A PACKING PROBLEM FOR MEASURABLE SETS

D. SANKOFF AND D. A. DAWSON

1. Introduction. Given a probability measure space $(\Omega, \mathfrak{F}, P)$ consider the following *packing problem*. What is the maximum number, $b(K, \Lambda)$, of sets which may be chosen from $\mathfrak{F}$ so that each set has measure K and no two sets have intersection of measure larger than $\Lambda < K$?

In this paper the packing problem is solved for any non-atomic probability measure space. Rather than obtaining the solution explicitly, however, it is convenient to solve the following *minimal paving problem*. In a non-atomic σ-finite measure space $(\Omega, \mathfrak{F}, \mu)$ what is the measure, $V(b, K, \Lambda)$, of the smallest set which is the union of exactly b subsets of measure K such that no subsets have intersection of measure larger than Λ?

2. A lower bound for $V(b, K, \Lambda)$**.** A subset $\mathfrak{G}$ of $\mathfrak{F}$ is an *admissible family* if for all $A, B \in \mathfrak{G}$

$$\mu(A) = K,$$
$$\mu(A \cap B) \leqslant \Lambda.$$

If $\{A_1, A_2, \ldots, A_b\}$, $b > 1$, is an admissible family let $B_r \equiv \{x|x$ belongs to exactly r sets$\}$, $r = 1, 2, \ldots, b$, and let $a_r \equiv \mu(B_r)$. Then the a_r satisfy the following:

$$a_r \geqslant 0, \qquad r = 1, 2, \ldots, b, \tag{2.1}$$

$$\sum_{i=1}^{b} i a_i = bK, \tag{2.2}$$

and

$$\sum_{i=2}^{b} \binom{i}{2} a_i + a' = \binom{b}{2} \Lambda \tag{2.3}$$

where $a' \geqslant 0$ is a slack variable.

A *feasible intersection measure* (FIM) is a set of $b > 1$ numbers satisfying (2.1), (2.2), and (2.3). If a FIM arises from an admissible family as above, it is said to be *realized* by that family.

The minimal paving problem is to minimize the quantity

$$V \equiv P\left(\bigcup_{i=1}^{b} A_i\right) = \sum_{i=1}^{b} a_i \tag{2.4}$$

in which the minimization is to be over the class of all admissible families of b sets. Instead of doing this directly, we first minimize the linear expression

$$V = \sum_{i=1}^{b} a_i$$

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over all feasible intersection measures and then show that the minimal FIM is realized by an admissible family. The minimization of V is a linear programming problem whose solution is given by the following theorem.

THEOREM 2.1. *The* FIM

$$\begin{aligned} a_i &= 0, \qquad i \neq r-1, r, \\ a_{r-1} &= b\{K - \Lambda(b-1)/(r-1)\}, \\ a_r &= \{b(r-2)/r\}\{\Lambda(b-1)/(r-2) - K\}, \end{aligned} \tag{2.5}$$

is minimal where

$$r = [\Lambda(b-1)/K] + 2 \tag{2.6a}$$

if $\Lambda(b-1)/K$ *is not an integer, and*

$$r = \Lambda(b-1)/K + 1 \tag{2.6b}$$

if $\Lambda(b-1)/K$ *is an integer.*

The minimal value for V *is*

$$V = 2bK/r - \Lambda b(b-1)/r(r-1). \tag{2.7}$$

Finally $a' = 0$.

Proof. V is bounded below by zero. If we set $a_1 = bK$, $a_i = 0$ for $i \geqslant 2$, and $a' = \binom{b}{2}$ it is clear that equations (2.1) to (2.3) have a solution. Since there are two equations (2.2) and (2.3) relating the a_i's there is a solution to the minimization problem for which $b - 1$ of the variables $a_1, a_2, \ldots, a_b, a'$ are equal to zero **(1**, p. 222**)**.

According to **(1**, Theorem 9.1**)**, to determine the minimal solution the equations (2.2), (2.3), and (2.4) must be put into canonical form. That is, one must eliminate a_r from either (2.2) or (2.3) and a_{r-1} from the other, adjusting the coefficients of a_{r-1} and a_r, respectively, to be one, and eliminate a_r and a_{r-1} from (2.4).

In canonical form equations (2.2), (2.3), and (2.4) become

$$-2a'/(r-1) + a_1 + \sum_{i=2}^{b} a_i\{i - i(i-1)/(r-1)\} = b\{K - \Lambda(b-1)/(r-1)\},$$

$$2a'/r - a_1(r-2)/r + \sum_{i=2}^{b} \{i(i-1)/r - i(r-2)/r\}a_i = \{b(r-2)/r\}\{\Lambda(b-1)/(r-2) - K\},$$

$$a'(2/(r-1) - 2/r) + a_1(r-2)/r + \sum_{i=2}^{b} \{(r-i)(r-i-1)/r(r-1)\}a_i = V - 2bK/r + \Lambda b(b-1)/r(r-1).$$

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Now for $r \geqslant 2$,

$$2/(r-1) - 2/r \geqslant 0$$

and

$$(r-2)/r \geqslant 0,$$

$$(r-i)(r-i-1)/r(r-1) \geqslant 0, \qquad i = 1, 2, \ldots, b.$$

By the theorem quoted above, it follows that if

$$b\{K - \Lambda(b-1)/(r-1)\} \geqslant 0 \tag{2.8}$$

and

$$\{b(r-2)/r\}\{\Lambda(b-1)/(r-2) - K\} \geqslant 0, \tag{2.9}$$

then a minimal solution is

$$\begin{aligned} a_{r-1} &= b\{K - \Lambda(b-1)/(r-1)\}, \\ a_r &= \{b(r-2)/r\}\{\Lambda(b-1)/(r-2) - K\}, \\ a_i &= 0, \qquad i \neq r-1, r, \\ a' &= 0, \end{aligned}$$

and

$$V = 2bK/r - \Lambda b(b-1)/r(r-1).$$

Conditions (2.8) and (2.9) may be rewritten as

$$(r-2)K/(b-1) \leqslant \Lambda \leqslant (r-1)K/(b-1).$$

In other words

$$r = [\Lambda(b-1)/K] + 2 \qquad \text{if } \Lambda(b-1)/K \text{ is not an integer,}$$

or

$$r = \Lambda(b-1)/K + 1 \qquad \text{if } \Lambda(b-1)/K \text{ is an integer.}$$

Rewriting (2.7),

$$V = 2bK/\{[\Lambda(b-1)/K] + 2\} \\ - \Lambda b(b-1)/\{[\Lambda(b-1)K] + 2\}\{[\Lambda(b-1)/K] + 1\}$$

in case (2.6a), or

$$V = bK^2/\{\Lambda(b-1) + K\} = bK/r$$

in case (2.6b). It should also be noted that in case (2.6b), a_{r-1} is equal to zero.

3. Realization of the lower bound by an admissible family. In this section an admissible family in a non-atomic σ-finite measure space is constructed whose FIM agrees with that obtained in Theorem 2.1. This will prove that $V(b, K, \Lambda)$ is given by (2.7).

In case (2.6b) we wish to find b sets $\{A_1, A_2, \ldots, A_b\}$ such that $\mu(A_i) = K$, $i = 1, 2, \ldots, b$, $\mu(A_i \cap A_j) = \Lambda$, $i \neq j$, and such that every point in

$$\bigcup_{i=1}^{b} A_i$$

belongs to exactly r of the sets. Take any $\binom{b}{r}$ disjoint sets of measure $V\Big/\binom{b}{r}$, say

$$\left\{B_i: i = 1, 2, \ldots, \binom{b}{r}\right\}$$

and set up a one-to-one correspondence between these sets and the set of b-tuples consisting of r ones and $b - r$ zeros. Let A_i be the union of all the sets B_j whose corresponding b-tuple contains a one in the ith position. Then since $V = bK/r$,

$$\mu(A_i) = \binom{b-1}{r-1} V\Big/\binom{b}{r} = K, \qquad i = 1, 2, \ldots, b,$$

and since $\Lambda = (r-1)K/(b-1)$,

$$\mu(A_i \cap A_j) = \binom{b-2}{r-2} V\Big/\binom{b}{r} = \Lambda, \qquad i \neq j.$$

This admissible family is called a $(b, V, r, K, \Lambda)^*$-*configuration* since it arises from the (b, v, r, k, λ)-*configuration* where

$$v = \binom{b}{r}, \quad k = \binom{b-1}{r-1}, \quad \text{and } \lambda = \frac{r(k-1)}{v-1}.$$

(b, v, r, k, λ)-configurations are known as balanced incomplete block designs in statistics and are studied by Ryser in **(3)**.

In case (2.6a) we proceed as follows. Given

$$\theta = \Lambda(b-1)/K + 2 - r, \qquad 0 < \theta < 1,$$

set

$$V' = a_r, \qquad K' = K\theta, \qquad \Lambda' = K\theta(r-1)/(b-1),$$

and

$$V'' = a_{r-1}, \qquad K'' = K(1-\theta), \qquad \Lambda'' = K(1-\theta)(r-2)/(b-1).$$

Then since $\Lambda'(b-1)/K'$ and $\Lambda''(b-1)/K''$ are both integers, namely $r-1$ and $r-2$ respectively, we can construct a $(b, V', r, K', \Lambda')^*$-configuration $\{A_i': i = 1, 2, \ldots, b\}$ and a $(b, V'', r-1, K'', \Lambda'')^*$-configuration

$$\{A_i'': i = 1, 2, \ldots, b\},$$

as above. Then it is easy to verify that the family of sets $\{A_i: i = 1, 2, \ldots, b\}$, where $A_i = A_i' + A_i''$, $i = 1, 2, \ldots, b$, is an admissible family whose FIM is given by equations (2.5).

PROPOSITION 3.1. (i) *$V(b, k, \Lambda)$ is non-decreasing in b.*
(ii) $\lim_{b\to\infty} V(b, K, \Lambda) = K^2/\Lambda$.

Proof. The proof of (i) is immediate, since if we discard a set from an admissibly family of b sets, the remaining sets form an admissible family of $b-1$ sets.

The second result can be verified by rewriting (2.7) as

$$V(b, K, \Lambda) = \frac{bK^2\theta}{\Lambda(b-1) + 2K - \theta K} + \frac{bK^2(1-\theta)}{\Lambda(b-1) + K - \theta K} \tag{3.1}$$

where $\theta = \Lambda(b-1)/K - [\Lambda(b-1)/K]$, $0 \leqslant \theta < 1$, and then taking the limit as b goes to infinity.

PROPOSITION 3.2. *Given an admissible family $\{A_i: i = 1, 2, \ldots, b\}$ as above, let $U(b, K, \Lambda)$ be the measure of*

$$\bigcap_{i=1}^{b} A_i.$$

If the measure space is a non-atomic probability measure space, then

$$U(b, K, \Lambda) \leqslant 1 - V(b, 1-K, 1-2K+\Lambda).$$

Proof. This follows from

$$\bigcap_{i=1}^{b} A_i = \left(\bigcup_{i=1}^{b} A_i^c\right)^c$$

and

$$\mu(A_i^c) = 1 - K, \qquad \mu(A_i^c \cap A_j^c) \leqslant 1 - 2K + \Lambda, \qquad i \neq j.$$

4. The packing problem. The packing problem involves calculating $b(K, \Lambda) \equiv \max\{b : V(b, K, \Lambda) \leqslant 1\}$. Since $V(b, K, \Lambda)$ is monotone non-decreasing in b, $b(K, \Lambda)$, when finite, can be found by computing $V(b, K, \Lambda)$ from Equation (3.1) for a finite number of values of b. Because of the rather complicated behaviour of quadratic expressions involving the function $[\cdot]$ it does not seem possible to obtain a simple expression for $b(K, \Lambda)$. In the following proposition, however, some bounds are given for $b(K, \Lambda)$.

PROPOSITION 4.1. *If $\Lambda < K^2$, $V(b, K, \Lambda)$ and $b(K, \Lambda)$ satisfy the following inequalities:*

$$\frac{bK^2}{\Lambda(b-1) + K} \leqslant V(b, K, \Lambda) \leqslant \frac{bK^2}{\Lambda(b-1) + K/4}, \tag{4.1}$$

$$\left[\frac{K/4 - \Lambda}{K^2 - \Lambda}\right] \leqslant b(K, \Lambda) \leqslant \left[\frac{K - \Lambda}{K^2 - \Lambda}\right]. \tag{4.2}$$

Proof. If the expression on the right-hand side of (3.1) is considered as a function of θ, it may be verified that the minimum occurs for $\theta = 0$ or 1 and the maximum occurs for

$$\theta = \frac{K + a - a\sqrt{1 - K^2/a^2}}{2K}$$

where $a = \Lambda(b-1) + K$. But then

$$\theta < \frac{K + a - a(1 - K^2/2a^2)}{2K} < 3/4,$$

and hence, from the expression (3.1), it is easy to verify that (4.1) is satisfied. The inequalities in (4.2) then follow from the inequalities in (4.1).

In the case $\theta = 0$, the lower bound in (4.1) is attained. If in addition $(K - \Lambda)/(K^2 - \Lambda)$ is an integer, then $V(b, K, \Lambda) = 1$ and the packing completely covers Ω.

Example. For the case $K = \alpha/2$, $\Lambda = \alpha/4$, $1 < \alpha < 2$,

$$V(b, K, \Lambda) = \begin{cases} \alpha b/(b+1), & b \text{ even}, \\ \alpha(b+1)/(b+2), & b \text{ odd}. \end{cases}$$

Moreover $b(K, \Lambda) = [1/(\alpha - 1)]_e$ where $[x]_e$ designates the greatest even integer less than x.

PROPOSITION 4.2. *$b(K, \Lambda)$ is finite if and only if $\Lambda < K^2$.*

Proof. This follows immediately from Proposition 3.1.

Let $\mathfrak{F}(\alpha)$, $0 < \alpha < 1$, be the class of sets in $\mathfrak{F}$ whose measure is α. $\mathfrak{F}(\alpha)$ is said to be ϵ-*approximated* by a subfamily $\mathfrak{G}$ if for every $A \in \mathfrak{F}(\alpha)$ there is a set $B \in \mathfrak{G}$ such that $\mu(A \triangle B) < \epsilon$. The following is a direct consequence of Proposition 4.2.

COROLLARY. *$\mathfrak{F}(\alpha)$ is ϵ-approximated by a finite subfamily if and only if $\epsilon > 2(\alpha - \alpha^2)$.*

PROPOSITION 4.3. *If $K^2 = \Lambda$ and $\mu(\Omega) = 1$, then there exists a countable admissible family which covers Ω.*

Proof. A sequence of sets is constructed inductively as follows. A_1 is any set of measure K and A_2 any set of measure K such that $\mu(A_1 \cap A_2) = \Lambda$. For any n, let $\mathfrak{A}_n$ designate the set of atoms in the Boolean algebra generated by $\{A_1, A_2, \ldots, A_n\}$. Given $\mathfrak{A}_n$ we obtain $\mathfrak{A}_{n+1}$ by subdividing each atom B in $\mathfrak{A}_n$ into two sets whose measures are $K\mu(B)$ and $(1 - K)\mu(B)$ respectively. Then let

$$A_{n+1} = \bigcup\{B_i\colon B_i \in \mathfrak{A}_{n+1}, \mu(B_i) = K\mu(C), B_i \subset C, C \in \mathfrak{A}_n\}. \tag{4.3}$$

It is clear that $\mu(A_{n+1}) = K$. Furthermore if

$$a_n \equiv \mu\left(\bigcup_{i=1}^{n} A_i\right),$$

$a_{n+1} = a_n + K(1 - a_n)$ and therefore $\lim_{n\to\infty} a_n = 1$.

To show that

$$\mu(A_n \cap A_m) = \Lambda = K^2 \tag{4.4}$$

for $m > n$ we proceed by induction. If $m = n + 1$, (4.4) follows from (4.3). If (4.4) is true for m, then

$$\begin{aligned} \mu(A_n \cap A_{m+1}) &= K(\mu(A_n \cap A_m)) + K(\mu(A_m - A_n)) \\ &= K^3 + K(K - K^2) = K^2. \end{aligned}$$

Hence the induction argument is complete.

Note that if $\Lambda = K^2$ and the probability measure space is non-separable there may actually be uncountably many sets of measure K whose pairwise intersections have measure Λ.

5. Connections with probability theory. Given a non-atomic probability measure space, a collection of sets $\{A_i\}$ is said to be *pairwise α-dependent* if

$$\mu(A_i|A_j) = \alpha\mu(A_i), \qquad j \neq i, \alpha > 1,$$

pairwise α-antidependent if

$$\mu(A_i|A_j) = \alpha\mu(A_i), \qquad j \neq i, \alpha < 1,$$

and *pairwise independent* if

$$\mu(A_i|A_j) = \mu(A_i), \qquad j \neq i.$$

Then Proposition 4.2 can be rephrased as follows.

PROPOSITION 5.1. *Every collection of pairwise α-antidependent events of probability $K > 0$ is finite. There exist infinite classes of pairwise α-dependent or pairwise independent events of probability $K > 0$.*

Proposition 3.1 (ii) is closely related to the strong form of the Borel–Cantelli lemma (**4**, p. 317) which states that if

$$\sum_{n=1}^{\infty} P(E_n) = \infty$$

and if for some $\alpha \geqslant 1$

$$P(E_n \cap E_m) \leqslant \alpha P(E_n)\cdot P(E_m),$$

then

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) \geqslant 1/\alpha.$$

In fact if $P(A_k) = K$, $k = 1, 2, 3, \ldots$, $P(A_k \cap A_m) \leqslant \alpha K^2$, $k \neq m$, $\alpha \geqslant 1$, then Proposition 3.1 implies that

$$P\left(\bigcup_{n}^{\infty} A_k\right) \geqslant K^2/\Lambda = 1/\alpha$$

for any n. Furthermore the construction given in Proposition 4.3 shows that this inequality cannot be replaced by a strict inequality. More general results of this nature will appear elsewhere.

The results of this paper also have an application to the statistical theory of hypothesis testing. Consider a finite collection of statistical tests $T_1, T_2, \ldots, T_n$ whose critical regions are of probability K and such that the probability that any two of the tests will simultaneously reject the null hypothesis is Λ. Then Theorem 2.1 and Proposition 3.2 may be used to obtain a lower bound for the probability that at least one test will reject the null hypothesis and an upper bound for the probability that every test will reject the null hypothesis. In other words, we obtain bounds for the significance levels of the new tests $T_1 \wedge T_2 \wedge \ldots \wedge T_n$ and $T_1 \vee T_2 \vee \ldots \vee T_n$.

Finally we consider another application to approximation theory. Recall that if $\mathfrak{Z}$ is the class of sets in $\mathfrak{F}$ of P-measure zero, then the space $\mathfrak{M} \equiv \mathfrak{F}/\mathfrak{Z}$ may be endowed with a metric ρ. The metric is defined by

$$\rho([A_1], [A_2]) = P(A_1 \triangle A_2)$$

where $[A_i]$ denotes the equivalence class of A_i, $i = 1, 2$. Given any subset $\Gamma \subset \mathfrak{M}$, the ϵ-capacity of Γ, $C_\epsilon(\Gamma)$, is defined to be the logarithm of the maximum number of points contained in Γ with the distance between each pair of points at least ϵ; refer to G. G. Lorentz **(2)**. If Γ_K is the subset of $\mathfrak{M}$ consisting of points corresponding to sets of P-measure K, then

$$C_\epsilon(\Gamma_K) = \log b(K, (2K - \epsilon)/2).$$

The Proposition 4.1 immediately yields bounds for the ϵ-capacity of the non-compact set Γ_K.

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Department of Mathematics,
McGill University,
Montreal, Quebec