Canad. Math. Bull. Vol. **60** (4), 2017 pp. 721–735
http://dx.doi.org/10.4153/CMB-2016-072-4

On Identities with Composition of Generalized Derivations

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Abstract. Let R be a prime ring with extended centroid C, Q maximal right ring of quotients of R, RC central closure of R such that $\dim_C(RC) > 4$, $f(X_1,\dots,X_n)$ a multilinear polynomial over C that is not central-valued on R, and $f(R)$ the set of all evaluations of the multilinear polynomial $f(X_1,\dots,X_n)$ in R. Suppose that G is a nonzero generalized derivation of R such that $G^2(u)u \in C$ for all $u \in f(R)$. Then one of the following conditions holds:

(i) there exists $a \in Q$ such that $a^2 = 0$ and either $G(x) = ax$ for all $x \in R$ or $G(x) = xa$ for all $x \in R$;
(ii) there exists $a \in Q$ such that $0 \neq a^2 \in C$ and either $G(x) = ax$ for all $x \in R$ or $G(x) = xa$ for all $x \in R$ and $f(X_1,\dots,X_n)^2$ is central-valued on R;
(iii) $\operatorname{char}(R) = 2$ and one of the following holds:
  (a) there exist $a, b \in Q$ such that $G(x) = ax + xb$ for all $x \in R$ and $a^2 = b^2 \in C$;
  (b) there exist $a, b \in Q$ such that $G(x) = ax + xb$ for all $x \in R$, $a^2, b^2 \in C$ and $f(X_1,\dots,X_n)^2$ is central-valued on R;
  (c) there exist $a \in Q$ and an X-outer derivation d of R such that $G(x) = ax + d(x)$ for all $x \in R$, $d^2 = 0$ and $a^2 + d(a) = 0$;
  (d) there exist $a \in Q$ and an X-outer derivation d of R such that $G(x) = ax + d(x)$ for all $x \in R$, $d^2 = 0$, $a^2 + d(a) \in C$ and $f(X_1,\dots,X_n)^2$ is central-valued on R.

Moreover, we characterize the form of nonzero generalized derivations G of R satisfying $G^2(x) = \lambda x$ for all $x \in R$, where $\lambda \in C$.

1 Introduction

Throughout the paper, unless specially stated, R always denotes a noncommutative prime ring of characteristic $\operatorname{char}(R)$ with center $Z(R)$. Let Q denote the maximal right ring of quotients of R and let C denote the center of Q. It is known that Q is also a prime ring and C is a field that is called the *extended centroid* of R (see [2] for more details). For $a, b \in R$, let $[a, b] = ab - ba$, the commutator of a and b, let $f(X_1,\dots,X_n)$ a multilinear polynomial over C that is not central-valued on R, $f(R)$ the set of all evaluations of the multilinear polynomial $f(X_1,\dots,X_n)$ in R and s_4 the standard polynomial in 4 variables.

By a derivation of R, we mean an additive map d from R into itself satisfying $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $G\colon R \to R$ is called a *generalized derivation* of R if there exists a derivation d of R such that $G(xy) =$

Received by the editors July 12, 2016; revised October 7, 2016.
Published electronically November 15, 2016.
First and second authors are supported by The Scientific and Technological Research Council of Turkey, TUBITAK, No:2211-A and No:110T586, respectively.
AMS subject classification: **16N60**, 16N25.
Keywords: prime ring, generalized derivation, composition, extended centroid, multilinear polynomial, maximal right ring of quotients.

$G(x)y+xd(y)$ for all $x, y \in R$, and d is called the *associated derivation* of G. Evidently, any derivation is a generalized derivation. For $b \in R$, the mapping $[b, x] = bx - xb$ is a derivation of R and is known as an *inner derivation induced by the element* b and is denoted by $ad(b)$. A derivation of R is called *outer* if it is not inner. A generalized derivation G is called *inner* if its associated derivation d is inner; otherwise, G is called outer.

By T.-K. Lee, [18, Theorem 4], a generalized derivation G of a semiprime ring R is of the form $G(x) = ax + d(x)$ for all $x \in R$, where $a \in Q$ and d is a derivation of Q. Moreover, a and d are uniquely determined by G and d is also called the *associated derivation* of G. For $b \in Q$, if $d(x) = bx - xb$ for all $x \in R$, then d is said to be X-inner derivation of R. Derivations that are not X-inner are known as X-outer. The notion of generalized derivations was introduced by Brešar [3], and these maps had been extensively studied in ring theory and operator algebras. Therefore, any investigation from the algebraic point of view might be interesting (see, for example, [14, 18, 20]).

A well-known result proved by Posner in [24] states that if a prime ring R has a nonzero derivation such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R must be commutative. In [4], Brešar proved that if d and g are derivations of a prime ring R such that $d(x)x - xg(x) \in Z(R)$ for all $x \in R$, then $d = 0 = g$ or R is commutative. T.-L. Wong [27] extended this result to multilinear polynomials. He prove that if R is a prime ring, $f(X_1, \dots, X_n)$ is a multilinear polynomial over C that is not central-valued on R, and d, δ are derivations of R such that $d(u)u - u\delta(u) \in Z(R)$ for all $u \in f(R)$, then either $d = \delta = 0$ or $\delta = -d$ and $f(X_1, \dots, X_n)^2$ is central-valued on R, except when $\operatorname{char}(R) = 2$ and R satisfies s_4. In [19], T.-K. Lee and W.-K. Shiue extended this result to polynomials. They prove that if R is a prime ring, $h(X_1, \dots, X_n)$ is a polynomial over C that is not central-valued on RC, $h(R)$ the set of all evaluations of the polynomial $h(X_1, \dots, X_n)$ in R, and d, δ are two derivations of R such that $d(u)u - u\delta(u) \in C$ for all $u \in h(R)$, then either $d = 0 = \delta$ or $\delta = -d$ and $h(X_1, \dots, X_n)^2$ is central-valued on RC, except when $\operatorname{char}(R) = 2$ and $\dim_C(RC) = 4$.

On the other hand, Albaş and Argaç [1] extended Posner's theorem to generalized derivations by proving that if R is a noncommutative prime ring with a nonzero generalized derivation G such that $[G(x), x] \in Z(R)$ for all $x \in R$, then there exists $q \in C$ such that $G(x) = qx$ for all $x \in R$. F. Rania [25] proved that if G is a generalized derivation of a prime ring R such that $G(u)u = 0$ for all $u \in L$, where L is a non-central Lie ideal of R, then $G = 0$. Ma and Xu [22] gave a generalization of the result of Brešar [4] for generalized derivations on Lie ideals. They proved that if D and G are generalized derivations of a prime ring R such that $D(x)x - xG(x) \in Z(R)$ for all $x \in L$, where L is a non-central Lie ideal of R, then either R satisfies s_4 or there exists $a \in Q$ such that $D(x) = xa$ and $G(x) = ax$ for all $x \in R$. Recently, in [11], Ç. Demir and the second author gave a generalization of the result of T.-L. Wong [27] for generalized derivations as follows: if R is a prime ring, $f(X_1, \dots, X_n)$ is a multilinear polynomial over C that is not central-valued on R and G is a generalized derivation of R such that $G(u)u \in C$ for all $u \in f(R)$, then $G(x) = ax$, where $a \in C$ and $f(X_1, \dots, X_n)^2$ is central-valued on R, except when $\operatorname{char}(R) = 2$ and R satisfies s_4.

We note that most of above results were investigated in the case when $\operatorname{char}(R) \neq 2$. Our aim here is to consider the composition of nonzero generalized derivations on multilinear polynomials in prime rings in the cases when both $\mathrm{char}(R) = 2$ and $\mathrm{char}(R) \neq 2$. Motivated by these results, we characterize the structure of rings satisfied generalized polynomial identities with composition of nonzero generalized derivations, and we also characterize the form of the generalized derivations involved in the identities. More precisely, we will prove the following theorems.

Main Theorem *Let R be a prime ring with $\dim_C(RC) > 4$, extended centroid C, Q maximal right ring of quotients of R, $f(X_1, \dots, X_n)$ a multilinear polynomial over C that is not central-valued on R and G a nonzero generalized derivation of R. Suppose that*

$$G^2\big(f(r_1, \dots, r_n)\big) f(r_1, \dots, r_n) \in C$$

for all $r_1, \dots, r_n \in R$. Then one of the following conditions holds:

(i) *there exists $a \in Q$ such that $a^2 = 0$ and either $G(x) = ax$ for all $x \in R$ or $G(x) = xa$ for all $x \in R$;*
(ii) *there exists $a \in Q$ such that $0 \neq a^2 \in C$ and either $G(x) = ax$ for all $x \in R$ or $G(x) = xa$ for all $x \in R$ and $f(X_1, \dots, X_n)^2$ is central-valued on R;*
(iii) $\mathrm{char}(R) = 2$ *and one of the following holds:*
  (a) *there exist $a, b \in Q$ such that $G(x) = ax + xb$ for all $x \in R$ and $a^2 = b^2 \in C$;*
  (b) *there exist $a, b \in Q$ such that $G(x) = ax + xb$ for all $x \in R$, $a^2, b^2 \in C$ and $f(X_1, \dots, X_n)^2$ is central-valued on R;*
  (c) *there exist $a \in Q$ and an X-outer derivation d of R such that $G(x) = ax + d(x)$ for all $x \in R$, $d^2 = 0$ and $a^2 + d(a) = 0$;*
  (d) *there exist $a \in Q$ and an X-outer derivation d of R such that $G(x) = ax + d(x)$ for all $x \in R$, $d^2 = 0$, $a^2 + d(a) \in C$ and $f(X_1, \dots, X_n)^2$ is central-valued on R.*

To prove the main theorem we need the following theorems.

Theorem 1.1 *Let R be a prime ring with a nonzero generalized derivation G. Suppose that $G^2(x) = \lambda x$ for all $x \in R$, where $\lambda \in C$. Then one of the following holds:*

(i) *there exists $a \in Q$ such that $a^2 = \lambda \in C$ and either $G(x) = ax$ for all $x \in R$ or $G(x) = xa$ for all $x \in R$;*
(ii) $\mathrm{char}(R) = 2$ *and one of the following holds:*
  (a) *there exist $a, b \in Q$ such that $G(x) = ax + xb$ for all $x \in R$ and $a^2, b^2 \in C$;*
  (b) *there exist $a \in Q$ and an X-outer derivation d of R such that $G(x) = ax + d(x)$, $d^2 = 0$ and $a^2 + d(a) \in C$.*

In view of Theorem 1.1, we have the following special case.

Corollary 1.2 *Let R be a prime ring with a nonzero generalized derivation G. Suppose that $G^2(x) = 0$ for all $x \in R$. Then one of the following holds:*

(i) *there exists $a \in Q$ such that $a^2 = 0$ and either $G(x) = ax$ for all $x \in R$ or $G(x) = xa$ for all $x \in R$;*
(ii) $\mathrm{char}(R) = 2$ *and one of the following holds:*
  (a) *there exist $a, b \in Q$ such that $G(x) = ax + xb$ for all $x \in R$ and $a^2 = b^2 \in C$;*

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(b) *there exist* $a \in Q$ *and an* X*-outer derivation* d *of* R *such that* $G(x) = ax + d(x)$, $d^2 = 0$ *and* $a^2 + d(a) = 0$.

Theorem 1.3 *Let* R *be a prime ring with a nonzero generalized derivation* G. *Suppose that*

$$G^2\big(f(x_1,\ldots,x_n)\big)f(x_1,\ldots,x_n) = 0$$

for all $x_1,\ldots,x_n \in R$. *Then one of the following holds:*

(i) *there exists* $a \in Q$ *such that* $a^2 = 0$ *and either* $G(x) = ax$ *for all* $x \in R$ *or* $G(x) = xa$ *for all* $x \in R$;

(ii) $\mathrm{char}(R) = 2$ *and one of the following holds:*

(a) *there exist* $a, b \in Q$ *such that* $G(x) = ax + xb$ *for all* $x \in R$ *and* $a^2 = b^2 \in C$;

(b) *there exist* $a \in Q$ *and an* X*-outer derivation* d *of* R *such that* $G(x) = ax + d(x)$, $d^2 = 0$ *and* $a^2 + d(a) = 0$.

Note that we have $G^2 = 0$ *in all conditions.*

The following is an example for a nonzero outer derivation d such that $d^2 = 0$ on a prime ring R with $\mathrm{char}(R) = 2$.

Example 1.4 Let F be a field and let $R := M_n(F[X])$ be the n by n matrix ring over $F[X]$, where $n > 2$ and $\mathrm{char}(F) = 2$, the derivative of a function f, denoted by f'. Define $d((f_{ij}(x))) = (f'_{ij}(x))$ for $(f_{ij}(x)) \in R$. Then it is clear that $d^2 = 0$ on R, as required.

2 Preliminaries

In this section, recall that unless specially stated R is a noncommutative prime ring with extended centroid C, Q the maximal right ring of quotients of R and $f(X_1,\ldots,X_n)$ a multilinear polynomial over C, which is not central-valued on R. In order to prove the main theorem we will frequently use the theory of generalized polynomial identities and differential identities (see [2, 9, 16, 18, 23]). In particular we need to recall the following facts.

Fact 2.1 ([9]) If R is a prime ring, then R and Q satisfy same generalized polynomial identities with coefficients in Q.

Fact 2.2 ([18]) Every generalized derivation G of R can be uniquely extended to a generalized derivation of Q. In particular, there exist $a \in Q$ and a derivation d of Q such that $G(x) = ax + d(x)$ for all $x \in R$.

The following result is one of the cornerstones of the theory of generalized polynomial identities. Its original version was proved by Martindale in [23, Theorem 2].

Fact 2.3 ([6, Theorem A.7.]) Let R be a prime ring with extended centroid C and let $a_i, b_i, c_j, d_j \in Q$ be such that $\Sigma_{i=1}^{n} a_i x b_i = \Sigma_{j=1}^{m} c_j x d_j$ for all $x \in R$. If $b_1,\ldots,b_n$ are linearly independent over C, then each a_i is a linear combination of $c_1,\ldots,c_m$. Similarly, if $a_1, \dots, a_n$ are linearly independent over C, then each b_i is a linear combination of $d_1, \dots, d_m$.

Fact 2.4 ([10, Lemma 1.5]) Let K be an infinite field and $m \geq 2$. If $A_1, \dots, A_k$ are not scalar matrices in $M_m(K)$, then there exists some invertible matrix $P \in M_m(K)$ such that each matrix $PA_1P^{-1}, \dots, PA_kP^{-1}$ has all non-zero entries.

Fact 2.5 Let $C\{X\}$ be the free C-algebra with the noncommutative indeterminates in $X := \{X_1, X_2, \dots\}$. We denote by $Q *_C C\{X\}$ the free product of C-algebras Q and $C\{X\}$ over C. Any element of $Q *_C C\{X\}$ can be written in the form $g = \sum_i \alpha_i m_i$, where the coefficients $\alpha_i \in C$ and the elements m_i's called monomials, $m_i = q_0 Y_1 q_1 \dots Y_h q_h$, with $q_i \in Q$ and $Y_i \in \{X_1, \dots, X_n\}$. The elements of $Q *_C C\{X\}$ are said to be *generalized polynomials* with coefficients in Q. Nontrivial generalized polynomial means a nonzero element of $Q *_C C\{X\}$. Let $g = g(X_1, \dots, X_n) \in Q *_C C\{X\}$, if $g(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in R$; then g is said to be a generalized polynomial identity (GPI) of R and also R is said to be a GPI-ring if R satisfies a nontrivial generalized polynomial identity (see [2], for more details).

Fact 2.6 We need to recall the following notation for a multilinear polynomial $f(X_1, \dots, X_n)$ over C:

$$f(X_1, \dots, X_n) = X_1 \cdots X_n + \sum_{\sigma \in S_n, \sigma \neq 1} \alpha_\sigma X_{\sigma(1)} \cdots X_{\sigma(n)}$$

for some $\alpha_\sigma \in C$, and S_n is the symmetric group of degree n.

3 Results

In this section, recall that unless specially stated, R is a noncommutative prime ring with extended centroid C, Q the maximal right ring of quotients of R, $f(X_1, \dots, X_n)$ a multilinear polynomial over C that is not central-valued on R, $f(R)$ the set of all evaluations of the multilinear polynomial $f(X_1, \dots, X_n)$ in R.

We start with proof of the Theorem 1.1.

Proof of the Theorem 1.1 By Fact 2.2, there exist $a \in Q$ and a derivation d of Q such that $G(x) = ax + d(x)$ for all $x \in R$. If $d = 0$, then $G(x) = ax$ for all $x \in R$, and so we have that $a^2 = \lambda \in C$, by primeness of R, as required. Therefore, we can assume that $d \neq 0$. By hypothesis, we get that

$$\big(a^2 + d(a)\big)x + 2ad(x) + d^2(x) - \lambda x = 0 \tag{3.1}$$

for all $x \in R$. Suppose that $\operatorname{char}(R) \neq 2$. In this case, we assume first that d is an X-inner derivation. Then there exists $q \in Q \smallsetminus C$ such that $d(x) = [q, x]$, $d^2(x) = [q, [q, x]]$ and so $G(x) = (a + q)x - xq$ for all $x \in R$. By (3.1), we obtain that

$$\big(a^2 + [q, a] + 2aq + q^2 - \lambda\big)x - 2(a + q)xq + xq^2 = 0$$

for all $x \in R$. It follows from $q \notin C$ that $2(a + q) \in C$, by Fact 2.3. Therefore, since $\operatorname{char}(R) \neq 2$, we get $a + q \in C$, which means that $G(x) = xa$ for all $x \in R$, and so $a^2 = \lambda \in C$, as required.

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We assume now that d is not X-inner. On the other hand, since $\operatorname{char}(R) \neq 2$ and $d \neq 0$, we know that d^2 is not a derivation. Then by applying Kharchenko's theorem in [16] to (3.1), we have that $(a^2 + d(a))x + 2ay + z - \lambda x = 0$ for all $x, y, z \in R$. Taking $x = 0 = y$, we get a contradiction.

Suppose now that $\operatorname{char}(R) = 2$. Then by (3.1) we obtain that

$$(3.2) \qquad \left(a^2 + d(a)\right)x + d^2(x) - \lambda x = 0$$

for all $x \in R$. Assume first that d is X-inner derivation. Then there exists $q \in Q \setminus C$ such that $d(x) = [q, x]$, $d^2(x) = [q^2, x]$, $G(x) = (a + q)x + xq$, and so $G^2(x) = (a^2 + d(a))x + d^2(x)$ for all $x \in R$. By (3.2), we have that

$$\left((a + q)^2 + \lambda\right)x - xq^2 = 0$$

for all $x \in R$. Thus, by Fact 2.3, we get $q^2 \in C$ and $(a + q)^2 \in C$, and so condition (ii)(a) holds, as required.

Now assume that d is not X-inner. On the other hand, since $\operatorname{char}(R) = 2$, it is clear that d^2 is a derivation. If d and d^2 are C-independent module X-inner derivations, then by applying Kharchenko's theorem to (3.2), we have that $(a^2 + d(a))x + y - \lambda x = 0$ for all $x, y \in R$. Taking $x = 0$, we get a contradiction. Thus, we can assume that d and d^2 are C-dependent module X-inner derivations. Then there exist $\mu \in C$ and $0 \neq c \in Q$ such that $d^2 = \mu d + ad(c)$. If $\mu = 0$, then $d^2 = ad(c)$, namely, $d^2(x) = [c, x]$ for all $x \in R$. By (3.2), R satisfies $(a^2 + d(a) + c - \lambda)x - xc$. Hence by Fact 2.3, $c \in C$ and $a^2 + d(a) \in C$, which means that $d^2 = 0$ and so the condition (b) of (ii) holds, as required. Finally we may assume that $\mu \neq 0$. Then by (3.2), we have that R satisfies $(a^2 + d(a))x + \mu d(x) + [c, x] - \lambda x$. By Kharchenko's theorem in [16], R satisfies $(a^2 + d(a))x + \mu y + [c, x] - \lambda x$. Taking $x = 0$, we get a contradiction. ∎

To prove the Main Theorem we need the following lemmas. For the first lemma, we will study the case where $R = M_m(F)$ is the algebra of $m \times m$ matrices over a field F. Here we will assume that there exist c, q elements of R such that $c(cx + xq)x + (cx + xq)qx = 0$ for all $x \in f(R)$. Notice that the set $f(R) = \{f(r_1, \dots, r_n) : r_1, \dots, r_n \in R\}$ is invariant under the action of all inner automorphisms of $R = M_m(F)$. Let us denote as usual by e_{ij} the matrix unit with 1 in (i, j)-entry and zero elsewhere. And then we will study the previous result when R is a prime ring.

Lemma 3.1 *Let F be a field of $\operatorname{char}(F) \neq 2$ and let $R = M_m(F)$ be the algebra of $m \times m$ matrices over F, $Z(R)$ the center of R. Assume that there exist $c, q \in R$ such that $c(cx + xq)x + (cx + xq)qx = 0$ for all $x \in f(R)$. Then $(c + q)^2 = 0$, and, moreover, either $c \in Z(R)$ or $q \in Z(R)$.*

Proof Denote $c = \sum_{ij} c_{ij} e_{ij}$ and $q = \sum_{ij} q_{ij} e_{ij}$, for suitable $c_{ij}, q_{ij} \in F$. First, we assume that F is an infinite field. To prove this lemma, we assume that c and q are non-central matrices. By Fact 2.4, there exists some invertible matrix $P \in M_m(F)$ such that $PcP^{-1} = c'$ and $PqP^{-1} = q'$ have all non-zero entries. We say $c' = \sum_{ij} c'_{ij} e_{ij}$ and $q' = \sum_{ij} q'_{ij} e_{ij}$, for suitable $c'_{ij}, q'_{ij} \in F$, the conjugates of elements c, q. Now let φ be an automorphism of $M_m(F)$ such that $\varphi(x) = PxP^{-1}$ for all $x \in R$. We note that $f(R)$ is invariant under the action of all inner automorphisms of R. Then we have that

$$0 = \varphi(c)\big(\varphi(c)x + x\varphi(q)\big)x + \big(\varphi(c)x + x\varphi(q)\big)\varphi(q)x$$
$$= c'(c'x + xq')x + (c'x + xq')q'x$$

for all $x \in f(R)$. Since $f(X_1,\dots,X_n)$ is a not-central polynomial for $M_m(F)$, then by [21], there exist $u_1,\dots,u_n \in M_m(F)$ and $0 \neq \alpha \in F$, such that $f(u_1,\dots,u_n) = \alpha e_{ij}$ with $i \neq j$. Therefore,

$$0 = c'(c'\alpha e_{ij} + \alpha e_{ij}q')\alpha e_{ij} + (c'\alpha e_{ij} + \alpha e_{ij}q')q'\alpha e_{ij} = 2\alpha^2 c' e_{ij} q' e_{ij} + \alpha^2 e_{ij}(q')^2 e_{ij}$$

and left multiplying by e_{jj}, we get $2e_{jj}c'e_{ij}q'e_{ij} = 0$. It implies the contradiction $c'_{ji}q'_{ji} = 0$. Hence either $c \in Z(R)$ or $q \in Z(R)$. First, we suppose that c is a central element of $M_m(F)$. Hence it follows from the hypothesis that $x(c+q)^2 x = 0$ for all $x \in f(R)$. In other words,

$$f(x_1,\dots,x_n)(c+q)^2 f(x_1,\dots,x_n) = 0$$

for all $x_1,\dots,x_n \in R$. In this case, since $f(X_1,\dots,X_n)$ is not central-valued on R, by the result of [11, Lemma 1] we get $(c+q)^2 = 0$. Similarly, in case q is a central element, we have that $(c+q)^2 = 0$.

Now let K be an infinite field which is an extension of the field F and let $\overline{R} = M_m(K) \cong R \otimes_F K$. The generalized polynomial

$$\varphi(x_1,\dots,x_n) = c\big(cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)q\big)f(x_1,\dots,x_n) + \big(cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)q\big)qf(x_1,\dots,x_n),$$

which is a generalized polynomial identity for R, is multi-homogeneous of multi-degree $(2,\dots,2)$ in the indeterminates $x_1,\dots,x_n$. Completing the linearization of $\varphi(x_1,\dots,x_n)$, we get the multilinear generalized polynomial $\phi(x_1,\dots,x_n,y_1,\dots,y_n)$ in $2n$ indeterminates such that

$$\phi(x_1,\dots,x_n,x_1,\dots,x_n) = 2^n\varphi(x_1,\dots,x_n).$$

Thus, the multilinear polynomial $\phi(x_1,\dots,x_n,y_1,\dots,y_n)$ is a generalized polynomial for R and $\overline{R}$ too. Since $\mathrm{char}(F) \neq 2$, we have that $\varphi(x_1,\dots,x_n) = 0$ for all $x_1,\dots,x_n \in \overline{R}$. Moreover, since the multilinear polynomial $f(x_1,\dots,x_n)$ is central-valued on R if and only if it is central valued on $\overline{R}$, we have that $f(x_1,\dots,x_n)$ is not central valued on $\overline{R}$. Hence, the required conclusion follows from the first argument. ∎

Lemma 3.2 *Let R be a prime ring with $\mathrm{char}(R) \neq 2$ and $c, q \in Q$ such that*

$$c(cx + xq)x + (cx + xq)qx = 0$$

for all $x \in f(R)$. If R is not a GPI-ring, then $(c+q)^2 = 0$, and, moreover, either $c \in Z(R)$ or $q \in Z(R)$.

Proof Since R does not satisfy any non-trivial generalized polynomial identity,

$$\varphi(X_1,\dots,X_n) = c\big(cf(X_1,\dots,X_n) + f(X_1,\dots,X_n)q\big)f(X_1,\dots,X_n) + \big(cf(X_1,\dots,X_n) + f(X_1,\dots,X_n)q\big)qf(X_1,\dots,X_n) \tag{3.3}$$

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is a trivial generalized polynomial identity for R. It follows from Fact 2.5 that $\varphi(X_1,\dots,X_n)$ is a zero element of $Q *_C C\{X\}$. Therefore by (3.3), we get that

$$c^2 f(X_1,\dots,X_n) + 2cf(X_1,\dots,X_n)q + f(X_1,\dots,X_n)q^2$$

is a zero element of $Q *_C C\{X\}$. This implies that either $c \in C$ or $q \in C$ by [9]. In any case, clearly it is obtained that $(c+q)^2 = 0$, as required. ■

Lemma 3.3 *Let R be a prime ring with* $\mathrm{char}(R) \neq 2$ *and $c, q \in Q$ such that*

$$c(cx + xq)x + (cx + xq)qx = 0$$

for all $x \in f(R)$. Then $(c+q)^2 = 0$, and, moreover, either $c \in Z(R)$ or $q \in Z(R)$.

Proof We consider the generalized polynomial

$$\begin{aligned}\varphi(x_1,\dots,x_n) = c\big(&cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)q\big)f(x_1,\dots,x_n)\\ &+ \big(cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)q\big)qf(x_1,\dots,x_n).\end{aligned}$$

By Lemma 3.2, if R is not a GPI-ring, then the proof is finished. So we can assume that R is a GPI-ring. Then by Martindale's result in [23], RC is a primitive ring with nonzero socle. There exists a vector space V over a division ring D such that RC is a dense subring of D-linear transformations of V over D. Assume first that V is finite-dimensional over D. Then RC is a simple ring that satisfies a non-trivial generalized polynomial identity. By [17], $RC \subseteq M_t(F)$ for suitable field F such that $\mathrm{char}(F) \neq 2$, and, moreover, $M_t(F)$ satisfies the same generalized identity of RC. Hence,

$$\begin{aligned}c\big(&cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)q\big)f(x_1,\dots,x_n)\\ &+ \big(cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)q\big)qf(x_1,\dots,x_n) = 0\end{aligned}$$

for all $x_1,\dots,x_n \in M_t(F)$. In this case, the conclusion follows by Lemma 3.1. Now consider the case where $\dim_D(V) = \infty$. Then by [26, Lemma 2], we get that RC satisfies the identity $c(cx+xq)x + (cx+xq)qx$. Linearizing this identity, we have that $(c(cx+xq) + (cx+xq)q)y + (c(cy+yq) + (cy+yq)q)x = 0$ for all $x, y \in RC$. By [5, Lemma 4.4], we see that $c(cx+xq) + (cx+xq)q = 0$. This implies that

$$c^2x + 2cxq + xq^2 = 0 \tag{3.4}$$

for all $x \in RC$. If $c \notin C$, then $\{1, c\}$ are linearly C-independent. Thus, in view of Fact 2.3 and (3.4), we have that $\{1, q\}$ are C-dependent, *i.e.*, $q \in C$. Hence by (3.4), we see that $(c+q)^2x = 0$ for all $x \in RC$. Namely, we get $(c+q)^2 = 0$. Similarly, in case $q \notin C$, we have that $c \in C$ and so $(c+q)^2 = 0$. ■

Proof of the Theorem 1.3 By Fact 2.2, there exist $a \in Q$ and a derivation d of Q such that $G(x) = ax + d(x)$ for all $x \in R$. If $d = 0$, then $G(x) = ax$ for all $x \in R$ and so the conclusion follows from Lemma 3.3. Therefore, we can assume that $d \neq 0$. Let $f^d(x_1,\dots,x_n)$, $f^{d^2}(x_1,\dots,x_n)$ be the polynomials obtained from $f(x_1,\dots,x_n)$ replacing each coefficient α_σ with $d(\alpha_\sigma)$ and $d^2(\alpha_\sigma)$, respectively.

First, we assume that $\mathrm{char}(R) \neq 2$. By using Fact 2.6, we have that

$$d\big(f(X_1,\dots,X_n)\big) = f^d(X_1,\dots,X_n) + \sum_i f(X_1,\dots,d(X_i),\dots,X_n),$$

and so,

$$\begin{aligned}
&d^2\big(f(X_1,\dots,X_n)\big)\\
&\quad = f^{d^2}(X_1,\dots,X_n) + \sum_i f^d(X_1,\dots,d(X_i),\dots,X_n)\\
&\qquad + \sum_i f^d(X_1,\dots,d(X_i),\dots,X_n) + \sum_i f(X_1,\dots,d^2(X_i),\dots,X_n)\\
&\qquad + \sum_{i\neq j} f(X_1,\dots,d(X_i),\dots,d(X_j),\dots,X_n).
\end{aligned}$$

By the hypothesis, we have that R satisfies

$$G\Big(af(X_1,\dots,X_n) + d\big(f(X_1,\dots,X_n)\big)\Big)f(X_1,\dots,X_n)$$

and hence,

$$\begin{aligned}
(3.5)\quad &\Big(a^2 f(X_1,\dots,X_n) + ad\big(f(X_1,\dots,X_n)\big)\Big)f(X_1,\dots,X_n)\\
&\qquad + \Big(d\big(af(X_1,\dots,X_n)\big) + d^2\big(f(X_1,\dots,X_n)\big)\Big)f(X_1,\dots,X_n).
\end{aligned}$$

Suppose first that d is X-inner derivation; then there exists $b \in Q$ such that $d(x) = [b,x]$ for all $x \in R$. In this case by (3.5) we have that R satisfies

$$\begin{aligned}
&\Big(a^2 f(X_1,\dots,X_n) + a\big[b, f(X_1,\dots,X_n)\big]\Big)f(X_1,\dots,X_n)\\
&\qquad + \Big(\big[b, af(X_1,\dots,X_n)\big] + \big[b,[b,f(X_1,\dots,X_n)]\big]\Big)f(X_1,\dots,X_n)
\end{aligned}$$

that is, R satisfies

$$\begin{aligned}
&(a+b)\Big((a+b)f(X_1,\dots,X_n) + f(X_1,\dots,X_n)(-b)\Big)f(X_1,\dots,X_n)\\
&\qquad + \Big((a+b)f(X_1,\dots,X_n) + f(X_1,\dots,X_n)(-b)\Big)(-b)f(X_1,\dots,X_n).
\end{aligned}$$

By Lemma 3.3, we have that G is either of the form $G(x) = ax$ or $G(x) = xa$; moreover, $a^2 = 0$, as required.

Now we consider that d is X-outer derivation. Then by (3.5), we have that R satisfies

$$\begin{aligned}
&\big(a^2 + d(a)\big)f(X_1,\dots,X_n)^2 + 2ad\big(f(X_1,\dots,X_n)\big)f(X_1,\dots,X_n)\\
&\qquad + d^2\big(f(X_1,\dots,X_n)\big)f(X_1,\dots,X_n);
\end{aligned}$$

that is, R satisfies

$$\begin{aligned}
(3.6)\quad &\big(a^2 + d(a)\big)f(X_1,\dots,X_n)^2\\
&\quad + 2a\Big(f^d(X_1,\dots,X_n) + \sum_i f(X_1,\dots,d(X_i),\dots,X_n)\Big)f(X_1,\dots,X_n)\\
&\quad + \Big(f^{d^2}(X_1,\dots,X_n) + 2\sum_i f^d(X_1,\dots,d(X_i),\dots,X_n)\\
&\qquad + \sum_i f(X_1,\dots,d^2(X_i),\dots,X_n)\\
&\qquad + \sum_{i\neq j} f(X_1,\dots,d(X_i),\dots,d(X_j),\dots,X_n)\Big)f(X_1,\dots,X_n).
\end{aligned}$$

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Since $d \neq 0$ and d is not X-inner, by applying Kharchenko's theorem in [16] to (3.6), we get that R satisfies the generalized polynomial

$$\begin{aligned}&\big(a^2 + d(a)\big) f(x_1,\dots,x_n)^2\\ &\quad + 2a\Big(f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n)\Big) f(x_1,\dots,x_n)\\ &\quad + \Big(f^{d^2}(x_1,\dots,x_n) + 2\sum_i f^d(x_1,\dots,y_i,\dots,x_n)\\ &\qquad + \sum_i f(x_1,\dots,z_i,\dots,x_n)\\ &\qquad + \sum_{i\neq j} f(x_1,\dots,y_i,\dots,y_j,\dots,x_n)\Big) f(x_1,\dots,x_n),\end{aligned}$$

which implies that R satisfies the blended component

$$f(x_1,\dots,z_i,\dots,x_n) f(x_1,\dots,x_n),$$

and so R satisfies $f(X_1,\dots,X_n)^2$. By [21], this implies that $f(X_1,\dots,X_n)$ is a polynomial identity for R, a contradiction.

Now we assume that char $R = 2$. Then it is clear that G^2 is a generalized derivation with associated derivation d^2. Namely, $G(x) = ax + d(x)$ and $G^2(x) = (a^2 + d(a))x + d^2(x)$. Therefore by the hypothesis, we note that $G^2(u)u = 0$ for all $u \in f(R)$, where G^2 is a generalized derivation of R. It follows from [11, Corollary 1] that $G^2(x) = 0$ for all $x \in R$. Hence we are done by the Corollary 1.2. ∎

Lemma 3.4 *Let F be a field of* char$(F) \neq 2$ *and let $R = M_m(F)$ be the algebra of $m \times m$ matrices over F, $Z(R)$ the center of R. Assume that there exist $c, q \in R$ such that*

$$c(cx + xq)x + (cx + xq)qx \in Z(R)$$

for all $x \in f(R)$. Then one of the following holds:

(i) *$c \in Z(R)$, and, moreover, either $(c + q)^2 = 0$ or $f(X_1,\dots,X_n)^2$ is central valued on R and $(c + q)^2 \in Z(R)$;*

(ii) *$q \in Z(R)$, and, moreover, either $(c + q)^2 = 0$ or $f(X_1,\dots,X_n)^2$ is central valued on R and $(c + q)^2 \in Z(R)$.*

Proof Let $c = \sum_{ij} c_{ij} e_{ij}$ and $q = \sum_{ij} q_{ij} e_{ij}$, for suitable $c_{ij}, q_{ij} \in F$. First, we assume that F is an infinite field. To prove this lemma, we assume that c and q are non-central matrices. We know from the proof of Lemma 3.1 that there exist $c' = \sum_{ij} c'_{ij} e_{ij}$ and $q' = \sum_{ij} q'_{ij} e_{ij}$ for suitable $c'_{ij}, q'_{ij} \in F$ have all non-zero entries; moreover, $c'(c'x + xq')x + (c'x + xq')q'x \in Z(R)$ for all $x \in f(R)$. Since $f(X_1,\dots,X_n)$ is not-central polynomial for $M_m(F)$, then by [21], there exist $u_1,\dots,u_n \in M_m(F)$ and $0 \neq \alpha \in F$ such that $f(u_1,\dots,u_n) = \alpha e_{ij}$ with $i \neq j$. Therefore,

$$\begin{aligned}&c'(c'\alpha e_{ij} + \alpha e_{ij} q')\alpha e_{ij} + (c'\alpha e_{ij} + \alpha e_{ij} q')q'\alpha e_{ij} =\\ &\qquad\qquad 2\alpha^2 c' e_{ij} q' e_{ij} + \alpha^2 e_{ij} (q')^2 e_{ij} \in Z(R).\end{aligned}$$

Then $[2\alpha^2 c' e_{ij} q' e_{ij} + \alpha^2 e_{ij} (q')^2 e_{ij}, e_{jj}] = 0$, and so we get $2 e_{jj} c' e_{ij} q' e_{ij} = 0$. Since char$(F) \neq 2$, we get that $c'_{ji} q'_{ji} = 0$. Then a contradiction follows from Fact 2.4.

Hence, we must have either $c \in Z(R)$ or $q \in Z(R)$. First, we assume that c is central. Then by the hypothesis, we have

$$f(x_1,\dots,x_n)(c+q)^2 f(x_1,\dots,x_n) \in Z(R)$$

for all $x_1,\dots,x_n \in R$. Then from above it follows that there exist $u_1,\dots,u_n \in M_m(F)$ and $0 \neq \alpha \in F$ such that $f(u_1,\dots,u_n) = \alpha e_{kl}$ with $k \neq l$. Moreover, since the set $f(R)$ is invariant under the action of all F inner-automorphisms of $M_m(F)$, for any $i \neq j$, there exist $r_1,\dots,r_n \in M_m(F)$ such that $f(r_1,\dots,r_n) = \alpha e_{ij}$. Hence, $\alpha^2 e_{ij}(c+q)^2 e_{ij} \in Z(R)$ for all $i \neq j$. This means that that(j,i)-entry of matrix $(c+q)^2$ is zero for all $i \neq j$; *i.e.,* $(c+q)^2$ is a diagonal matrix.

Now let φ be an inner-automorphism of $M_m(F)$. Since the set $f(R)$ is invariant under the action of all inner-automorphisms, the element $\varphi((c+q)^2)$ must satisfy the same conclusions that are satisfied by $(c+q)^2$. Therefore, $\varphi((c+q)^2)$ must be a diagonal matrix. Set $a = (c+q)^2$. In particular, choose $\varphi(x) = (1+e_{ij})x(1-e_{ij})$, for any $i \neq j$. Then $\varphi(a) = a + e_{ij}a - ae_{ij} - e_{ij}ae_{ij} = a + (a_{jj} - a_{ii})e_{ij}$. Since a and $\varphi(a)$ are diagonal, it follows that $a_{ii} = a_{jj}$ for all $i \neq j$; that is, $a = (c+q)^2$ must be central. This implies that $(c+q)^2 f(x_1,\dots,x_n)^2 \in Z(R)$. By the primeness of R, one obtains $(c+q)^2 = 0$ or $f(x_1,\dots,x_n)^2 \in Z(R)$ for all $x_1,\dots,x_n \in R$. Hence, we get the conclusion (i).

Now suppose that q is central. Then by the hypothesis we have

$$(c+q)^2 f(x_1,\dots,x_n)^2 \in Z(R)$$

for all $x_1,\dots,x_n \in R$. If $f(x_1,\dots,x_n)^2 \in Z(R)$ for all $x_1,\dots,x_n \in R$, then $(c+q)^2 \in Z(R)$, and we are done. Thus, we can assume that $f(X_1,\dots,X_n)^2$ is not central on R. Let A be the additive subgroup generated by the evaluations of $f(R)^2$. In [8], it is proved that if $\operatorname{char}(R) \neq 2$ and $f(X_1,\dots,X_n)^2$ is not central-valued, then A contains a noncentral Lie ideal L of R. Moreover, it is well known that, in case $\operatorname{char}(R) \neq 2$, we also have $[R,R] \subseteq L \subseteq A$, by [12]. Thus, $(c+q)^2 x \in Z(R)$ for all $x \in [R,R]$. Fix $x = e_{ij} \in [R,R]$ with $i \neq j$; then $(c+q)^2 e_{ij} \in Z(R)$. Again, set $a = (c+q)^2$. Thus, we get that $[ae_{ij}, e_{ij}] = 0$, which means that $a_{ji} = 0$ for any $i \neq j$, that is, a is a diagonal matrix: $a = \sum_{ii} a_{ii}e_{ii}$. Moreover, it follows from $ae_{ij} \in Z(R)$ that $[ae_{ij}, e_{ji}] = 0$, which means that $ae_{ii} - e_{ji}ae_{ij} = 0$ for any $i \neq j$. Left multiplying by e_{ii}, we get $a_{ii} = 0$ for all i. It means that $a = (c+q)^2 = 0$, and we have conclusion (ii).

Now let K be an infinite field that is an extension of the field F and let $\overline{R} = M_m(K) \cong R \otimes_F K$. Notice that the multilinear polynomial $f(X_1,\dots,X_n)$ is central-valued on R if and only if it is central valued on $\overline{R}$. The generalized polynomial

$$\begin{aligned}\phi(x_1,\dots,x_{n+1}) = \Big[c\big(cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)q\big) f(x_1,\dots,x_n) \\ + \big(cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)q\big) qf(x_1,\dots,x_n), x_{n+1}\Big]\end{aligned}$$

which is a generalized polynomial identity for R, is multihomogeneous of multidegree $(2,\dots,2)$ in the indeterminates $x_1,\dots,x_{n+1}$.

Completing the linearization of $\phi(x_1,\dots,x_{n+1})$, we have the multilinear generalized polynomial $\theta(x_1,\dots,x_{n+1},y_1,\dots,y_{n+1})$ in 2^{n+1} indeterminates such that

$$\theta(x_1,\dots,x_{n+1},x_1,\dots,x_{n+1}) = 2^{n+1}\phi(x_1,\dots,x_n,x_{n+1}).$$

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Clearly the multilinear polynomial $\theta(x_1,\ldots,x_{n+1},y_1,\ldots,y_{n+1})$ is a generalized polynomial for R and $\overline{R}$ too. It follows from $\operatorname{char}(F)\neq 2$ that

$$\phi(r_1,\ldots,r_{n+1})=0$$

for all $r_1,\ldots,r_{n+1}\in\overline{R}$. Hence, the required conclusion follows from the first argument. ∎

We are now in a position to prove the main theorem.

Proof of the Main Theorem By Fact 2.2, there exist $a\in Q$ and a derivation d of Q such that $G(x)=ax+d(x)$ for all $x\in R$. If

$$G^2\big(f(x_1,\ldots,x_n)\big)f(x_1,\ldots,x_n)=0$$

for all $x_1,\ldots,x_n\in R$, then we are done by the Theorem 1.3. Otherwise, there exist $r_1,\ldots,r_n\in R$ such that $G^2(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)\neq 0$. Therefore, we can also assume $G^2(Q)\neq 0$.

First, we assume that $\operatorname{char}(R)\neq 2$. By assumption, we have that R possesses a central differential polynomial identity

$$\Big(a^2f(x_1,\ldots,x_n)+ad\big(f(x_1,\ldots,x_n)\big)\Big)f(x_1,\ldots,x_n)\\ +\Big(d\big(af(x_1,\ldots,x_n)\big)+d^2\big(f(x_1,\ldots,x_n)\big)\Big)f(x_1,\ldots,x_n).$$

Then by [7, Theorem 1], we get that R is a PI-ring and hence a GPI-ring, and by Fact 2.1, so is Q.

Suppose first that d is X-inner. Then there exists $b\in Q$ such that $d(x)=[b,x]$ for all $x\in R$, so we have $G(x)=(a+b)x-xb$. Since R is a GPI-ring by Martindale's theorem in [23], Q is primitive ring. It follows from Kaplansky's well-known theorem [15, Theorem 6.1.10], that Q is a finite dimensional central simple algebra over C. In view of [17, Lemma 2], there exists a suitable field F of $\operatorname{char}(F)\neq 2$ such that $Q\subseteq M_t(F)$ for some positive integer t, and moreover, Q and $M_t(F)$ satisfy the same GPI. Then by Lemma 3.4, we are done. Thus, we can assume that $d\neq 0$.

Now we suppose that d is not X-inner. Then by Fact 2.6 and the hypothesis, we have that

$$\big(a^2+d(a)\big)f(x_1,\ldots,x_n)^2+2ad\big(f(x_1,\ldots,x_n)\big)f(x_1,\ldots,x_n)\\ +d^2\big(f(x_1,\ldots,x_n)\big)f(x_1,\ldots,x_n)\in C;$$

that is,

$$\begin{aligned}(3.7)\quad &\big(a^2+d(a)\big)f(x_1,\ldots,x_n)^2\\ &+2a\Big(f^d(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)\Big)f(x_1,\ldots,x_n)\\ &+\Big(f^{d^2}(x_1,\ldots,x_n)+2\sum_i f^d(x_1,\ldots,d(x_i),\ldots,x_n)\\ &\quad+\sum_i f(x_1,\ldots,d^2(x_i),\ldots,x_n)\\ &\quad+\sum_{i\neq j} f(x_1,\ldots,d(x_i),\ldots,d(x_j),\ldots,x_n)\Big)f(x_1,\ldots,x_n)\in C\end{aligned}$$

for all $x_1, \ldots, x_n \in R$.

Since $d \neq 0$ and d is not X-inner, by applying Kharchenko's theorem to (3.7), we get that

$$
\begin{aligned}
&\big(a^2 + d(a)\big) f(x_1, \ldots, x_n)^2 \\
&\quad + 2a\Big(f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)\Big) f(x_1, \ldots, x_n) \\
&\quad + \Big(f^{d^2}(x_1, \ldots, x_n) + 2\sum_i f^d(x_1, \ldots, y_i, \ldots, x_n) \\
&\qquad + \sum_i f(x_1, \ldots, z_i, \ldots, x_n) \\
&\qquad + \sum_{i \neq j} f(x_1, \ldots, y_i, \ldots, y_j, \ldots, x_n)\Big) f(x_1, \ldots, x_n) \in C
\end{aligned}
$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n \in R$. In particular,

$$f(x_1, \ldots, z_i, \ldots, x_n) f(x_1, \ldots, x_n) \in C$$

for all $i = 1, \ldots, n$. Let $c \in Q \setminus C$, then we have

$$
\begin{aligned}
&\big[c, f(x_1, \ldots, x_n)\big] f(x_1, \ldots, x_n) = \\
&\qquad\qquad \sum_i f(x_1, \ldots, [c, x_i], \ldots, x_n) f(x_1, \ldots, x_n) \in C.
\end{aligned}
$$

In other words $[c, f(x_1, \ldots, x_n)] f(x_1, \ldots, x_n) \in C$ for all $x_1, \ldots, x_n \in R$. Since $\operatorname{char}(R) \neq 2$ and $c \notin C$, we get the contradiction that $f(X_1, \ldots, X_n)$ is central valued on R by [19, Theorem 2].

Finally, we assume that $\operatorname{char}(R) = 2$. Then it is clear that G^2 is a generalized derivation with associated derivation d^2. Namely, $G(x) = ax + d(x)$ and $G^2(x) = (a^2 + d(a))x + d^2(x)$. Therefore, by the hypothesis, we note that $G^2(u)u \in C$ for all $u \in f(R)$, where G^2 is a generalized derivation of R. It follows from $G^2 \neq 0$ and [11, Lemma 3] that $f(X_1, \ldots, X_n)^2$ is central valued on R and there exists $\lambda \in C$ such that $G^2(x) = \lambda x$ for all $x \in R$. Hence, we are done by Theorem 1.1. ■

Corollary 3.5 *Let R be a prime ring with $\operatorname{char}(R) \neq 2$, let L be a non-central Lie ideal of R, and let G be a nonzero generalized derivation of R. Suppose that $G^2(x)x \in Z(R)$ for all $x \in L$. Then $G(x) = ax$ or $G(x) = xa$ and moreover either $a^2 = 0$ or $[x, y]^2 \in C$ and $a^2 \in C$.*

Proof Since $\operatorname{char}(R) \neq 2$ and L is a non-central Lie ideal of R, we recall that we also have $[R, R] \subseteq L$. Therefore, we can assume that $G^2([x, y])[x, y] \in Z(R)$ for any $x, y \in R$. By the main theorem, there exists $a \in Q$ such that either $G(x) = ax$ or $G(x) = xa$, and, moreover, either $a^2 = 0$ or $[x, y]^2$ is central valued on R. If $[x, y]^2 \in C$ for all $x, y \in R$, then R satisfies s_4 by [13]. Now the proof is complete. ■

The following example shows that the condition $\operatorname{char}(R) \neq 2$ cannot be omitted in Corollary 3.5.

Example 3.6 Let $R = \mathrm{M}_3(\mathbb{Z}_2)$ and $\delta(x) = [b,\ x]$ for all $x \in R$ where

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a noncentral element of R. Since

$$b^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Z(R) \quad \text{and} \quad \mathrm{char}(R) = 2,$$

we get that

$$G^2(x)x = G(G(x))x = G([b,x])x = \big[b,[b,x]\big]x = b^2x^2 + 2bxbx + xb^2x = 0$$

for all $x \in R$.

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https://doi.org/10.4153/CMB-2016-072-4 Published online by Cambridge University Press