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ZERO DENSITY THEOREMS FOR FAMILIES OF DIRICHLET *L*-FUNCTIONS

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Abstract

We prove some zero density theorems for certain families of Dirichlet L-functions. More specifically, the subjects of our interest are the collections of Dirichlet L-functions associated with characters to moduli from certain sparse sets and of certain fixed orders.

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1. Introduction

It goes without saying that the locations of the nontrivial zeros of Dirichlet L-functions are of fundamental importance in analytic number theory. Let χ be a Dirichlet character of conductor q. Suppose that $\rho = \beta + i\gamma$ with β, $\gamma \in \mathbb{R}$ is a nontrivial zero of the Dirichlet L-function $L(s,\chi)$. Let $\sigma > 1/2$ and $T > 0$. Set

$$N(\sigma, T, \chi) = \#\{\rho : L(\rho, \chi) = 0,\ \beta \geq \sigma,\ |\gamma| \leq T\}.$$

The generalised Riemann hypothesis (GRH) asserts that $\beta = 1/2$ for all ρ, that is, $N(\sigma, T, \chi) = 0$ for all $\sigma > 1/2$ and $T > 0$.

Although the GRH is currently still an unresolved conjecture, there have been many upper bounds over the past century for $N(\sigma, T, \chi)$ in the literature, both individually and on average as χ runs over a family of characters. We refer the reader to [14, Ch. 10] and [16, Ch. 12] for discussions of these results. In brief, these estimates, dubbed zero density theorems, amount to saying that the zeros lying off the critical line should at least be very rare.

The aim of this paper is to extend these zero density results to various special collections of Dirichlet characters, more specifically, families of primitive Dirichlet characters to moduli from certain sparse sets and of certain fixed orders.

Our first result is on sparse sets of moduli. Let $\mathcal{Q}$ be a set of natural numbers contained in $(Q_0, Q_0 + Q]$. Using the nomenclature of [1], we define, for each $t \in \mathbb{N}$, the set

$$\mathcal{Q}_t = \{q \in \mathbb{N} : tq \in \mathcal{Q}\}.$$

Suppose that, for $t \in \mathbb{N}$ and $0 \le Q_0 \le Q$, there is a $\Phi \ge 1$ such that the bound

$$\max_{Q_0/t \le v \le (Q_0+Q)/t} |\{q \in \mathcal{Q}_t \cap (v, v+u] : q \equiv l \bmod k\}| \le \Big(1 + \frac{|\mathcal{Q}_t| tu}{Qk}\Big)\Phi \tag{1.1}$$

holds for $(k, l) = 1$. In this case, we say that the set $\mathcal{Q}$ is well distributed. We now state our results for sparse sets of moduli.

Theorem 1.1. *Let $T > 1$, $\varepsilon > 0$ and $\mathcal{Q} \subset (Q_0, Q_0 + Q]$, with $|\mathcal{Q}| \le Q^{1/2}$, be a well-distributed set of natural numbers such that (1.1) holds with $\Phi \ll (QT)^{\varepsilon}$. Then, for sufficiently large T and any σ with $\frac{1}{2} \le \sigma \le 1$,*

$$\sum_{q \in \mathcal{Q}} \sideset{}{^*}\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll (QT)^{\varepsilon} \min(\eta_{\mathcal{Q},T}, |\mathcal{Q}|(QT)^{3(1-\sigma)/(2-\sigma)}, (|\mathcal{Q}|Q^3T^2)^{(1-\sigma)/\sigma})$$

where

$$\eta_{\mathcal{Q},T} = T^{3(1-\sigma)/(2-\sigma)} \begin{cases} |\mathcal{Q}|^{3(3-4\sigma)/(5-4\sigma)} Q & \text{if } \frac{1}{2} \le \sigma \le \frac{3}{4} \\ (|\mathcal{Q}|^{4\sigma-3} Q^{12\sigma-7})^{(1-\sigma)/(9\sigma-4(\sigma^2+1))} & \text{otherwise.} \end{cases}$$

Here the implied constant depends on ε alone.

As in [1], one can easily check that the set of perfect k-powers, with $k \ge 2$, form a well-distributed sparse set. Thus we readily get the following corollary from Theorem 1.1.

Corollary 1.2. *For $k \ge 3$, sufficiently large Q, $T > 0$, and any $\varepsilon > 0$, we have*

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod q^k} N(\sigma, T, \chi) \ll (QT)^{\varepsilon} \min((Q^{3k+2-(3k+1)\sigma} T^{3(1-\sigma)})^{1/(2-\sigma)}, (Q^{3k+1}T^2)^{(1-\sigma)/\sigma}),$$

where the implied constant depends on ε and k at most.

Corollary 1.2 also holds for $k = 2$. But for square moduli, we have the following result which is better.

Theorem 1.3. *For sufficiently large Q, $T > 0$ and any $\varepsilon > 0$, we have*

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod q^2} N(\sigma, T, \chi) \ll (QT)^{\varepsilon} \min(\eta_{Q,T}, (Q^7T^2)^{(1-\sigma)/\sigma})$$

where

$$\eta_{Q,T} = \begin{cases} Q^{(17-16\sigma)/2(2-\sigma)} T^{3(1-\sigma)/(2-\sigma)} & \text{if } \frac{1}{2} \le \sigma \le \frac{3}{4} \\ Q^{(1-\sigma)(28\sigma-17)/(9\sigma-4(\sigma^2+1))} T^{3(1-\sigma)/(2-\sigma)} & \text{otherwise,} \end{cases}$$

and the implied constant depends on ε alone.

Our result on fixed order characters is as follows.

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THEOREM 1.4. *Let $j \in \{2,3,4,6\}$ and $C_j(Q)$ be the collection of primitive Dirichlet characters of order j and conductor $q \le Q$. Then, for $T \gg 1$, we have*

$$\sum_{\chi \in C_2(Q)} N(\sigma,T,\chi) \ll (QT)^{\varepsilon} \min((Q^3T^4)^{(1-\sigma)/(2-\sigma)}, (QT)^{3(1-\sigma)/\sigma}). \tag{1.2}$$

If $j = 3$ or 6 and $T \gg Q^{2/3}$, we have

$$\sum_{\chi \in C_j(Q)} N(\sigma,T,\chi) \ll (QT)^{\varepsilon} \min(Q^{(125-108\sigma)/(90-72\sigma)}T^{(49-44\sigma)/(22-8\sigma)}, (QT)^{7(1-\sigma)/2\sigma}). \tag{1.3}$$

Finally,

$$\sum_{\chi \in C_4(Q)} N(\sigma,T,\chi) \ll (QT)^{\varepsilon} \min(Q^{(41-36\sigma)/(30-24\sigma)}T^{(49-44\sigma)/(22-8\sigma)}, (QT)^{7(1-\sigma)/2\sigma}) \tag{1.4}$$

for $T \gg Q^{1/2}$. Here the implied constants depend on ε.

2. The setup

Our plan of attack goes along similar lines to those in [16, Ch. 12]. Let $\alpha > 0$ be some fixed constant and C be a family of primitive Dirichlet characters, all with conductors not exceeding Q. Now define $\mathcal{R}$ to be a finite set of (ρ,χ) such that $L(\rho,\chi) = 0$ for some $\chi \in C$, where $\beta \ge \sigma > \frac{1}{2}$ and $|\gamma| \le T$ for all $(\rho,\chi) \in \mathcal{R}$ and $|\gamma - \gamma'| \ge \alpha \log QT$ for some constant α and all distinct (ρ,χ) and $(\rho',\chi) \in \mathcal{R}$.

Let $\{a_n\}$ be a arbitrary sequence of complex numbers. We define

$$R(\chi) = \sum_{n \le N} a_n\chi(n) \quad \text{and} \quad S(s,\chi) = \sum_{n \le N} \frac{a_n\chi(n)}{n^s}.$$

If $\{A_l\}$ and $\{B_l\}$ are sequences of nonnegative real numbers and $L \in \mathbb{N}$, then we set

$$\Delta(Q,N) = \sum_{l \le L} Q^{A_l}N^{B_l} \quad \text{and} \quad \Delta_T(Q,N) = \sum_{l \le L} Q^{A_l}N^{B_l}T^{1-B_l}.$$

With these conditions and notation, we can show, using the same arguments as those for (12.28) or (12.29) in [16], that it is possible to choose the elements of $\mathcal{R}$ so that, for any $\varepsilon > 0$,

$$\sum_{\chi \in C} N(\sigma,T,\chi) \ll (QT)^{\varepsilon}(|\mathcal{R}| + 1), \tag{2.1}$$

where the implied constant depends on ε only. Consequently, our attention is shifted to estimating the size of $\mathcal{R}$.

We define, for $X \ge 2$

$$M_X(s,\chi) = \sum_{n \le X} \frac{\mu(n)\chi(n)}{n^s},$$

where μ is the Möbius function. We note here that the Dirichlet series of $\mathfrak{M}_X(s,\chi) = L(s,\chi)M_X(s,\chi)$ has coefficients $\mathfrak{m}_{X,n}\chi(n)$ with

$$\mathfrak{m}_{X,n} = \sum_{\substack{d|n \\ d\le X}} \mu(d).$$

Thus $\mathfrak{m}_{X,1} = 1$, $\mathfrak{m}_{X,n} = 0$ for $2 \le n \le X$, and $|\mathfrak{m}_{X,n}| \le \tau(n)$ for $n > X$, with τ denoting the divisor function.

Now we consider the Dirichlet series with coefficients $\mathfrak{m}_{X,n}\chi(n)e^{-n/Y}$ where $1 \ll X \ll Y \ll (QT)^K$ for some sufficiently large $K \ge 1$. From (12.25) and (12.26) of [16], for sufficiently large $\alpha = 3A$, each $(\rho,\chi) \in \mathcal{R}$ satisfies at least one of the inequalities

$$\Big|\sum_{X<n\le Y^2} \mathfrak{m}_{X,n}\chi(n)n^{-\rho}e^{-n/Y}\Big| \ge \frac{1}{6} \tag{2.2}$$

and

$$\frac{1}{2\pi}\Big|\int_{-A\log(QT)}^{A\log(QT)} \mathfrak{M}_X\Big(\frac{1}{2}+i\gamma+iu,\chi\Big)Y^{1/2-\beta+iu}\Gamma\Big(\frac{1}{2}-\beta+iu\Big)\,du\Big| \ge \frac{1}{6}. \tag{2.3}$$

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the sets consisting of all elements of $\mathcal{R}$ satisfying (2.2) and (2.3), respectively. Hence,

$$|\mathcal{R}| \le |\mathcal{R}_1| + |\mathcal{R}_2| \tag{2.4}$$

and it suffices to estimate from above the sizes of $\mathcal{R}_1$ and $\mathcal{R}_2$.

Along similar lines to the treatment in [16], we obtain

$$|\mathcal{R}_1| \ll (\log Y)^3 \sum_{(\rho,\chi)\in\mathcal{R}_1} \Big|\sum_{n=U}^{2U} \mathfrak{m}_{X,n}\chi(n)n^{-\rho}e^{-n/Y}\Big|^2, \tag{2.5}$$

for some U with $X \le U \le Y^2$. For $\mathcal{R}_2$, we get

$$|\mathcal{R}_2| \ll Y^{2/3-4\sigma/3}(QT)^{\varepsilon}\Big(\sum_{(\rho,\chi)\in\mathcal{R}_2}\Big|L\Big(\frac{1}{2}+it_\rho,\chi\Big)\Big|^4\Big)^{1/3}\Big(\sum_{(\rho,\chi)\in\mathcal{R}_2}\Big|M_X\Big(\frac{1}{2}+it_\rho,\chi\Big)\Big|^2\Big)^{2/3}. \tag{2.6}$$

Here, for each $(\rho,\chi) \in \mathcal{R}_2$, t_ρ is defined to be the real number in the interval $[\gamma - A\log(QT), \gamma + A\log(QT)]$ for which $|\mathfrak{M}_X(\frac{1}{2}+it_\rho,\chi)|$ is maximum.

Now we are led to estimate sums of the form

$$\sum_{\chi\in C}\sum_{s\in\mathcal{S}_\chi}\Big|\sum_{n\le N}\frac{a_n\chi(n)}{n^s}\Big|^2, \tag{2.7}$$

where $\mathcal{S}_\chi$ is a set of complex numbers. To that end, various kinds of large sieve inequalities will play an indispensable role. We refer the reader to [16, 18] and [14, Ch. 7] for more extensive discussions on the large sieve, a subject of independent interest.

We first write down a general result for sums of the form (2.7).

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LEMMA 2.1. *Let C be an arbitrary set of primitive Dirichlet characters with conductors at most Q, and $\mathcal{S}_\chi$ be a finite set of complex numbers $s = \sigma + it$. Suppose T_0, T, $\sigma_0 > \delta > 0$ are such that $T_0 + \delta/2 \le |t| \le T_0 + T - \delta/2$ for all $s \in \mathcal{S}_\chi$, $1/2 \le \sigma_0 \le \sigma \le 1$ for all $s \in \mathcal{S}_\chi$ and $|t - t'| \ge \delta$ for distinct s, $s' \in \mathcal{S}_\chi$. If the bound*

$$\sum_{\chi \in C} |R(\chi)|^2 \ll \Delta(Q, N) \sum_{n \le N} |a_n|^2$$

holds, then we have

$$\sum_{\chi \in C} \sum_{s \in \mathcal{S}_\chi} |S(s,\chi)|^2 \ll \Big(\frac{1}{\delta} + \log N\Big) \Delta_T(Q, N) \sum_{n \le N} \frac{|a_n|^2}{n^{2\sigma_0}} \Big(1 + \log \frac{\log 2N}{\log 2n}\Big).$$

PROOF. The proof is rather standard and thus we only give a sketch here. Let

$$S_u(s,\chi) = \sum_{2 \le n \le u} a_n \chi(n) n^{-s}.$$

Partial summation and Cauchy's inequality give

$$|S_u(s,\chi)|^2 \ll |a_1|^2 + |S(\sigma_0 + it, \chi)|^2 + \int_2^N |S_u(\sigma_0 + it, \chi)|^2 \frac{du}{u \log u}.$$

Using [16, Lemma 1.4], we get

$$\sum_{\chi \in C} \sum_{t \in \mathcal{T}_\chi} |S(it,\chi)|^2 \ll \Big(\frac{1}{\delta} + \log N\Big) \sum_{\chi \in C} \int_{T_0}^{T_0+T} |S(it,\chi)|^2 \, dt \tag{2.8}$$

where $\mathcal{T}_\chi = \{t : s = \sigma + it \in \mathcal{S}_\chi\}$. Now arguing along similar lines to the proof of [7, Theorem 2], we arrive at

$$\sum_{\chi \in C} \int_{T_0}^{T_0+T} |S(it,\chi)|^2 \, dt \ll \Delta_T(Q, N) \sum_{n \le N} |a_n|^2. \tag{2.9}$$

The desired bound follows easily by combining all the bounds above. □

From this discussion, we have the following general result which can be used to derive a zero density result for any collection of primitive Dirichlet characters if the corresponding large sieve inequality and bound for the fourth moment of L-functions are available.

THEOREM 2.2. *Let C be a finite family of primitive Dirichlet characters, none of which have conductors greater than Q, and suppose that*

$$\sum_{\chi \in C} |R(\chi)|^2 \ll \Delta(Q, N) \sum_{n \le N} |a_n|^2 \quad \textit{and} \quad \sum_{\chi \in C} \int_{-T}^{T} \Big|L\Big(\frac{1}{2} + it, \chi\Big)\Big|^4 dt \ll \mathfrak{L}$$

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hold. Then, for any σ with $\frac{1}{2} \leq \sigma \leq 1$, and X, Y satisfying $1 \ll X \ll Y \ll (QT)^K$ for some absolute constant K, there is a U with $X \ll U \ll Y^2$ such that

$$\sum_{\chi \in C} N(\sigma, T, \chi) \ll (QT)^{\varepsilon}(\mathfrak{L}^{1/3}\Delta_T(Q, X)^{2/3}Y^{2(1-2\sigma)/3} + \Delta_T(Q, U)U^{1-2\sigma}e^{-2U/Y}),$$

where the implied constant depends on ε alone.

PROOF. We take $\delta = 3A \log QT$ in Lemma 2.1, where A is as in (2.3), and obtain

$$\sum_{(\rho,\chi)\in\mathcal{R}_1} \left|\sum_{n=U}^{2U} \mathfrak{m}_{X,n}\chi(n)n^{-\rho}e^{-n/Y}\right|^2 \ll (QT)^{\varepsilon}\Delta_T(Q, U)U^{1-2\sigma}e^{-2U/Y} \tag{2.10}$$

and

$$\sum_{(\rho,\chi)\in\mathcal{R}_2} \left|M_X\left(\frac{1}{2} + it, \chi\right)\right|^2 \ll (QT)^{\varepsilon}\Delta_T(Q, X). \tag{2.11}$$

Using similar methods to [16, Theorem 10.3], we can show that

$$\sum_{(\rho,\chi)\in\mathcal{R}_2} \left|L\left(\frac{1}{2} + it, \chi\right)\right|^4 \ll (QT)^{\varepsilon}\mathfrak{L}. \tag{2.12}$$

Now, from (2.5) and (2.10) we obtain a bound for $|\mathcal{R}_1|$, and (2.6), (2.11) and (2.12) give rise to a majorant for $|\mathcal{R}_2|$. The result now follows from (2.1) and (2.4). □

Our second general result below does not require any large sieve-type bound.

THEOREM 2.3. *Let $C, Q, T, \mathfrak{L}$ be as in Theorem 2.2. Then, for any σ with $\frac{1}{2} < \sigma \leq 1$ and any $\varepsilon > 0$,*

$$\sum_{\chi \in C} N(\sigma, T, \chi) \ll (QT)^{\varepsilon}((\mathfrak{L}Q^2T)^{(1-\sigma)/\sigma} + (Q^2T)^{(1-\sigma)/(2\sigma-1)}),$$

where the implied constant depends on ε alone.

PROOF. The proof follows the same arguments as [16, (12.14)]. The only difference is that we do not insert any specific bound for the fourth moment of L-functions. □

3. Proof of Theorems 1.1 and 1.3

Before proving Theorems 1.1 and 1.3, we need the following lemma.

LEMMA 3.1. *Let $\mathcal{Q}$ be as above. Then, for any $T \geqslant 2$ and $\varepsilon > 0$,*

$$\sum_{q\in\mathcal{Q}} \sideset{}{^*}\sum_{\chi \bmod q} \int_{-T}^{T} \left|L\left(\frac{1}{2} + it, \chi\right)\right|^4 dt \ll |\mathcal{Q}|(QT)^{1+\varepsilon},$$

where the implied constant depends on ε alone.

PROOF. This result follows readily from [16, Theorem 10.1]. □

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We now proceed with the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. From [1, Theorem 2] we have

$$\sum_{q\in\mathcal{Q}} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2 \ll (QN)^{\varepsilon}(|\mathcal{Q}|Q + N + QN^{1/2}) \sum_{n\le N} |a_n|^2. \tag{3.1}$$

Moreover, the classical large sieve inequality gives (see the discussion around (5.4) and (5.5) in [19]),

$$\sum_{q\in\mathcal{Q}} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2 \ll \min(|\mathcal{Q}|(Q+N), Q^2 + N) \sum_{n\le N} |a_n|^2. \tag{3.2}$$

Using (3.1), Lemma 3.1 and Theorem 2.2, we obtain

$$\sum_{q\in\mathcal{Q}} \sideset{}{^*}\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll (QT)^{\varepsilon}\Big((|\mathcal{Q}|QT)^{1/3}(|\mathcal{Q}|QT + X + QT^{1/2}X^{1/2})^{2/3}Y^{2(1-2\sigma)/3}$$
$$+ |\mathcal{Q}|QTX^{1-2\sigma} + Y^{2-2\sigma} + QT^{1/2}\begin{cases} Y^{3/2-2\sigma} & \text{if } \frac{1}{2} \le \sigma \le \frac{3}{4} \\ X^{3/2-2\sigma} & \text{otherwise} \end{cases}\Big).$$

On taking

$$X = |\mathcal{Q}|^2 T \quad \text{and} \quad Y = |\mathcal{Q}|^{6/(5-4\sigma)} T^{3/2(2-\sigma)}$$

in the case $1/2 \le \sigma \le 3/4$, and

$$X = |\mathcal{Q}|^{(2\sigma-2)/(9\sigma-4(\sigma^2+1))} Q^{(4\sigma-2)/(9\sigma-4(\sigma^2+1))} T$$

and

$$Y = |\mathcal{Q}|^{(4\sigma-3)/(18\sigma-8(\sigma^2+1))} Q^{(12\sigma-7)/(18\sigma-8(\sigma^2+1))} T^{3/2(2-\sigma)}$$

in the other case, we obtain

$$\sum_{q\in\mathcal{Q}} \sideset{}{^*}\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll (QT)^{\varepsilon} \begin{cases} |\mathcal{Q}|^{3(3-4\sigma)/(5-4\sigma)} QT^{3(1-\sigma)/(2-\sigma)}, \\ (|\mathcal{Q}|^{4\sigma-3} Q^{12\sigma-7})^{(1-\sigma)/(9\sigma-4(\sigma^2+1))} T^{3(1-\sigma)/(2-\sigma)}, \end{cases} \tag{3.3}$$

respectively, in the case $1/2 \le \sigma \le 3/4$ and in the other case. Now, using (3.2) and Lemma 3.1 in Theorem 2.2, we also have

$$\sum_{q\in\mathcal{Q}} \sideset{}{^*}\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll (QT)^{\varepsilon}((|\mathcal{Q}|QT)^{1/3} \min(|\mathcal{Q}|QT + |\mathcal{Q}|X, Q^2T + X)^{2/3} Y^{2(1-2\sigma)/3}$$
$$+ \min(|\mathcal{Q}|QTX^{1-2\sigma} + |\mathcal{Q}|Y^{2-2\sigma}, Q^2TX^{1-2\sigma} + Y^{2-2\sigma})).$$

We first take

$$X = QT \quad \text{and} \quad Y = X^{3/2(2-\sigma)}$$

and then

$$X = Q^2T \quad \text{and} \quad Y = X^{3/2(2-\sigma)},$$

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and on comparing these results, we arrive at

$$\sum_{q\in\mathcal{Q}}\ \sideset{}{^*}\sum_{\chi \bmod q} N(\sigma,T,\chi) \ll (QT)^{\varepsilon}\min(|\mathcal{Q}|Q^{3(1-\sigma)/(2-\sigma)}, Q^{6(1-\sigma)/(2-\sigma)})T^{3(1-\sigma)/(2-\sigma)}. \tag{3.4}$$

Our desired result follows from comparing (3.3) with (3.4) and Theorem 2.3. □

Now, considering the case in which $\mathcal{Q}$ is the set of k-power moduli, from [3, Theorem 1], we have, for any integer $k \geq 3$ and any $Q, \varepsilon > 0$,

$$\sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi \bmod q^k} |R(\chi)|^2 \ll (QN)^{\varepsilon}(Q^{k+1}+N+Q^kN^{1/2})\sum_{n\leq N}|a_n|^2, \tag{3.5}$$

which is just a special case of (3.1). Thus (3.5) will lead to a result already contained in Theorem 1.1 and gives precisely Corollary 1.2. We note here that (3.5) has been improved in certain ranges by a number of authors: Halupczok [10, 11], Munsch [17], Halupczok and Munsch [12], and Baker *et al.* [5]. Unfortunately, using the method here, the results in [5, 10, 11, 12] do not lead to any outcome better than Corollary 1.2.

In the case of square moduli, the best available large sieve inequality is found in [4],

$$\sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi \bmod q^2} |R(\chi)|^2 \ll (QN)^{\varepsilon}(Q^3+N+\min(N\sqrt{Q}, Q^2\sqrt{N}))\sum_{n\leq N}|a_n|^2,$$

and from this we can derive Theorem 1.3, which is better than what Theorem 1.1 gives in certain regions.

PROOF OF THEOREM 1.3. By Theorem 2.2,

$$\begin{aligned}
&\sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi \bmod q^2} N(\sigma,T,\chi)\\
&\ll (QT)^{\varepsilon}\Big((Q^3T)^{1/3}(Q^3T+X+\min(Q^{1/2}X, Q^2T^{1/2}X^{1/2}))^{2/3}Y^{2(1-2\sigma)/3}+Q^3TX^{1-2\sigma}\\
&\qquad + Y^{2-2\sigma}+\min\Big(Q^{1/2}Y^{2-2\sigma}, Q^2\begin{cases} Y^{3/2-2\sigma} & \text{if } \sigma\leq \frac{3}{4}\\ X^{3/2-2\sigma} & \text{otherwise}\end{cases}\Big)\Big).
\end{aligned}$$

To get the desired result, we simply take

$$X = Q^2T \quad\text{and}\quad Y = Q^{15/4(2-\sigma)}T^{3/2(2-\sigma)}$$

if $\sigma \leq \frac{3}{4}$, and

$$X = Q^{(10\sigma-6)/(9\sigma-4(\sigma^2+1))}T \quad\text{and}\quad Y = Q^{(28\sigma-17)/(18\sigma-8(\sigma^2+1))}T^{3/2(2-\sigma)}$$

in the latter case. Hence,

$$\sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi \bmod q^2} N(\sigma,T,\chi) \ll (QT)^{\varepsilon}\begin{cases} Q^{(17-16\sigma)/2(2-\sigma)}T^{3(1-\sigma)/(2-\sigma)} & \text{if } \frac{1}{2}\leq\sigma\leq\frac{3}{4}\\ Q^{(1-\sigma)(28\sigma-17)/(9\sigma-4(\sigma^2+1))}T^{3(1-\sigma)/(2-\sigma)} & \text{otherwise.}\end{cases} \tag{3.6}$$

The result follows on comparing (3.6) with Theorem 1.1. □

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In [19], an optimal conjectural large sieve inequality for power moduli is given and yields the bound

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q^k} |R(\chi)|^2 \ll Q^{\varepsilon}(Q^{k+1}+N)\sum_{n\le N}|a_n|^2. \tag{3.7}$$

If (3.7) holds, then

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q^k} N(\sigma,T,\chi) \ll (Q^{k+1}T)^{3(1-\sigma)/(2-\sigma)+\varepsilon}$$

holds for all positive Q and T.

4. Proof of Theorem 1.4

To establish Theorem 1.4, we require, in view of (2.5) and (2.6), bounds for

$$\sum_{\chi\in C}|R(\chi)|^2 \quad \text{and} \quad \sum_{\chi\in C}\Big|L\Big(\frac{1}{2}+it,\chi\Big)\Big|^4,$$

where C is the family of characters under consideration.

For $C_2(Q)$, using [13, Corollary 3], we get

$$\sum_{\chi\in C_2(Q)}|R(\chi)|^2 \ll (QN)^{\varepsilon}(QN+N^2)\max_{n\le N}|a_n|^2. \tag{4.1}$$

By setting $\sigma = 1/2$ in Theorem 2 of [13], for $T > 1$ and $|t| \le T$,

$$\sum_{\chi\in C_2(Q)}\Big|L\Big(\frac{1}{2}+it,\chi\Big)\Big|^4 \ll (QT)^{1+\varepsilon}. \tag{4.2}$$

Using (4.1) and Lemma 2.1 with some minor changes (the bound (4.1) is formally different from what is in the condition of Lemma 2.1), we get

$$\sum_{(\rho,\chi)\in\mathcal{R}_1}\Big|\sum_{n=U}^{2U}\mathfrak{m}_{X,n}\chi(n)n^{-\rho}e^{-n/Y}\Big|^2 \ll (QT)^{\varepsilon}(QTX^{1-2\sigma}+Y^{2-2\sigma}). \tag{4.3}$$

Now (2.8) and (2.9), together with [13, Corollary 1], produce the bound

$$\sum_{(\rho,\chi)\in\mathcal{R}_2}\Big|M_X\Big(\frac{1}{2}+it_\rho,\chi\Big)\Big|^2 \ll (QT)^{\varepsilon}(QT+X). \tag{4.4}$$

Substituting (4.2), (4.3) and (4.4) into (2.5) and (2.6), we get

$$\sum_{\chi\in C_2(Q)} N(\sigma,T,\chi) \ll (QT)^{\varepsilon}((QT^2)^{1/3}(QT+X)^{2/3}Y^{2(1-2\sigma)/3}+QTX^{1-2\sigma}+Y^{2-2\sigma}).$$

Setting

$$X = QT \quad \text{and} \quad Y = (Q^3T^4)^{1/(2(2-\sigma))},$$

we arrive at the first term in the minimum in (1.2).

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In the case of $C_j(Q)$ with $j = 3$, 4 and 6, we use the results in [2, 9]. The only minor obstruction is that one requires the sum over n in $R(\chi)$ to be over square-free n. This is easily handled by rewriting $n = kl^2$ with k square-free and applying Cauchy's inequality and then utilising the large sieve inequalities for cubic, quartic and sextic characters. For $j = 3$ and 6, from Theorems 1.4 and 1.5 of [2],

$$\sum_{\chi\in C_j(Q)} |R(\chi)|^2 \ll (QN)^\varepsilon \min\{Q^{5/2}N^{1/2} + N^{3/2}, Q^{11/9} + Q^{2/3}N\} \sum_{n\le N} |a_n|^2. \tag{4.5}$$

With $j = 4$, from Lemma 2.10 of [9], which is an improvement of [8, Theorem 1.2], we arrive at the bound

$$\sum_{\chi\in C_4(Q)} |R(\chi)|^2 \ll (QN)^\varepsilon \min\{Q^{3/2}N^{1/2} + N^{3/2}, Q^{7/6} + Q^{2/3}N\} \sum_{n\le N} |a_n|^2. \tag{4.6}$$

The results in [2, 9] have more terms in the minimum than those given in (4.5) and (4.6). Here, we only cite what we will use later.

If $j = 3$ or 6, then for all $T \gg Q^{2/3}$,

$$\sum_{\chi\in C_j(Q)} \left|L\left(\frac{1}{2} + it, \chi\right)\right|^4 \ll (QT)^{3/2+\varepsilon}. \tag{4.7}$$

The bound (4.7) also holds if $j = 4$ and $T \gg Q^{1/2}$. The proof of (4.7) uses the same arguments as in [13, Theorem 2]. The only difference is that, instead of (4.1), one uses (4.5) or (4.6) with the first terms in the minima at the appropriate places.

Now proceeding in the same way as for $C_2(Q)$, using (4.7) and the second terms in the minima given in the bounds (4.5) and (4.6), we deduce

$$\begin{aligned}\sum_{\chi\in C_j(Q)} N(\sigma, T, \chi) \ll (QT)^\varepsilon((Q^{3/2}T^{5/2})^{1/3}&(Q^{11/9}T + Q^{2/3}X)^{2/3}Y^{2(1-2\sigma)/3}\\ &+ Q^{11/9}T\max(X^{3/2-2\sigma}, Y^{3/2-2\sigma}) + Q^{2/3}Y^{5/2-2\sigma})\end{aligned} \tag{4.8}$$

for $j = 3$, 6, and

$$\begin{aligned}\sum_{\chi\in C_4(Q)} N(\sigma, T, \chi) \ll (QT)^\varepsilon((Q^{3/2}T^{5/2})^{1/3}&(Q^{7/6}T + Q^{2/3}X)^{2/3}Y^{2(1-2\sigma)/3}\\ &+ Q^{7/6}T\max(X^{3/2-2\sigma}, Y^{3/2-2\sigma}) + Q^{2/3}Y^{5/2-2\sigma}).\end{aligned} \tag{4.9}$$

Taking

$$X = Q^{5/27}T^{1/2} \quad \text{and} \quad Y = Q^{5/(45-36\sigma)}T^{12/(11-4\sigma)}$$

in (4.8) and

$$X = Q^{2/9}T^{1/2} \quad \text{and} \quad Y = Q^{2/(15-12\sigma)}T^{12/(11-4\sigma)}$$

in (4.9), we get the first terms in the minima in (1.3) and (1.4). The second terms in the minima in (1.2), (1.3) and (1.4) are derived from Theorem 2.3 and either (4.2) or (4.7). This concludes the proof of Theorem 1.4.

Jutila [15, Theorem 2] previously gave the bound

$$\sum_{\chi \in C_2(Q)} N(\sigma, T, \chi) \ll (QT)^{(7-6\sigma)/(6-4\sigma)+\varepsilon} \tag{4.10}$$

without the advantage of the mean value estimate (4.1). After proving (4.1), Heath-Brown [13, Theorem 3] was able to improve the Q-aspect of (4.10) to

$$\sum_{\chi \in C_2(Q)} N(\sigma, T, \chi) \ll (QT)^{\varepsilon} Q^{3(1-\sigma)/(2-\sigma)} T^{(3-2\sigma)/(2-\sigma)}. \tag{4.11}$$

However, (4.11) was obtained by first bounding the number of zeros in the subregions

$$\{\rho : \sigma \leq \beta < \sigma + (\log QT)^{-1}, \tau \leq \gamma < \tau + (\log QT)^{-1}\} \quad \text{with } |\tau| \leq T$$

and then summing trivially over these subregions to obtain a bound for the total number of zeros in the rectangle $\{\rho : \sigma \leq \beta \leq 1, |\gamma| \leq T\}$. By considering the whole rectangle from the start and employing Lemma 2.1 to average over the ρ in the rectangle, we are able to improve the T-aspect of (4.11) in our result (1.2). Moreover, (1.2) is an improvement of (4.10) when $Q^{-4+11\sigma-6\sigma^2} \gg T^{-10+21\sigma-10\sigma^2}$, which is true for all $Q, T > 1$ when $\sigma \geq (21 - \sqrt{41})/20 \approx 0.7298$.

We end the paper with the following remark. Recent heuristics in [6] gave rise to some surprising revelations on the true optimal bound in the large sieve inequality for cubic Hecke characters, based on which, as well as its quartic analogue, the estimates in (4.5) and (4.6) are derived. Thus it gives one pause in conjecturing what the best possible form of the large sieve inequality for cubic and quartic Dirichlet characters should be. Consequently, unlike Theorem 1.1, it is unclear what the best possible unconditional bounds one can hope for in (1.3) and (1.4) may be using the methods of this paper.

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