(j_1, \dots, j_k), (\epsilon_1, \dots, \epsilon_k)$$

be any admissible sequence (i.e., $0 \leqq j_{l-1} \leqq j_l - \epsilon_l$, $2 \leqq l \leqq k$ and $j_l - j_{l-1} \equiv \epsilon_l \bmod(2)$, $2 \leqq l \leqq k$), and write

$$J_l = (p-1)(j_{k-l+1}, \dots, j_k), (\epsilon_{k-l+1}, \dots, \epsilon_k).$$

Then write x_l for Q_{J_l} and note that $x_k = Q_J$. We assume

$$P^{p^r}_* Q_J = Q_{J(s-1)} + \text{ others},$$

and show that then $J = J(s)$.

Now $J(s-1)$ has only one Bockstein in the $k-i+1$ position from the left so that J can have only one Bockstein say in the l^{th} position from the left with $l \neq k-i+1$ (this follows immediately from the form of the Nishida relations).

Case (i): $l < k-i+1$. Then repeated applications of the Nishida relations yield

$$P^{p^r}_* Q_J = \sum (-1)^{p^r+m_k} c_1 \dots c_k Q_{(p-1)K} + \text{ others},$$

with

$$\begin{gathered} K = (j_1 - 2p^r + 2pm_1, \dots, j_{l-1} - 2m_{l-2} + 2pm_{l-1}, \\ j_l - 2m_{l-1} + 2pn_l + 1, \dots, j_{k-i} - 2n_{k-i-1} + 2pn_{k-i} + 1, \\ j_{k-i+1} - 2n_{k-i} + 2pm_{k-i+1}, \dots, j_{k-1} - 2m_{k-2} + 2pm_k, j_k - 2m_{k-1}) \quad \text{and} \end{gathered}$$

$$c_q = \binom{(j_q - 2m_{q-1} + |x_{k-q}|)\left(\frac{p-1}{2}\right)}{m_{q-1} - pm_q},$$

$$1 \leqq q \leqq l-1 \text{ and } k-1+1 \leqq q \leqq k$$

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(set $m_0 = p^r$, $m_{k-i} = n_{k-i}$, and note $m_k = 0$),

$$c_q = \binom{(j_q - 2n_{q-1} + |x_{k-q}|)\left(\frac{p-1}{2}\right) - 1}{n_{q-1} - pn_{q^{-1}}}, \quad 1 \leqq q \leqq k-1$$

(set $n_{l-1} = m_{l-1}$). The sum is over m_q, n_q with

$$pm_q \leqq m_{q-1} \quad \text{and} \quad pn_q + 1 \leqq n_{q-1}.$$

Since $m_0 = p^r$, it follows that $m_q \leqq p^{r-q}$, $1 \leqq q \leqq l-1$ and $k-i+1 \leqq q \leqq k$, and $n_q \leqq p^{r-q}$, $l \leqq q \leqq k-i$. We have used m's to indicate when a Nishida relation of the first kind has been used and n's to indicate that the second term of a Nishida relation of the second kind has been used.

We want $j_k - 2m_{k-1} = 2(p^s + \cdots + p + 2)$ so that

$$c_k = \binom{(j_k - 2m_{k-1})\left(\frac{p-1}{2}\right)}{m_{k-1}} = \binom{p^{s+1} + p - 2}{m_{k-1}}.$$

Consequently, in order to have $c_k \not\equiv 0 \operatorname{mod}(p)$ we must have $m_{k-1} = 0, 1, \ldots, p-2$, or p^{s+1} (here we use Lucas' lemma: see [**9**, Lemma 2.6]). Of course, $m_{k-1} \leqq p^{s+1}$ from above.

Subcase (a). $m_{k-1} = p^{s+1}$. This maximal choice of m_{k-1} forces $m_q = p^{r-q}$ and $n_q = p^{r-q}$. However, no such choice of n_q's can give nonzero c_q's; for example we require

$$p^{s+i+1} \geqq n_{k-i-1} \leqq pn_{k-i} + 1 = pp^{s+i} + 1,$$

so that $c_{k-i-1} = 0$.

Subcase (b). $m_{k-1} = 0, 1, \ldots, p-2$. Then

$$j_k = 2(p^s + \cdots + p + 2 + m_{k-1}).$$

In order for $c_1 \not\equiv 0 (\operatorname{mod} p)$ we must have

$$j_1 + |x_{k-1}| \geqq 2p^r.$$

But J is admissible so in particular

$$j_1 \leqq \cdots \leqq j_k = 2(p^s + \cdots + p + 2 + m_{k-1}), \quad \text{and}$$

$$|x_2| = \sum_{q=2}^{k} p^{q-1}(j_q).$$

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Then it is not hard to see that $2p^r > j_1 + |x_2|$ to that $c_1 \equiv 0 (\mathrm{mod}\, p)$.

Case (ii). $l = k - i + 1$. Then repeated applications of the Nishida relations yield

$$P_*^{p^r} Q_j \sum (-1)^{p^r + m_k} c_1 \ldots c_k Q_{(p-1)K} + \text{ others,}$$

with

$$K = (j_1 - 2p^r + 2pm_1, \ldots, j_{k-1} - 2m_{k-2} + 2pm_{k-1}, j_k - 2m_{k-1}) \quad \text{and}$$

$$c_q = \binom{(j_q - 2m_{q-1} + |x_{k-q}|)\left(\frac{p-1}{2}\right) - 1}{m_{q-1} - pm_q}$$

(from the first term of a Nishida relation of the second kind). As above, we obtain $m_q \leqq p^{r-q}$, $1 \leqq q \leqq k$. Also, we have

$$j_k - 2m_{k-1} = 2(p^s + \cdots + p + 2)$$

and so again, for $c_k \not\equiv 0 (\mathrm{mod}\, p)$, we have

$$m_{k-1} = 0, \ldots, p-2 \text{ or } p^{s+1}.$$

Subcase (a). $m_{k-1} = p^{s+1}$. This maximal choice of m_{k-1} forces $m_q = p^{r-q}$, $1 \leqq q \leqq k$. Then it is straightforward to see $J = J(s)$ and $c_q \not\equiv 0 (\mathrm{mod}\, p)$, $1 \leqq q \leqq k$.

Subcase (b). $m_{k-1} = 0, \ldots, p-2$. We see as in (i)(b) that $c_1 \equiv 0 (\mathrm{mod}\, p)$ for such choices of m_{k-1}.

Finally we compute the action of the Steenrod algebra on the e's and d's.

THEOREM 3.2.

(a) $\qquad P^{p^r} e_s = 0 \quad$ *if* $r < k$;

$$P^{p^{k+r}} e_s = (Q^I)^* e_{s-1}^p \quad \textit{for}$$
$$I = (p^{k+r} - p^{k+r-1}, \ldots, p^{r+2} - p^{r+1}, p^{r+1} - 1).$$

Note that $Q^I = Q_J$ *for*

$$J = 2(p-1)(1, \ldots, 1, p^r + \cdots + p + 1) \quad \textit{and}$$

$$Q_J^* = a_1^{p^{r+1} - p} a_k^{p-1} + \textit{ others.}$$

[$Sq^{2^r} e_s = 0$ *if* $r < k$, $Sq^{2^{k+r}} e_s = (Q^I)^* e_{s-1}^2$ *for* $I = (2^{k+r-1}, \ldots, 2^{r+2}, 2^{r+1} - 1)$. *Note that* $Q^I = Q_J$ *for*

$$J = (1, \ldots, 1, 2^{r+1} - 1) \quad \textit{and}$$

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$$Q_J^* = a_1^{2^{r+1}-1} a_k + \textit{ others.}]$$

(b) $\quad P^{\Delta_r} e_s = 0 \quad \textit{if } r < k;$

$$P^{\Delta_{k+r}} e_s = (Q^I)^* e_{s-r}^{p^r} \quad \textit{for } I = (p^{k+r-1}, \dots, p^r).$$

Note that $Q^I = Q_J$ *for*

$$J = 2(p-1)(p^r, \dots, p^r) \quad \textit{and} \quad Q_J^* = a_k^{p^r}.$$

$[Sq^{\Delta_r} e_s = 0$ *if* $r < k;$

$$Sq^{\Delta_{k+r}} e_s = (Q^I)^* e_{s-r}^{2^r} \quad \textit{for } I = (2^{k+r-1}, \dots, 2^r).$$

Note that $Q^I = Q_J$ *for* $J = (2^r, \dots, 2^r)$ *and* $Q_J^* = a_k^{2^r}.]$
If $p > 2$

(c) $\quad P^{p^r} d_s = 0 \quad \textit{if } r < k;$

$$P^{p^{k+r}} d_s = \begin{cases} 0 & , r < s-1 \\ (Q^I)^* d_{s-1} & , r \leqq s-1 \end{cases} \quad \textit{for}$$

$$I = (p^{k+r} - p^{k+r-1}, \dots, p^{r+1} - p^r, p^r - p^{s-1}).$$

Note that $Q^I = Q_J$ *for*

$$J = 2(p-1)(1, \dots, 1, p^{r-1} + \dots + p^{s-1}) \quad \textit{and}$$

$$Q_J^* = a_1^{p^{r+1} - p^{s-1} - p - 1} a_k^{p-1} + \textit{ others.}$$

(d) $\quad P^{\Delta_r} d_s = 0 \quad \textit{if } r < k;$

$$P^{\Delta_{k+r}} d_s = \delta_s^r (Q^I)^* d_o \quad \textit{for } I = (p^{r+k-1}, \dots, p^r).$$

Note that $Q^I = Q_J$ *for* $J = 2(p-1)(p^r, \dots, p^r)$ *and* $Q_J^* = a_k^{p^r}.$

(e) $\quad T^r e_s = T^r d_s = 0, \forall r, s.$

First we need a series of lemmas.

LEMMA 3.3. (*Summary of* **[6]**). (a) $P_*^I P_*^J = P_*^{\Delta_s} +$ *others if and only if*

$$I = \Delta_r, \quad J = p^r \Delta_{s-r}, \quad 0 < r < s.$$

In particular $P_*^j P_*^J = P_*^{\Delta_s} +$ *others if and only if* $j = 1$ *and* $J = p\Delta_{s-1}.$

(b) $\quad P_*^{p^{s-1}} T_*^s = T_*^{s-1} P^{p^s} + T_*^s.$

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(c) $\quad P^{\Delta_r}T^0_* = T^0 P^{\Delta_r}_* + T^r_*.$

Proof. Only (a) needs a proof. We use Milnor's description of the multiplication in terms of matrices. P^{Δ_s} appears in the Milnor expansion of $P^J P^I$ if and only if the rightmost $(s-r, r+1)$-matrix below appears, and this can only be obtained from the matrix on the left.

$$\begin{bmatrix} * & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ p^r & 0 & \cdots & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

LEMMA 3.4. (a) $P^{p^r}_* Q^a = Q^0 P^j_* +$ *others, if and only if*

$$a = p^r - j, \quad j = j_0 p^s + j_1 p^{s-1} + \cdots + j_m p^{s-m},$$
$$0 < j_0 < \cdots < j_m < p,$$
$$pj \leqq p^r, \quad s \leqq r-1.$$

(b) $P^{\Delta_{r+1}}_* Q^a = cQ^{a-p^r} P^{\Delta_r}_* +$ *others* $(r \geqq 1, c \neq 0)$. *None of the terms labeled "others" has the form $Q^j P^{\Delta_s}_*$ for any j, s.*

In particular,

$$P^{\Delta_{r+1}}_* Q^{p^r} = Q^0 P^{\Delta_r}_* + \textit{others} \quad \textit{and}$$
$$P^{p^r}_* Q^{p^{r-1}} = Q^0 P^1_*.$$

(c) *If j is as in* (a) *then $P^j_* Q^a = Q^0 P^l_*$ if and only if*

$$l = l_0 p^t + l_1 p^{t-1} + \cdots + l_m p^{t-m},$$
$$0 < l_0 < \cdots < l_m < p, \quad pl \leqq j, \quad l \leqq r-1.$$

Proof.

(a) $$P^{p^r}_* Q^a = \sum_j (-1)^{p^r+j} \binom{(a-p^r)(p-1)}{p^r - pj} Q^{a-p^r+j} P^j_*.$$

So $a - p^r + j = 0$ if and only if $a = p^r - j$ so the coefficient is

$$\binom{(-j)(p-1)}{p^r - pj} \equiv \binom{(p^{r+1}-j)(p-1)}{p^r - pj} \bmod p.$$

The proof that this is non-zero only when j is of the stated form is by brute force; see Appendix A, Lemma A.

(c) $$P^j_* Q^a = \sum_l (-1)^{j+l} \binom{(a-j)(p-1)}{j - pl} Q^{a-j+l} P^l_*.$$

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So $a-j+l=0$ if and only if $a=j-l$, so the coefficient is

$$\binom{(-l)(p-1)}{j-pl} \equiv \binom{(p^r-l)(p-1)}{j-pl} \bmod p$$

and again brute force is required. The proof is similar to the proof of Lemma A, consequently we omit it.

(b) Here we use induction on r. For $r=1$, we have

$$\begin{aligned}
P_*^{\Delta_2}Q^a &= P_*^1P_*^pQ^a + P_*^{p+1}Q^a = \binom{(a-p)(p-1)}{p} P_*^1 Q^{a-p} \\
&\quad + P_*^1 Q^{a-p+1}P^1 + \binom{(a-(p+1)(p-1)}{p+1} Q^{a-(p+1)} \\
&\quad + \binom{(a-(p+1))(p-1)}{1} Q^{a-p}P^1 \\
&= \left(\binom{(a-p)(p-1)}{1} + \binom{(a-p-1)(p-1)}{1}\right) Q^{a-p}P_*^1 \\
&\quad + \text{other term,}
\end{aligned}$$

and

$$(a-p)(p-1)+(a-p-1)(p-1) \equiv (1-2a) \bmod p \not\equiv 0 (\bmod p) \quad \text{and}$$

$$1-2a \equiv 1 (\bmod p) \quad \text{if } a=p.$$

Now

$$\begin{aligned}
P_*^{\Delta_{r+1}}Q^a &= P_*^{p^r}P_*^{\Delta_r}Q^a + P^{\Delta_r}P_*^{p^r}Q^a = P_*^{p^r}(Q^{a-p^{r-1}}P_*^{\Delta_{r-1}} + \text{ other terms}) \\
&\quad + \sum_j (-1)^{p^r+j} \binom{(a-p^r(p-1)}{p^r-pj} P_*^{\Delta_r} Q^{a-p^r+j} P_*^j \\
&= \sum_j (-1)^{p^r+j} \binom{(a-p^{r-1}-p^r)(p-1)}{p^r-pj} Q^{a-p^{r-1}-p^r+j} P^j P_*^{\Delta_{r-1}} \\
&\quad + \sum_j (-1)^{p^r+j} \binom{(a-p^r)(p-1)}{p^r-pj} Q^{a-p^{r-1}-p^r+j} P_*^{\Delta_{r-1}} P_*^j + \text{ others}
\end{aligned}$$

Check the $j=p^{r-1}$ terms to get

$$Q^{a-p^r}(P_*^{p^{r-1}}P_*^{\Delta_{r-1}} + P_*^{\Delta_{r-1}}P_*^{p^{r-1}}) + \text{ others}$$

$$= Q^{a-p^r}P_*^{\Delta_r} + \text{others.}$$

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We may assume by induction that $P_*^{\Delta_r}Q^a$ has no terms of the form $Q^jP^{\Delta_s}$ other than the given one; in $P_*^{\Delta_r}Q^a$ no such term can arise from

$$P_*^{\Delta_r}P_*^{p^r}Q^a = \sum_j c_j P_*^{\Delta_r}Q^{a-p^{r+j}}P_*^j$$

by Lemma 3.3. But Lemma 3.3 also says that no such term can arise from $P_*^{p^r}P_*^{\Delta_r}Q^a$ using the induction hypothesis.

LEMMA 3.5. (a) $P_*^{p^{r+k}}Q^I = Q\dots Q^0P_*^{p^s}$, *$I$ admissible,* $0 \leqq s \leqq r$, *if and only if*

$$I = (p^{r+k} - p^{r+k-1}, \dots, p^{r+1} - p^s).$$

(b) $P_*^{\Delta_{k+r}}Q^I = Q^0\dots Q^0P_*^{\Delta_r}$, *$I$ admissible, if and only if*

$$I = (p^{k+r-1}, \dots, p^r).$$

Proof. (a) The proof is an easy induction on the length k of I, the case $k = 1$ following from Lemma 3.4 above (the general case uses Appendix A). (b) follows from Lemma 3.4(b) above.

Proof of Theorem 3.2. We have

$$\langle P^Rx, Q^IP_*^S\rangle = \langle x, P_*^RQ^IP_*^S\rangle$$

so we must show

$$P_*^RQ^IP_*^S = x^* + \text{ others}.$$

We note that Lemma 3.4 implies the first statements in (a), (b), (c) and (d) that only operations of the form $P^{p^{k+r}}$ or $P^{\Delta_{k+r}}$ need be considered. We note also that (e) follows easily by induction on r.

(a) By Lemmas 3.5(a) and 3.3(a) we have

$$P_*^{p^{k+r}}Q^IP_*^S = Q^{(0,\dots,0)}P_*^{\Delta_s} + \text{ others } (k \text{ zeros})$$

if and only if

$$S = P^{\Delta_{s-1}} \text{ and } I = (p^{k+r-1}, \dots, p^{r+2} - p^{r+1}, p^{r+1} - 1).$$

(b) By Lemmas 3.5(b) and 3.3(b) we have

$$P_*^{\Delta_{k+r}}Q^IP_*^S = Q^{(0,\dots,0)}P_*^{\Delta_s} + \text{ others } (k \text{ zeros})$$

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if and only if

$$S = P^{r\Delta_{s-r}} \quad \text{and} \quad I = (p^{k+r-1}, \dots, p^r).$$

(c) Follows from Lemmas 3.5(b) and 3.3(a).
(d) Follows from Lemmas 3.5(b) and 3.3(b).

4. Applications. In this section we take $p = 2$ throughout so, for example, $H_*(X)$ means $H_*(X, \mathbf{F}_2)$. Our original interest in M arose from our study, with F. R. Cohen and F. P. Peterson, of atomic spaces see [**5**, **2**, **3**]. In this connection we recall that an unstable A-module N is said to be A-atomic if given an A-map $f : N \longrightarrow N$ which is an isomorphism in lowest non-trivial degree, then f is an isomorphism. A natural class of examples is provided by $H^*(\mathbf{R}P^{2^s})$ for $s \geqq 0$. Roughly speaking, all cells are joined to the bottom one via Steenrod operations and since f commutes with these operations, if f is an isomorphism on $H^1(\mathbf{R}P^{2^s})$ then f is forced to be an isomorphism in all dimensions. We will see that the spaces $Q\mathbf{R}P^{2^s}$ behave the same with respect to M as the spaces QS^s behave with respect to A.

Let x_l denote the non-zero element of $H_l(\mathbf{R}P^{2^s})$ considered as an element of $H_l(Q\mathbf{R}P^{2^s})$ via the injection induced by

$$\mathbf{R}P^{2^s} \longrightarrow Q\mathbf{R}P^{2^s}.$$

Then A_* acts by

$$Sq^r_*(x_s) = \binom{s-r}{r} x_{s-r}.$$

It is not hard to see that for every l, $1 \leqq l \leqq 2^s$, we can choose an element y_l of A with

$$y_l(x_{2^s}) = x_l.$$

In fact,

$$y_{2^s} = 1, \quad y_{2^s-1} = Sq^1_*, \dots, y_1 = Sq^1_* \dots Sq^{2^{s-1}}_* \text{ or } y_1 = Sq^{\Delta_{s-1}}_*.$$

Now, according to J. P. May's paper in [**4**, Theorem 4.2, p. 40] we have

$$H_*(Q\mathbf{R}P^{2^s}) = \mathbf{F}_2[Q_I x_l,\ 1 \leqq l \leqq 2^s, I \text{ admissible } i_1 > 0].$$

Let $\Sigma^{2^s} M$ be M regarded so that

$$(\Sigma 2^s M)_l = M_{l+2^s}.$$

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Our remark above implies that the (degree zero) A-module map

$$\phi : \Sigma^{2^s} M \longrightarrow H_*(Q\mathbf{R}P^{2^s})$$

given by

$$\phi(Q_I Sq_*^J) = Q_I Sq_*^J(x_{2^s})$$

is onto the indecomposables; in fact

$$\phi(Q_I y_I) = Q_I x_I.$$

We denote by ϕ_k the map

$$\phi| : \Sigma^{2^s} M[k] \longrightarrow H_*(Q\mathbf{R}P^{2^s});$$

for example $\phi_0(y_I) = x_I$.

If A_n denotes the subalgebra of A_* generated by $Sq_*^{2^{n-1}}, \ldots, Sq_*^2, Sq_*^1$, then we note that $y_I \in A_s$. Thus if I has length k, we have that

$$Q_I y_I \in \Sigma^{2^s} M[k, s],$$

where $\Sigma^{2^s} M[k,s]$ denotes $\Sigma^{2^s} R[k] \otimes A_s$. So

$$(Q_I y_I)^* = Q_I^* y_I^* \in \Sigma^{2^s} M[k,s]^* \quad \text{with}$$

$$M[k,s]^* = M[k]^*/(e_{s+1}, e_{s+2}, \ldots) = \hat{\mathbf{F}}_2[a_1, \ldots, a_k, e_1, \ldots, e_s].$$

The map

$$\phi^* : H^*(Q\mathbf{R}P^{2^s}) \longrightarrow \underset{k \geqq 0}{\pi} \Sigma^{2^s} M[k,s]^*$$

is an injection of A_s-modules. The element of lowest degree in $\operatorname{im}(\phi_k^*) \subset M[k,s]^*$ is $a_{kk} e_s$ (the dual element is $Q_{(1,\ldots,1)} Sq_*^{\Delta_s}$) and

$$|a_{kk} e_s| = 2^s + [(2^k - 1) - (2^s - 1)] = 2^k - 1.$$

In fact, $a_{kk} e_s$ also has polynomial degree, $d(a_{kk} e_s) = 2$. If $x = (x_k)_{k \geqq 0} \in \operatorname{im}(\phi^*)$ is homogeneous (i.e., $|x_k| = |x_l|$ for all k, l) then it follows that x can have only finitely many non-zero components.

LEMMA 4.1. *$y \in M[k,s]^*$ is a square if and only if*

$$y \in \bigcap_{l=1}^{k+s} \underline{\ker}(Sq^{\Delta_l}),$$

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where $\underline{\ker}(Sq^{\Delta_l})$ *denotes the kernel of* Sq^{Δ_l} *restricted to* $M[k, s]$.

Proof. The Milnor elements act as derivations so, recalling 1.4 and 3.2 (b), we have that

$$a^R e^I \in \bigcap_{l=1}^{k} \underline{\ker}(Sq^{\Delta_l})$$

if and only if $R = 2S$ i.e., if and only if a^R is a square (so $a^R \in \ker(Sq^{\Delta_l})$ for all l, $1 \leqq l < \infty$). Now we have from 3.2 (b) that

$$Sq^{\Delta_{k+l}} e^I = \sum_{j=1}^{s} i_j a_k^{2^l} \frac{(e_{j-l})^{2^l}}{e_j} e^I$$

where

$$e^I = e_1^{i_1} \dots e_s^{i_s} \quad \text{and } e_{j-l} = 0 \quad \text{for } j - l < 0.\, e_0 = 1,$$

so

$$SQ^{\Delta_{k+l}} e^I = 0$$

if and only if $i_j \equiv 0(2)$ for $l \leqq j \leqq s$. Thus

$$a^R e^I \in \bigcap_{l=1}^{k} \underline{\ker}(Sq^{\Delta_l}) \bigcap \underline{\ker}(Sq^{\Delta_{k+s}})$$

if and only if a^R is a square and $i_s \equiv 0(2)$, and furthermore such elements are a vector space basis for

$$\bigcap_{l=1}^{k} \underline{\ker}(Sq^{\Delta_l}) \bigcap \underline{\ker}(Sq^{\Delta_{k+s}}).$$

It then follows that

$$a^R e^I \in \bigcap_{l=1}^{k} \underline{\ker}(Sq^{\Delta_l}) \bigcap \underline{\ker}(Sq^{\Delta_{k+s}}) \bigcap \underline{\ker}(Sq^{\Delta_{k+s-1}})$$

if and only if a^R is a square and $i_s \equiv i_{s-1} \equiv 0(2)$, and that such elements form a vector space basis for this set. Continuing, we obtain the result.

LEMMA 4.2. *Suppose* $x = (x_k) \in \mathrm{Im}(\phi)^*$, *then there is an* $\alpha \in A$ *such that* $\alpha(x)$ *is a non-zero square and*

$$d(\alpha(x)) \leqq d(x) + 1$$

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where $d(x)$ denotes polynomial degree.

Proof. The proof is word for word that of [**8**, Lemma 11, p. 366].

THEOREM 4.3. *$Q\mathbf{R}P^{2^s}$ is* mod 2 *H-atomic.*

Proof. Suppose

$$f : Q\mathbf{R}P^{2^s} \to Q\mathbf{R}P^{2^s}$$

is an H-map with $f_*(x_1) = x_1$. We must show that f is a mod 2 homotopy equivalence. Of course, because f is an H-map we need only show that f_* is an isomorphism on the indecomposables $QH_*(Q\mathbf{R}P^{2^s})$ (or dually, on primitives $PH^*(Q\mathbf{R}P^{2^s})$). However, f is only an H-map, not an infinite loop space map, so the Dyer-Lashof operations serve only a book-keeping function.

We have an epimorphism of A-modules

$$\Sigma^{2^s} M[*, s] \to H_*(Q\mathbf{R}P^{2^s}) \to QH_*(Q\mathbf{R}P^{2^s}),$$

where

$$M[*, s] = \bigoplus_{k \geqq 0} M[k, s],$$

with kernel $\Sigma^{2^s} Q_\epsilon M[*, s]$, where

$$Q_\epsilon M[k, s] = \langle Q_1 Sq_*^J | i_1 = 0 \text{ or } i_1 = 1 \rangle.$$

Now $Q_\epsilon M[*, s]$ is an A_s-subcoalgebra of $M[k, s]$. We have

$$M[*, s]/Q_\epsilon M[*, s] = \bigoplus_k M[k, s]/Q_\epsilon M[k, s]$$

so that dualizing we have an A-module isomorphism

$$PH^*(Q\mathbf{R}^{2^s}) = \pi_k (M[k, s]/Q_\epsilon M[k, s])^*.$$

It is not difficult to see that

$$(M[k, s]/Q_\epsilon M[k, s])^* \subset (a_{kk}) \subset M[k, s]^*,$$

see [**2** or **8**].

Now we have

$$f^* : PH^*(Q\mathbf{R}P^{2^s}) \to PH^*(Q\mathbf{R}P^{2^s})$$

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and we want to show it is an isomorphism; since $PH^*(Q\mathbf{R}P^{2^s})$ has finite type, it is enough to show f^* is a monomorphism.

Suppose $x \in \ker(f^*)$; then there is an α in A as in Lemma 4.2 such that

$$0 \neq \alpha(x) \in \ker(f^*), \quad d(\alpha x) \leqq d(x) + 1, \quad \text{and } \alpha(x) = y^2.$$

Although $\phi^* \circ f^* \circ \phi^{*-1}$ need not commute with multiplication in general, it does commute with squaring since squaring is a Steenrod operation. Thus

$$y \in \ker(f^*) \quad \text{and} \quad d(y) \leqq \frac{d(x)+1}{2}.$$

Continuing with this process, we eventually obtain a $z \in \ker(f^*)$ such that $d(z) = 1$. In other words

$$a_{kk} \in \ker(f^*) \quad \text{for some } k.$$

We now show that $a_{kk} \notin \ker(f_*)$ for any k, finishing the proof. Now a_{kk} is dual to

$$Q_1^k x_{2^s} = Q_1 \dots Q_1 x_{2^s}$$

in $H_*(Q\mathbf{R}P^{2^s})$ so we show $Q_1^k x_{2^s}$ is mapped to itself by f_*. If $k \geqq 2$, we have

$$Sq_*^1 Q_1^k x_{2^s} = Q_0 Q_1^{k-1} x_{2^s} = (Q_1^{k-1} x_{2^s})^2$$

and no other term in our basis maps to this under Sq_*^1. So we are finished by induction when we show $Q_1 x_{2^s}$ and x_{2^s} are mapped to themselves by f_*. Easy proofs of this are given in [**2**, §4].

Appendix A. This appendix is devoted to the proof of Lemma A.

LEMMA A.

$$c(r,j) = \binom{(p^{r+1} - j)(p-1)}{p^r - pj} \not\equiv 0 (\operatorname{mod} p)$$

if and only if

$$j = j_0 p^s + \dots + j_m p^{s-m} \quad \textit{for } 0 < j_0 < \dots < j_m < p \textit{ and } s \leqq r - 1.$$

Proof. Suppose that

$$j \equiv ip^t + j_0 p^s + \dots + j_m p^{s-m} (\operatorname{mod} p^{t+1})$$

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where we allow $0 \leqq i < p$ with $t \geqq s+2$, but $0 < j_l < p$ for $0 \leqq l \leqq m$. Note that every j is of this form for some choice of t, and m. We are trying to characterize those j for which

$$c(r, j) \not\equiv 0(\bmod p).$$

Consequently we require $p^r \geqq j$ so that $t \leqq r-1$ if $i > 0$ while $s \leqq r-1$ if $i = 0$.

In the case $i > 0$ we have that $(p^{r+1} - j)(p-1)$ is congruent, modulo p^{t+2}, to

$$(*) \qquad (p-i-1)p^{t+1} + (i-1)p^t + (p-1)(p^{t-1} + \cdots + p^{s+2}) + (p-j_0)p^{s+1} + (j_0 - j_1)p^s + \cdots + (j_{m-1} - j_m)p^{s-m+1} + j_m p^{s-m}.$$

This need not be the p-adic expansion of $(p^{r+1})(p-1)$. However, (*) also gives the case $i = 0$ if we set $t = r$.

On the other hand, $p^r - pj$ is congruent, modulo p^{t+2}, to

$$(**) \qquad (p-i)p^{t+1} + (p-1)(p^r + \cdots + p^{s+2}) + (p-j_0-1)p^{s+1} + (p-j_1-1)p^s + \cdots + (p-j_{m-1}-1)p^{s-m+2} + (p-j_m)p^{s-m+1},$$

which is the first few terms in the p-adic expansion of $p^r - pj$.

We note that (**) also gives the case $i = 0$ if we set $t = r-2$.

Now if $j_{m-1} \geqq j_m$ then the factor

$$\binom{j_{m-1} - j_m}{p - j_m}$$

occurs in $c(r, j)$ by Lucas' lemma. But this factor is zero because $j_{m-1} < p$. Consequently, to ensure that this factor is not congruent to zero modulo p we need $j_{m-1} < j_m$. In this case the last three terms of (*) are rewritten as

$$(j_{m-2} - j_{m-1} - 1)p^{s-m+2} + (p + j_{m-1} - j_m)p^{s-m+1} + j_m p^{s-m}$$

so that the last two terms occur in the p-adic expansion of $(p^{r+1} - j)(p-1)$. This guarantees that the factor associated with p^{s-m+1} is non zero modulo p. Similar arguments force in turn $j_{l-1} < j_l$, $1 \leqq l \leqq m$ so that none of the factors associated with $p^s, \ldots, p^{s-m}$ are zero modulo p. Similar arguments force in turn $j_{l-1} < j_l$, $1 \leqq l \leqq m$ so that none of the factors associated with $p^s, \ldots, p^{s-m}$ are zero modulo p. Thus, in the case $i = 0$, we have shown that

$$c(r, j) \equiv 0(\bmod p)$$

unless j is of the stated form in which case

$$c(r, j) \not\equiv 0(\bmod p).$$

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It remains to show that

$$c(r, j) \equiv 0(\operatorname{mod} p) \quad \text{if } i > 0.$$

The argument given above guarantees that

$$c(r, j) \equiv 0(\operatorname{mod} p)$$

unless we have $0 < j_0 < \cdots < j_m < p$. In this case, the p-adic expansion of $(p^{r+1} - j)(p - 1)$ is

$$(p - i - 1)^{t+1} + (i - 1)p^t + (p - 1)(p^{t-1} + \cdots + p^{s+2}) + (p - j_0 - 1)p^{s+1}$$

$$+(p + j_0 - j_1)p^s + \cdots + (p + j_{m-1} - jm)p^{s-m+1} + j_m p^{s-m}.$$

Consequently, the factor associated to p^t is $\binom{i-1}{p-1}$ which is zero since $i < p$.

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