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HARMONICITY OF A FOLIATION AND OF AN ASSOCIATED MAP

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A foliation on a Riemannian manifold (M, g) is harmonic if all the leaves are minimal submanifolds. We give a new characterisation of the harmonicity of a foliation on (M, g) by the harmonicity of an associated bundle map of (TM, g^C), where g^C is the complete lift metric of g to the tangent bundle as introduced by Yano and Ishihara.

1. INTRODUCTION AND STATEMENT OF THE RESULT

A foliation $\mathcal{F}$ on a Riemannian manifold is *harmonic* if all the leaves of $\mathcal{F}$ are minimal submanifolds of (M, g). The reason for this terminology (introduced in [7]) is that the local submersions defining $\mathcal{F}$ in distinguished charts are harmonic maps [**2, 3, 4**] precisely when $\mathcal{F}$ is a harmonic foliation [**7**, Theorem 2.28 and Theorem 3.3, (i), (ii)]. A foliation $\mathcal{F}$ on a manifold M is (geometrically) *taut* if a metric g exists on M which turns $\mathcal{F}$ into a harmonic foliation. There is a simple topological (cohomological) criterion characterising the tautness of a foliation $\mathcal{F}$. For simplicity, we assume throughout this note that the tangent bundle L and the normal bundle $Q = TM/L$ of $\mathcal{F}$ are oriented (and hence also M is oriented). The dimension of the leaves is denoted p, $0 < p < n = \dim M$.

RUMMLER – SULLIVAN CRITERION FOR TAUTNESS. [**12, 13**]. Let g_L be a Riemannian (fiber) metric on L with volume form ω_L along the leaves. Then $\mathcal{F}$ is harmonic for a metric g on M restricting to g_L on L if and only if ω_L is the restriction of a p-form χ on M satisfying

$$d\chi(X_1, \cdots, X_{p+1}) = 0$$

if p of the vector fields $X_1, \cdots, X_{p+1}$ are tangent to $\mathcal{F}$.

For $p = n-1$ this condition simply states that χ is a closed form. For Riemannian foliations [**11, 14, 15**], this criterion takes a particularly simple form (see [**8, 9, 16**]).

Many examples of harmonic foliations were given in [**7**]. They include totally geodesic foliations, foliations of Kähler manifolds by complex submanifolds, codimension one foliations orthogonal to a divergence free unit vector field [**7**, Proposition 3.9],

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or equivalently, defined by the vanishing of a co-closed one-form of unit length, and Roussarie's foliation of $\Gamma \setminus SL(2,\mathbb{R})$ with $\Gamma \subset SL(2,\mathbb{R})$ discrete and cocompact [**7**, Proposition 3.34]. The last example is certainly not Riemannian since its Godbillon-Vey class is non-trivial.

In this note we give a new characterisation of the harmonicity of a foliation $\mathcal{F}$ on (M,g). We shall recall the definition of the Gray-O'Neill tensor T and the mean curvature form κ of $\mathcal{F}$ in Section 2. Define a (0,2)-tensor field by

$$\Phi(E,F) = \kappa(T_E F). \tag{1.1}$$

Let $\varphi : TM \to TM$ be the associated endomorphism field given by

$$\Phi(E,F) = g(\varphi(E),F). \tag{1.2}$$

Consider the complete lift metric g^C on TM of Yano and Ishihara [**17**]. This is the semi-Riemannian metric of signature (n,n) defined by

$$\begin{cases} g^C(X^H,Y^H) = & g^C(X^V,Y^V) = 0, \\ g^C(X^H,Y^V) = & g^C(X^V,Y^H) = g(X,Y)^V. \end{cases} \tag{1.3}$$

Here the horizontal and vertical lifts of tangent vector fields X,Y on M refer to the decomposition of the tangent space of TM at every point in horizontal vectors with respect to the Levi Civita connection ∇ associated to g and canonical vertical vectors. For vector fields X,Y on M the function $g(X,Y)^V$ on TM is the pull-back of $g(X,Y)$ under the projection $TM \to M$.

With these definitions out of the way, we can state our main result as follows.

THEOREM. *Let $\mathcal{F}$ be a foliation on the Riemannian manifold (M,g). Then $\mathcal{F}$ is harmonic if and only if the map $\varphi : (TM,g^C) \to (TM,g^C)$, viewed as a map of semi-Riemannian manifolds, is a harmonic map.*

In fact, we shall prove moreover that if this map is harmonic, it necessarily reduces to the 0-map on each fiber, that is, $\varphi = \pi$, the projection $TM \to M$.

2. Preliminaries

Let (M,g) be an n-dimensional (oriented) Riemannian manifold, and g^C the complete lift of the metric g to TM as defined by (1.3). This metric can also be considered as the horizontal lift g^H of g when this is considered with respect to the Levi Civita connection ∇ associated to g [**17**]. Note that by definition (1.3) horizontal and vertical lifts of vector fields on M are null vectors for g^C.

Now, consider a $(1,1)$-tensor field on M as a map $\varphi : (TM,g^C) \to (TM,g^C)$. The following characterisation of its harmonicity was given in [**5**].

PROPOSITION. *φ is harmonic if and only if $\nabla^*\varphi = 0$, where ∇^* denotes the formal adjoint of ∇.*

$\nabla^*\varphi$ is the vector field defined as follows. In local coordinates with local components φ_i^k and where (g^{ij}) denotes the inverse matrix of (g_{ij}), we have

$$(\nabla^*\varphi)^k = -\sum_{i,j} g^{ij}\nabla_i\varphi_j^k. \tag{2.1}$$

Next, we recall the definition of the Gray–O'Neill tensor T of a foliation $\mathcal{F}$ on (M,g) [**6, 10**]. Let $\pi : TM \to L^\perp$ be the orthogonal projection to $L^\perp \cong Q$ with respect to g, and $\pi^\perp : TM \to L$ the orthogonal projection to L. Then

$$T_E F = \pi\left(\nabla_{\pi^\perp E}\pi^\perp F\right) + \pi^\perp(\nabla_{\pi^\perp E}\pi F) \tag{2.2}$$

for vector fields E, F on M. The conventions adapted below are $U, V, W \in \Gamma L$ and $X, Y, Z \in \Gamma L^\perp$. Clearly $T_E = T_{\pi^\perp E}$. Moreover, we have [**6**]

(2.3a) $T_X U = 0,$ $T_X Y = 0;$

(2.3b) $T_U V = \pi(\nabla_U V),$ $T_U X = \pi^\perp(\nabla_U X);$

(2.3c) $T_U V = T_V U;$

(2.3d) T_U is alternating, in particular $g(T_U V, X) = -g(V, T_U X)$.

The *mean curvature vector field* or *tension field* τ of $\mathcal{F}$ on (M,g) is given by [**14**, 6.16]

$$\tau = \sum_{i=1}^{p} T_{U_i}U_i = \sum_{i=1}^{p} \pi(\nabla_{U_i}U_i) \tag{2.4}$$

for a (local) orthonormal frame $U_1, \cdots, U_p$ of L. Note that we have suppressed the usual factor p^{-1}. The *mean curvature one-form* κ is defined by

$$\kappa(E) = g(\tau, E)$$

and satisfies $\kappa(U) = 0$ for $U \in \Gamma L$. The harmonicity of $\mathcal{F}$ is characterised by $\tau = 0$ or $\kappa = 0$.

3. PROOF OF THE THEOREM

We return now to the $(0,2)$-tensor field Φ on (M,g) defined by (1.1) and its associated endomorphism field φ. Instead of evaluating the vector field $\nabla^*\varphi$, we calculate the divergence of Φ. Note that by (2.3) Φ is a symmetric tensor and thus $\operatorname{div}\Phi$ a one-form given as follows (see [**1**, p.34]). Let $U_1, \cdots, U_p$ and $X_1, \cdots, X_q$ be (local) orthonormal frames of L and $L^\perp$. Then (up to a for our purpose irrelevant conventional sign)

$$(3.1)\qquad (\operatorname{div}\Phi)(E) = \sum_{i=1}^{p} (\nabla_{U_i}\Phi)(U_i, E) + \sum_{\alpha=1}^{q} (\nabla_{X_\alpha}\Phi)(X_\alpha, E).$$

But

$$\Phi(E, F) = \kappa(T_E F) = g(T_E F, \tau),$$

so that

$$(3.2)\qquad (\nabla_{U_i}\Phi)(U_i, E) = g\Big(\big(\nabla_{U_i}T\big)_{U_i}E, \tau\Big) + g\big(T_{U_i}E, \nabla_{U_i}\tau\big),$$

$$(3.3)\qquad (\nabla_{X_\alpha}\Phi)(X_\alpha, E) = g\big((\nabla_{X_\alpha}T)_{X_\alpha}E, \tau\big) + g\big(T_{X_\alpha}E, \nabla_{X_\alpha}\tau\big).$$

Next, we use the formulas of Gray [**6**] for T and the integrability tensor A (O in Gray's notation) given by

$$(3.4)\qquad A_E F = \pi^\perp(\nabla_{\pi E}\pi F) + \pi\big(\nabla_{\pi E}\pi^\perp F\big).$$

(O'Neill's formula apparatus for the tensors T and A is developed in [**10**] for the special context of Riemannian submersions only, while we need here Gray's more general context.) Then for $E = Y \in \Gamma L^\perp$ we have by (2.3) that $T_{X_j}Y = 0$, and by [**6**, (2.5)]

$$(\nabla_{X_\alpha}T)_{X_\alpha}Y = -T_{A_{X_\alpha}X_\alpha}Y = -T_{\pi^\perp(\nabla_{X_\alpha}X_\alpha)}Y$$

which by (2.3b) is a vertical vector field. It follows that

$$(3.5)\qquad g\big((\nabla_{X_\alpha}T)_{X_\alpha}Y, \tau\big) = 0.$$

Moreover, by [**6**, (2.4)]

$$g\Big(\big(\nabla_{U_i}T\big)_{U_i}Y, \tau\Big) = -g\Big(\big(\nabla_{U_i}T\big)_{U_i}\tau, Y\Big) = -g\Big(\pi\Big(\big(\nabla_{U_i}T\big)_{U_i}\tau\Big), Y\Big).$$

But by [**6**, (2.9)] we have

$$\pi\Big(\big(\nabla_{U_i}T\big)_{U_i}\tau\Big) = 0,$$

so that

$$(3.6)\qquad g\Big(\big(\nabla_{U_i}T\big)_{U_i}Y, \tau\Big) = 0.$$

It follows that for $Y \in \Gamma L^\perp$

$$(\operatorname{div}\Phi)(Y) = \sum_{i=1}^{p} g(T_{U_i}Y, \nabla_{U_i}\tau). \tag{3.7}$$

Note that $T_{U_i}Y$ is vertical by (2.3b), so that

$$(\operatorname{div}\Phi)(Y) = \sum_{i=1}^{p} g(T_{U_i}Y, \pi^\perp\nabla_{U_i}\tau) = \sum_{i=1}^{p} g(T_{U_i}Y, T_U\tau).$$

Applied to the horizontal mean curvature vector field τ itself, we find the expression

$$(\operatorname{div}\Phi)(\tau) = \sum_{i=1}^{p} g(T_{U_i}\tau, T_{U_i}\tau) = \sum_{i=1}^{p} |\pi^\perp(\nabla_{U_i}\tau)|^2. \tag{3.8}$$

From this, the result in the theorem is now clear. Obviously $\tau = 0$ implies the vanishing of $\operatorname{div}\Phi$ and hence $\nabla^*\varphi = 0$. Conversely, the vanishing of $\operatorname{div}\Phi$ implies $T_{U_i}\tau = 0$ for $i = 1, \cdots, p$. It follows that

$$g(T_{U_i}\tau, U_i) = -g(\tau, T_{U_i}U_i) = 0.$$

Then we see by means of (2.4) that $\operatorname{div}\Phi = 0$ implies $g(\tau,\tau) = 0$ and hence $\tau = 0$.

Note that this calculation also shows that in the case of vanishing τ the tensor Φ itself vanishes according to definition (1.1). Thus $\operatorname{div}\Phi = 0$ implies $\Phi = 0$, and then the associated endomorphism φ reduces to the canonical projection $TM \to M$.

Remark. For the case $p = 1$ the harmonicity of $\mathcal{F}$ means that all leaves are totally geodesic. For the case of codimension $q = 1$ it is of interest to compare (3.8) with the divergence formulas in [**14**, p.92]. Thus, let Z be the unit normal vector field to the (oriented and transversally oriented) foliation $\mathcal{F}$. Then by [**14**, (7.34) and (7.36)]

$$\operatorname{div} Z = -g(\tau, Z)$$

and the vanishing of this expression characterises the harmonicity of $\mathcal{F}$. The divergence formula (3.8), valid in arbitrary codimension, and whose vanishing again characterises harmonicity, in contrast involves covariant derivatives of τ along the leaves.

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