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Abstract

In 1971, Griffiths used a generating function to define polynomials in d variables orthogonal with respect to the multinomial distribution. The polynomials possess a duality between the discrete variables and the degree indices. In 2004, Mizukawa and Tanaka related these polynomials to character algebras and the Gelfand hypergeometric series. Using this approach, they clarified the duality and obtained a new proof of the orthogonality. In the present paper, we interpret these polynomials within the context of the Lie algebra $\mathfrak{sl}_{d+1}(\mathbb{C})$. Our approach yields yet another proof of the orthogonality. It also shows that the polynomials satisfy d independent recurrence relations each involving d^2+d+1 terms. This, combined with the duality, establishes their bispectrality. We illustrate our results with several explicit examples.

1. Introduction

We start by introducing some basic notation which will be used throughout the paper. The first important object is the set $\mathcal{K}_d$ which parametrizes families of multivariate Krawtchouk polynomials. In order to motivate the definition of $\mathcal{K}_d$, let us first consider positive real numbers $\{p_j\}_{j=0}^{d}$ such that $p_0+p_1+\cdots+p_d=1$ and let $P=\mathrm{diag}(p_0,p_1,\ldots,p_d)$ be the corresponding diagonal matrix. Let $w_0,w_1,\ldots,w_d$ denote mutually orthogonal vectors in $\mathbb{R}^{d+1}$ with respect to the inner product $\langle\alpha,\beta\rangle=\alpha^tP\beta$ such that $w_0=(1,1,\ldots,1)^t$ and the 0th coordinate of w_j is 1 for $j=1,2,\ldots,d$. If we denote by U the matrix with columns $w_0,w_1,\ldots,w_d$, then $Q=U^tPU$ is a non-singular diagonal matrix whose $(0,0)$th entry is equal to 1. If we set $\tilde{P}=p_0Q^{-1}$, then $\tilde{P}=\mathrm{diag}(p_0,\tilde{p}_1,\tilde{p}_2,\ldots,\tilde{p}_d)$ is also diagonal and

$$\frac{1}{p_0}PU\tilde{P}U^t=I_{d+1},$$

where I_{d+1} denotes the identity $(d+1)\times(d+1)$ matrix. The above construction is closely related to the one by Griffiths [Gri71]. It shows how to build matrices U and $\tilde{P}$ starting from P. In the present paper, we will continue to work with the matrices $P,\tilde{P},U$ but we would like to put P and $\tilde{P}$ on the same footing. Moreover, we would like to work over $\mathbb{C}$ instead of $\mathbb{R}$. This leads to the following formal definition which extracts the important properties of the matrices P, $\tilde{P}$ and U needed in the paper.

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DEFINITION 1.1. Let $\mathcal{K}_d$ denote the set of 4-tuples $(\nu, P, \tilde{P}, U)$, where ν is a nonzero complex number and $P, \tilde{P}, U$ are $(d+1)\times(d+1)$ matrices with complex entries satisfying the following conditions.

(i) $P = \mathrm{diag}(p_0, p_1, \ldots, p_d)$ and $\tilde{P} = \mathrm{diag}(\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_d)$ are diagonal and $p_0 = \tilde{p}_0 = 1/\nu$.

(ii) $U = (u_{i,j})_{0\leqslant i,j\leqslant d}$ is such that $u_{0,j} = u_{j,0} = 1$ for all $j = 0, 1, \ldots, d$, that is,

$$U = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & u_{1,1} & u_{1,2} & \ldots & u_{1,d} \\ \vdots & & & & \\ 1 & u_{d,1} & u_{d,2} & \ldots & u_{d,d} \end{pmatrix}. \tag{1.1}$$

(iii) The following matrix equation holds

$$\nu P U \tilde{P} U^t = I_{d+1}. \tag{1.2}$$

We note that the points in $\mathcal{K}_d$ arise naturally from the so-called character algebras [BI84]. In particular, Hecke algebras of Gelfand pairs or, more generally, the Bose–Mesner algebras of commutative association schemes are examples of character algebras. From the above definition, it is easy to see that p_j and $\tilde{p}_j$ are nonzero numbers such that

$$\sum_{j=0}^{d} p_j = \sum_{j=0}^{d} \tilde{p}_j = 1.$$

Griffiths [Gri71] used a generating function to construct d-variable Krawtchouk polynomials for every point $\kappa \in \mathcal{K}_d$ and positive integer N. His construction is as follows. For $m = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_0^d$ and $\tilde{m} = (\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_d) \in \mathbb{N}_0^d$ such that $m_1 + m_2 + \cdots + m_d \leqslant N$ and $\tilde{m}_1 + \tilde{m}_2 + \cdots + \tilde{m}_d \leqslant N$, define polynomials $\mathcal{P}(m, \tilde{m}; \kappa, N) = \mathcal{P}(m, \tilde{m})$ in the variables $\tilde{m}_1, \ldots, \tilde{m}_d$ with degree indices $m_1, \ldots, m_d$ by

$$\prod_{i=0}^{d}\left(1 + \sum_{j=1}^{d} u_{i,j} z_j\right)^{\tilde{m}_i} = \sum_{\substack{m\in\mathbb{N}_0^d \\ m_1+\cdots+m_d\leqslant N}} \frac{N!}{m_0! m_1! \cdots m_d!} \mathcal{P}(m, \tilde{m}) z_1^{m_1} \cdots z_d^{m_d},$$

where $m_0 = N - m_1 - m_2 - \cdots - m_d$ and $\tilde{m}_0 = N - \tilde{m}_1 - \tilde{m}_2 - \cdots - \tilde{m}_d$.

Mizukawa and Tanaka [MT04] gave an explicit formula for $\mathcal{P}(m, \tilde{m})$ in terms of the Gelfand hypergeometric series

$$\mathcal{P}(m, \tilde{m}) = \sum_{A=(a_{i,j})\in\mathcal{M}_{d,N}} \frac{\prod_{j=1}^{d}(-m_j)_{\sum_{i=1}^{d} a_{i,j}} \prod_{i=1}^{d}(-\tilde{m}_i)_{\sum_{j=1}^{d} a_{i,j}}}{(-N)_{\sum_{i,j=1}^{d} a_{i,j}}} \prod_{i,j=1}^{d} \frac{\omega_{i,j}^{a_{i,j}}}{a_{i,j}!}, \tag{1.3}$$

where $\omega_{i,j} = 1 - u_{i,j}$. In the above formula, $\mathcal{M}_{d,N}$ denotes the set of all $d \times d$ matrices $A = (a_{i,j})$ with non-negative integer entries such that $\sum_{i,j=1}^{d} a_{i,j} \leqslant N$. One advantage of (1.3) is the transparent symmetry between m and $\tilde{m}$.

In the present paper, we define two Cartan subalgebras for $\mathfrak{sl}_{d+1}(\mathbb{C})$, denoted H and $\tilde{H}$, which depend on κ. We display an antiautomorphism $\mathfrak{a}$ of $\mathfrak{sl}_{d+1}(\mathbb{C})$ that fixes each element of H and each element of $\tilde{H}$. We consider a certain finite-dimensional irreducible $\mathfrak{sl}_{d+1}(\mathbb{C})$-module V consisting of homogeneous polynomials in $d+1$ variables of total degree N. We define a non-degenerate symmetric bilinear form $\langle\, ,\rangle$ on V such that $\langle \beta\cdot\xi, \eta\rangle = \langle \xi, \mathfrak{a}(\beta)\cdot\eta\rangle$ for all $\beta \in \mathfrak{sl}_{d+1}(\mathbb{C})$ and $\xi, \eta \in V$. We define also two bases for V; one diagonalizes H and the other diagonalizes $\tilde{H}$. Both

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bases are orthogonal with respect to $\langle\,,\rangle$. We show that when $\langle\,,\rangle$ is applied to a vector in each basis, the result is a trivial factor times one of the polynomials $\mathcal{P}(m,\tilde{m})$ defined in (1.3). Thus, the transition matrices between the bases are described by the polynomials (1.3). From these results, we recover the previously known orthogonality relation proved in [Gri71, MT04].

Our approach naturally leads to two sets of d recurrence relations for the polynomials (1.3), parametrized by the Cartan algebras H and $\tilde{H}$. Equivalently, we can interpret these recurrence relations as two commutative algebras of difference operators, each generated by d algebraically independent operators diagonalized by the polynomials $\mathcal{P}(m,\tilde{m})$. One of the algebras consists of difference operators acting on the variables m and the other one on the variables $\tilde{m}$. Following the terminology established in the literature starting with [DG86], we can say that the polynomials $\mathcal{P}(m,\tilde{m})$ solve a discrete–discrete bispectral problem. The two algebras can be connected via a natural (bispectral) involution $\mathfrak{b}$ on $\mathcal{K}_d$ defined as follows:

$$\mathfrak{b}:\kappa=(\nu,P,\tilde{P},U)\to\mathfrak{b}(\kappa)=(\nu,\tilde{P},P,U^t).$$

This map exchanges the roles of m and $\tilde{m}$ and corresponds to a symmetry between the degree parameters and the variables of the polynomials.

At the end of the paper, we illustrate how our theory applies to multivariate Krawtchouk polynomials, which have appeared in different applications in the literature. The first example explains how the present work extends our joint paper with Terwilliger [IT] related to the bivariate polynomials defined in [HR08]. The next example concerns the polynomials defined by Milch [Mil68] more than 40 years ago within the context of multidimensional growth birth and death processes. We fix the matrix P and we exhibit explicit matrices $\tilde{P}$ and U satisfying the conditions in Definition 1.1. Our constructions lead to the bispectral commutative algebras of difference operators obtained recently in [GI10, § 5.4]. The bispectral involution $\mathfrak{b}$ above corresponds to the mapping $\mathfrak{f}$ in [GI10, p. 450, (5.22)]. In the last example we explain how the polynomials discussed in [DS02] fit within our framework.

The results of the present paper yield a family of solutions to [IT, Problem 7.1]. Each of these solutions can be viewed as a rank-d generalization of a Leonard pair. The Leonard pairs are defined and classified in [Ter01]. For more information on Leonard pairs, see [Ter06].

2. Cartan subalgebras of $\mathfrak{sl}_{d+1}(\mathbb{C})$

For $i,j\in\{0,1,\dots,d\}$, let $e_{i,j}$ denote the $(d+1)\times(d+1)$ matrix that has (i,j)th entry 1 and all other entries 0. We denote by H the standard Cartan subalgebra of $\mathfrak{sl}_{d+1}(\mathbb{C})$ consisting of all diagonal matrices with basis $\{\phi_1,\phi_2,\dots,\phi_d\}$, where

$$\phi_i=e_{i,i}-\frac{1}{d+1}I_{d+1}\quad\text{for } i=1,2,\dots,d. \tag{2.1}$$

Let R denote the $(d+1)\times(d+1)$ matrix given by

$$R=\tilde{\theta}\tilde{P}U^t, \tag{2.2}$$

where $\tilde{\theta}\in\mathbb{C}$ is such that $\det(R)=1$. We consider the automorphism Ad_R on $\mathfrak{sl}_{d+1}(\mathbb{C})$ defined by $\mathrm{Ad}_R(\beta)=R\beta R^{-1}$ for every $\beta\in\mathfrak{sl}_{d+1}(\mathbb{C})$. We denote by $\tilde{\phi}_k$ and $\tilde{e}_{i,j}$ the images of ϕ_k and $e_{i,j}$ under Ad_R, that is, we set

$$\tilde{\phi}_k=R\phi_kR^{-1},\quad \tilde{e}_{i,j}=Re_{i,j}R^{-1} \tag{2.3}$$

for $k=1,2,\dots,d$ and $i,j\in\{0,1,\dots,d\}$. Let

$$\tilde{H}=\operatorname{span}\{\tilde{\phi}_1,\tilde{\phi}_2,\dots,\tilde{\phi}_d\} \tag{2.4}$$

denote the conjugated Cartan subalgebra of $\mathfrak{sl}_{d+1}(\mathbb{C})$. From (1.2) and (2.2), we see that

$$R^{-1}=\theta PU, \quad \text{where } \theta=\frac{\nu}{\tilde{\theta}}. \tag{2.5}$$

Using the above equations, we can easily expand $\tilde{\phi}_i$ in terms of the basis $\{\phi_j,e_{k,l}\}$, $j\in\{1,2,\dots,d\}$, $k\neq l\in\{0,1,\dots,d\}$, of $\mathfrak{sl}_{d+1}(\mathbb{C})$ as follows:

$$\tilde{\phi}_i=\nu\sum_{0\leqslant k\neq l\leqslant d}p_i\tilde{p}_k u_{i,k}u_{i,l}e_{k,l}+\sum_{j=1}^{d}p_i(\nu\tilde{p}_j u_{i,j}^2-1)\phi_j \tag{2.6}$$

for $i=1,2,\dots,d$. Likewise, we can also expand ϕ_i in terms of the dual basis $\{\tilde{\phi}_j,\tilde{e}_{k,l}\}$, $j\in\{1,2,\dots,d\}$, $k\neq l\in\{0,1,\dots,d\}$, of $\mathfrak{sl}_{d+1}(\mathbb{C})$:

$$\phi_i=\nu\sum_{0\leqslant k\neq l\leqslant d}\tilde{p}_i p_k u_{k,i}u_{l,i}\tilde{e}_{k,l}+\sum_{j=1}^{d}\tilde{p}_i(\nu p_j u_{j,i}^2-1)\tilde{\phi}_j \tag{2.7}$$

for $i=1,2,\dots,d$.

Next we define an antiautomorphism $\mathfrak{a}$ on $\mathfrak{sl}_{d+1}(\mathbb{C})$ by

$$\mathfrak{a}(\beta)=\tilde{P}\beta^t\tilde{P}^{-1} \text{ for every } \beta\in\mathfrak{sl}_{d+1}(\mathbb{C}). \tag{2.8}$$

Note that $\mathfrak{a}$ is an involution, that is, $\mathfrak{a}\circ\mathfrak{a}=\mathrm{Id}$. Using the fact that P and $\tilde{P}$ are diagonal matrices and (1.2), one can check the following lemma.

Lemma 2.1. *We have*

$$\mathfrak{a}(\phi_i)=\phi_i, \quad \mathfrak{a}(\tilde{\phi}_i)=\tilde{\phi}_i \quad \text{for all } i=1,2,\dots,d, \tag{2.9}$$

$$\mathfrak{a}(e_{i,j})=\frac{\tilde{p}_j}{\tilde{p}_i}e_{j,i}, \quad \mathfrak{a}(\tilde{e}_{i,j})=\frac{p_j}{p_i}\tilde{e}_{j,i} \quad \text{for all } 0\leqslant i\neq j\leqslant d. \tag{2.10}$$

In particular, $\mathfrak{a}$ preserves the Cartan subalgebras H and $\tilde{H}$.

Lemma 2.2. *The Cartan subalgebras H and $\tilde{H}$ together generate $\mathfrak{sl}_{d+1}(\mathbb{C})$.*

Proof. Consider

$$\phi_0=-\sum_{j=1}^{d}\phi_j=e_{0,0}-\frac{1}{d+1}I_{d+1}\in H$$

and

$$\tilde{\phi}_0=-\sum_{j=1}^{d}\tilde{\phi}_j=Re_{0,0}R^{-1}-\frac{1}{d+1}I_{d+1}\in\tilde{H}.$$

Using (2.2), (2.5) and the explicit form of the matrices P, $\tilde{P}$ and U, we see that

$$\begin{aligned}\tilde{\phi}_0&=\nu\tilde{P}U^t e_{0,0}PU-\frac{1}{d+1}I_{d+1}\\&=\begin{pmatrix}\tilde{p}_0&\tilde{p}_0&\tilde{p}_0&\dots&\tilde{p}_0\\\tilde{p}_1&\tilde{p}_1&\tilde{p}_1&\dots&\tilde{p}_1\\\vdots&&&&\\\tilde{p}_d&\tilde{p}_d&\tilde{p}_d&\dots&\tilde{p}_d\end{pmatrix}-\frac{1}{d+1}I_{d+1}.\end{aligned}$$

From the last equation, it follows easily that for $0 \leqslant i \neq j \leqslant d$ we have

$$e_{i,j} = \frac{[\phi_j, [\phi_i, [\phi_j, \tilde{\phi}_0]]] - [\phi_i, [\phi_j, \tilde{\phi}_0]]}{2\tilde{p}_i},$$

completing the proof. □

3. An $\mathfrak{sl}_{d+1}(\mathbb{C})$-module

Let $x_0, x_1, \ldots, x_d$ denote mutually commuting variables. We set $x = (x_0, x_1, \ldots, x_d)$ and we denote by $v_0, v_1, \ldots, v_d$ the standard basis of $\mathbb{C}^{d+1}$. Throughout the paper, we use multi-index notation. For instance, $\mathbb{C}[x]$ stands for the $\mathbb{C}$-algebra consisting of all polynomials in $x_0, x_1, \ldots, x_d$ and, if $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_d) \in \mathbb{N}_0^{d+1}$, then

$$x^\lambda = x_0^{\lambda_0} x_1^{\lambda_1} \cdots x_d^{\lambda_d}, \quad \lambda! = \lambda_0!\lambda_1! \cdots \lambda_d!,$$

$|\lambda| = \lambda_0 + \lambda_1 + \cdots + \lambda_d$, etc. We shall denote by $p = (p_0, p_1, \ldots, p_d)$ and $\tilde{p} = (\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_d)$ the vectors whose components are the entries of the diagonal matrices P and $\tilde{P}$ in Definition 1.1.

Consider the representation $\rho : \mathfrak{sl}_{d+1}(\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}[x])$ of $\mathfrak{sl}_{d+1}(\mathbb{C})$ defined by

$$\begin{aligned} \rho(e_{i,j}) &= x_i \partial_{x_j}, \\ \rho(e_{i,i} - e_{j,j}) &= x_i \partial_{x_i} - x_j \partial_{x_j} \end{aligned} \tag{3.1}$$

for $i \neq j$. Thus, every element of $\mathfrak{sl}_{d+1}(\mathbb{C})$ acts as a derivation. Let

$$\mathbb{I} = \{\lambda \in \mathbb{N}_0^{d+1} : |\lambda| = N\}$$

and let V denote the subspace of $\mathbb{C}[x]$ consisting of homogeneous polynomials of total degree N, that is,

$$V = \operatorname{span}\{x^\lambda : \lambda \in \mathbb{I}\}. \tag{3.2}$$

Since the operators in (3.1) preserve the total degree of the polynomials, we can consider V as an $\mathfrak{sl}_{d+1}(\mathbb{C})$-submodule of $\mathbb{C}[x]$ and it is easy to see that this module is irreducible. In particular, from Lemma 2.2 it follows that there is no proper subspace W of V such that $HW \subseteq W$ and $\tilde{H}W \subseteq W$.

Since

$$\phi_i \cdot x^\lambda = \left(\lambda_i - \frac{N}{d+1}\right) x^\lambda \quad \text{for } i = 1, 2, \ldots, d, \tag{3.3}$$

$$e_{i,j} \cdot x^\lambda = \lambda_j x^{\lambda + v_i - v_j} \quad \text{for } 0 \leqslant i \neq j \leqslant d, \tag{3.4}$$

we see that

$$V = \bigoplus_{\lambda \in \mathbb{I}} V_\lambda, \quad \text{where } V_\lambda = \operatorname{span}\{x^\lambda\} \tag{3.5}$$

is the weight decomposition of V with respect to H. To describe the weight decomposition of V with respect to $\tilde{H}$, we need to make a change of variables. Let us define $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_d)$ by

$$\tilde{x} = xR, \tag{3.6}$$

where R is the matrix given in (2.2). From (2.3), (3.3) and (3.4), it follows that

$$\tilde{\phi}_i \cdot \tilde{x}^\lambda = \left(\lambda_i - \frac{N}{d+1}\right) \tilde{x}^\lambda \quad \text{for } i = 1, 2, \ldots, d, \tag{3.7}$$

$$\tilde{e}_{i,j} \cdot \tilde{x}^\lambda = \lambda_j \tilde{x}^{\lambda + v_i - v_j} \quad \text{for } 0 \leqslant i \neq j \leqslant d. \tag{3.8}$$

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Thus,

$$V = \bigoplus_{\lambda \in \mathbb{I}} \tilde{V}_\lambda, \quad \text{where } \tilde{V}_\lambda = \operatorname{span}\{\tilde{x}^\lambda\} \tag{3.9}$$

is the weight decomposition of V with respect to $\tilde{H}$.

Remark 3.1. We comment on how H and $\tilde{H}$ act on the weight spaces of the other one. A pair of elements λ and μ in $\mathbb{I}$ will be called *adjacent* whenever $\lambda - \mu$ is a permutation of $(1, -1, 0, 0, \ldots, 0) \in \mathbb{C}^{d+1}$. Then H and $\tilde{H}$ act on the weight spaces of the other one as follows. For all $\lambda \in \mathbb{I}$,

$$\tilde{H} V_\lambda \subseteq V_\lambda + \sum_{\substack{\mu \in \mathbb{I} \\ \mu \operatorname{adj} \lambda}} V_\mu, \quad H \tilde{V}_\lambda \subseteq \tilde{V}_\lambda + \sum_{\substack{\mu \in \mathbb{I} \\ \mu \operatorname{adj} \lambda}} \tilde{V}_\mu.$$

We conclude the section by writing explicit formulas for the entries of $\tilde{x}$, which will be needed later. Using (3.6), the notation in Definition 1.1 and (2.2), we obtain

$$\tilde{x}_0 = \tilde{\theta} \sum_{j=0}^{d} \tilde{p}_j x_j, \tag{3.10}$$

$$\tilde{x}_k = \tilde{\theta} \tilde{p}_0 x_0 + \tilde{\theta} \sum_{j=1}^{d} u_{k,j} \tilde{p}_j x_j \quad \text{for } k = 1, 2, \ldots, d. \tag{3.11}$$

Recall that $\omega_{i,j} = 1 - u_{i,j}$ and, therefore, subtracting (3.10) from (3.11), we can rewrite (3.11) as follows:

$$\tilde{x}_k = \tilde{x}_0 - \tilde{\theta} \sum_{j=1}^{d} \tilde{p}_j \omega_{k,j} x_j \quad \text{for } k = 1, 2, \ldots, d. \tag{3.12}$$

4. A bilinear form

We define a symmetric bilinear form $\langle \, , \rangle$ on V by

$$\langle x^n, x^m \rangle = \delta_{n,m} \frac{n!}{\tilde{p}^n} \theta^N \quad \text{for all } n, m \in \mathbb{I}. \tag{4.1}$$

Using the explicit formulas for the action of $\mathfrak{a}$ on the basis $\{\phi_j, e_{k,l}\}$, $j \in \{1, 2, \ldots, d\}$, $k \neq l \in \{0, 1, \ldots, d\}$, of $\mathfrak{sl}_{d+1}(\mathbb{C})$ in Lemma 2.1, it is easy to check that

$$\langle \beta \cdot \xi, \eta \rangle = \langle \xi, \mathfrak{a}(\beta) \cdot \eta \rangle \quad \text{for all } \beta \in \mathfrak{sl}_{d+1}(\mathbb{C}), \quad \xi, \eta \in V. \tag{4.2}$$

We prove next that the vectors $\tilde{x}^n$ are also mutually orthogonal with respect to $\langle \, , \rangle$.

Lemma 4.1. *For $n, m \in \mathbb{I}$, we have*

$$\langle \tilde{x}^n, \tilde{x}^m \rangle = \delta_{n,m} \frac{n!}{p^n} \tilde{\theta}^N. \tag{4.3}$$

Proof. For $n \neq m$, (4.3) follows immediately from (4.2), the fact that $\mathfrak{a}$ preserves $\tilde{H}$ and (3.7). Thus, it remains to consider the case $n = m$, that is, we need to show that

$$\|\tilde{x}^n\|^2 = \langle \tilde{x}^n, \tilde{x}^n \rangle = \frac{n!}{p^n} \tilde{\theta}^N. \tag{4.4}$$

We prove that (4.4) is true by induction on $n_1+n_2+\cdots+n_d$. If $n_1+n_2+\cdots+n_d=0$, then $n=(N,0,0,\ldots,0)$ and we need to show that $\|\tilde{x}_0^N\|^2=N!\tilde{\theta}^N/p_0^N$. Taking the Nth power of (3.10) and using (4.1), we find

$$\begin{aligned}\|\tilde{x}_0^N\|^2&=\tilde{\theta}^{2N}\left\|\left(\sum_{j=0}^{d}\tilde{p}_jx_j\right)^N\right\|^2=\tilde{\theta}^{2N}\left\|\sum_{\alpha\in\mathbb{I}}\frac{N!}{\alpha!}\tilde{p}^{\alpha}x^{\alpha}\right\|^2\\&=\tilde{\theta}^{2N}\sum_{\alpha\in\mathbb{I}}\frac{(N!)^2}{(\alpha!)^2}\tilde{p}^{2\alpha}\|x^{\alpha}\|^2=\tilde{\theta}^{2N}\sum_{\alpha\in\mathbb{I}}\frac{(N!)^2}{\alpha!}\tilde{p}^{\alpha}\theta^N\\&=\tilde{\theta}^N(\tilde{\theta}\theta)^NN!\sum_{\alpha\in\mathbb{I}}\frac{N!}{\alpha!}\tilde{p}^{\alpha}=\frac{\tilde{\theta}^N}{p_0^N}N!,\end{aligned}$$

completing the proof in this case. Suppose now $n_1+n_2+\cdots+n_d>0$. Then $n_j>0$ for some $j\in\{1,\ldots,d\}$ and we consider the equality

$$\langle\tilde{e}_{0,j}\cdot\tilde{x}^n,\tilde{x}^{n+v_0-v_j}\rangle=\langle\tilde{x}^n,\mathfrak{a}(\tilde{e}_{0,j})\cdot\tilde{x}^{n+v_0-v_j}\rangle.$$

From (3.8), it follows that the left-hand side equals $n_j\|\tilde{x}^{n+v_0-v_j}\|^2$. From (2.10) and (3.8), we see that the right-hand side is $(p_j/p_0)(n_0+1)\|\tilde{x}^n\|^2$. Therefore, we have

$$\|\tilde{x}^n\|^2=\frac{p_0}{p_j}\frac{n_j}{(n_0+1)}\|\tilde{x}^{n+v_0-v_j}\|^2,$$

completing the proof by induction. □

5. The Krawtchouk polynomials and $\mathfrak{sl}_{d+1}(\mathbb{C})$

We are now ready to prove that the transition matrices between the bases $\{x^n:n\in\mathbb{I}\}$ and $\{\tilde{x}^n:n\in\mathbb{I}\}$ can be expressed in terms of the multivariate Krawtchouk polynomials (1.3). To state the theorem, we shall use the following convention: for a vector $w=(w_0,w_1,\ldots,w_d)\in\mathbb{C}^{d+1}$, we denote $w'=(w_1,\ldots,w_d)\in\mathbb{C}^d$.

Theorem 5.1. *With the above convention, we have*

$$\frac{\tilde{x}^{\tilde{n}}}{\tilde{\theta}^N}=N!\sum_{n\in\mathbb{I}}\mathcal{P}(n',\tilde{n}')\frac{\tilde{p}^n}{n!}x^n\quad\text{for }\tilde{n}\in\mathbb{I}\tag{5.1a}$$

and

$$\frac{x^n}{\theta^N}=N!\sum_{\tilde{n}\in\mathbb{I}}\mathcal{P}(n',\tilde{n}')\frac{p^{\tilde{n}}}{\tilde{n}!}x^{\tilde{n}}\quad\text{for }n\in\mathbb{I}.\tag{5.1b}$$

Proof. Let us denote $\omega_i=(\omega_{i,1},\omega_{i,2},\ldots,\omega_{i,d})$. Using (3.12), we find

$$\begin{aligned}\frac{\tilde{x}_i^{\tilde{n}_i}}{\tilde{\theta}^{\tilde{n}_i}}&=\sum_{\substack{c_i\in\mathbb{N}_0^d\\|c_i|\leqslant\tilde{n}_i}}\frac{\tilde{n}_i!}{c_i!(\tilde{n}_i-|c_i|)!}\left(\frac{\tilde{x}_0}{\tilde{\theta}}\right)^{\tilde{n}_i-|c_i|}(-1)^{|c_i|}(\tilde{p}')^{c_i}\omega_i^{c_i}(x')^{c_i}\\&=\sum_{c_i\in\mathbb{N}_0^d}\frac{(-\tilde{n}_i)_{|c_i|}}{c_i!}\left(\frac{\tilde{x}_0}{\tilde{\theta}}\right)^{\tilde{n}_i-|c_i|}(\tilde{p}')^{c_i}\omega_i^{c_i}(x')^{c_i}\quad\text{for }i=1,2,\ldots,d.\end{aligned}$$

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If we set $c_i = (a_{i,1}, a_{i,2}, \dots, a_{i,d})$ and denote by A the matrix with entries $(a_{i,j})$, then, multiplying the above equations, we obtain

$$\frac{\tilde{x}^{\tilde{n}}}{\tilde{\theta}^N} = \sum_{A\in\mathcal{M}_{d,N}} \left(\frac{\tilde{x}_0}{\tilde{\theta}}\right)^{N-\sum_{i,j=1}^d a_{i,j}} \prod_{i=1}^d (-\tilde{n}_i)_{\sum_{j=1}^d a_{i,j}} \prod_{j=1}^d (\tilde{p}_j x_j)^{\sum_{i=1}^d a_{i,j}} \prod_{i,j=1}^d \frac{\omega_{i,j}^{a_{i,j}}}{a_{i,j}!}. \tag{5.2}$$

Using now (3.10), we get

$$\left(\frac{\tilde{x}_0}{\tilde{\theta}}\right)^{N-\sum_{i,j=1}^d a_{i,j}} = \sum_{\substack{\gamma\in\mathbb{N}_0^{d+1} \\ |\gamma|=N-\sum_{i,j=1}^d a_{i,j}}} \frac{(N-\sum_{i,j=1}^d a_{i,j})!}{\gamma!} \prod_{k=0}^d (\tilde{p}_k x_k)^{\gamma_k}. \tag{5.3}$$

Note that

$$\frac{(N-\sum_{i,j=1}^d a_{i,j})!}{\gamma!} = \frac{N!}{\gamma_0!\prod_{j=1}^d(\gamma_j+\sum_{i=1}^d a_{i,j})!} \frac{\prod_{j=1}^d(-\gamma_j-\sum_{i=1}^d a_{i,j})_{\sum_{i=1}^d a_{i,j}}}{(-N)_{\sum_{i,j=1}^d a_{i,j}}}.$$

If we change the variables as follows: $n_0 = \gamma_0$ and $n_j = \gamma_j + \sum_{i=1}^d a_{i,j}$ for $j = 1, 2, \dots, d$ and we substitute the last formula in (5.3), we obtain

$$\begin{aligned}\left(\frac{\tilde{x}_0}{\tilde{\theta}}\right)^{N-\sum_{i,j=1}^d a_{i,j}} &= \frac{1}{(-N)_{\sum_{i,j=1}^d a_{i,j}}} \sum_{n\in\mathbb{I}} \frac{N!}{n!} \prod_{j=1}^d (-n_j)_{\sum_{i=1}^d a_{i,j}} \\ &\quad \times (\tilde{p}_0 x_0)^{n_0} \prod_{j=1}^d (\tilde{p}_j x_j)^{n_j-\sum_{i=1}^d a_{i,j}}.\end{aligned} \tag{5.4}$$

The proof of (5.1a) now follows immediately by substituting (5.4) into (5.2). The proof of (5.1b) can be obtained along the same lines, or one can use the duality. □

Corollary 5.2. *For $n, \tilde{n} \in \mathbb{I}$, we have*

$$\mathcal{P}(n', \tilde{n}') = \frac{1}{\nu^N N!}\langle x^n, \tilde{x}^{\tilde{n}}\rangle. \tag{5.5}$$

As an immediate consequence of Theorem 5.1 and Corollary 5.2, we obtain a new proof of the orthogonality relations established in [Gri71, MT04].

Corollary 5.3. *For $\tilde{n}, \tilde{k} \in \mathbb{I}$, we have*

$$N! \sum_{n\in\mathbb{I}} \mathcal{P}(n', \tilde{n}')\mathcal{P}(n', \tilde{k}')\frac{\tilde{p}^n}{n!} = \delta_{\tilde{n},\tilde{k}} \frac{\tilde{n}!}{N!\,\nu^N\, p^{\tilde{n}}} \tag{5.6a}$$

and similarly for $n, k \in \mathbb{I}$, we have

$$N! \sum_{\tilde{n}\in\mathbb{I}} \mathcal{P}(n', \tilde{n}')\mathcal{P}(k', \tilde{n}')\frac{p^{\tilde{n}}}{\tilde{n}!} = \delta_{n,k} \frac{n!}{N!\,\nu^N\, \tilde{p}^n}. \tag{5.6b}$$

Proof. Equation (5.6a) follows if we consider the inner product of each side of (5.1a) with $\tilde{x}^{\tilde{k}}$ and then we use (4.3) and (5.5). The proof of (5.6b) is similar. □

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6. Bispectral difference equations

In this section, we show that the Cartan subalgebras H and $\tilde{H}$ lead naturally to recurrence relations for the multivariate Krawtchouk polynomials. We work below with $\mathbb{C}^d$ and we denote by $v_1, v_2, \ldots, v_d$ its standard basis.

Theorem 6.1. *Fix $m = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_0^d$ and $\tilde{m} = (\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_d) \in \mathbb{N}_0^d$ such that $|m| \leqslant N$ and $|\tilde{m}| \leqslant N$. Then, for every $i \in \{1, 2, \ldots, d\}$, we have*

$$\begin{aligned}\left(m_i - \frac{N}{d+1}\right)\mathcal{P}(m, \tilde{m}) = {} & \sum_{l=1}^{d} \tilde{p}_i u_{l,i} \tilde{m}_l \mathcal{P}(m, \tilde{m} - v_l) \\ & + \nu \sum_{k=1}^{d} \tilde{p}_i p_k u_{k,i} (N - |\tilde{m}|)\mathcal{P}(m, \tilde{m} + v_k) \\ & + \sum_{j=1}^{d} \tilde{p}_i (\nu p_j u_{j,i}^2 - 1)\left(\tilde{m}_j - \frac{N}{d+1}\right)\mathcal{P}(m, \tilde{m}) \\ & + \nu \sum_{1 \leqslant k \neq l \leqslant d} \tilde{p}_i p_k u_{k,i} u_{l,i} \tilde{m}_l \mathcal{P}(m, \tilde{m} + v_k - v_l) \end{aligned} \tag{6.1a}$$

and

$$\begin{aligned}\left(\tilde{m}_i - \frac{N}{d+1}\right)\mathcal{P}(m, \tilde{m}) = {} & \sum_{l=1}^{d} p_i u_{i,l} m_l \mathcal{P}(m - v_l, \tilde{m}) \\ & + \nu \sum_{k=1}^{d} p_i \tilde{p}_k u_{i,k} (N - |m|)\mathcal{P}(m + v_k, \tilde{m}) \\ & + \sum_{j=1}^{d} p_i (\nu \tilde{p}_j u_{i,j}^2 - 1)\left(m_j - \frac{N}{d+1}\right)\mathcal{P}(m, \tilde{m}) \\ & + \nu \sum_{1 \leqslant k \neq l \leqslant d} p_i \tilde{p}_k u_{i,k} u_{i,l} m_l \mathcal{P}(m + v_k - v_l, \tilde{m}). \end{aligned} \tag{6.1b}$$

Proof. For $m, \tilde{m} \in \mathbb{N}_0^d$ such that $|m| \leqslant N$ and $|\tilde{m}| \leqslant N$, we consider $n = (N - |m|, m_1, m_2, \ldots, m_d) \in \mathbb{I}$ and $\tilde{n} = (N - |\tilde{m}|, \tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_d) \in \mathbb{I}$. From Lemma 2.1 and (4.2), it follows that

$$\langle \phi_i \cdot x^n, \tilde{x}^{\tilde{n}} \rangle = \langle x^n, \phi_i \cdot \tilde{x}^{\tilde{n}} \rangle. \tag{6.2}$$

From (3.3) and Corollary 5.2, we see that the left-hand side of (6.2) is equal to $\nu^N N!$ times the left-hand side of (6.1a). Now we use (2.7) and Lemma 2.1 to evaluate the right-hand side of (6.2), which gives $\nu^N N!$ times the right-hand side of (6.1a). The proof of (6.1b) is similar. □

Remark 6.2. Theorem 6.1 establishes the bispectrality of the polynomials $\mathcal{P}(m, \tilde{m})$. Indeed, (6.1a) show that the polynomials $\mathcal{P}(m, \tilde{m})$ are common eigenfunctions of d difference operators acting on the variables $\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_d$, with coefficients independent of $m_1, m_2, \ldots, m_d$, while (6.1b) indicate that the same polynomials are common eigenfunctions of d difference operators acting on the variables $m_1, m_2, \ldots, m_d$, with coefficients independent of $\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_d$.

Formulas (6.1) show that the polynomials $\mathcal{P}(m, \tilde{m})$ form a remarkable basis. A random basis of polynomials orthogonal with respect to the the multinomial distribution will not satisfy such

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simple recurrence relations; see [DX01] for the general theory. If we add (6.1a) for $i=1,2,\dots,d$ and use the matrix equation (1.2), we obtain

$$\begin{aligned}-|m|\mathcal{P}(m,\tilde m) &= p_0\sum_{l=1}^{d}\tilde m_l\mathcal{P}(m,\tilde m-v_l)+\sum_{l=1}^{d}p_l(N-|\tilde m|)\mathcal{P}(m,\tilde m+v_l)\\ &\quad+\sum_{1\leqslant k\neq l\leqslant d}p_k\tilde m_l\mathcal{P}(m,\tilde m+v_k-v_l)\\ &\quad+\Bigl(\sum_{j=1}^{d}\tilde m_jp_j+p_0(N-|\tilde m|)-N\Bigr)\mathcal{P}(m,\tilde m). \qquad (6.3)\end{aligned}$$

Note that the coefficients on the right-hand side in the above partial difference equation depend only on p, N and $\tilde m$, while the eigenvalue on the left-hand side depends only on the total degree $|m|$ (if we think of $\mathcal{P}(m,\tilde m)$ as a polynomial of $\tilde m$). This equation is *universal*, since every basis of polynomials orthogonal with respect to the multinomial distribution

$$\frac{p_0^{N-|\tilde m|}p_1^{\tilde m_1}p_2^{\tilde m_2}\cdots p_d^{\tilde m_d}}{(N-|\tilde m|)!\tilde m_1!\tilde m_2!\cdots\tilde m_d!}$$

will satisfy (6.3); see [IX07].

7. Explicit examples

7.1 The bivariate Krawtchouk polynomials introduced by Hoare and Rahman

Let $d=2$ and define

$$\begin{aligned}u_{1,1}&=1-\frac{(\mathfrak{p}_1+\mathfrak{p}_2)(\mathfrak{p}_1+\mathfrak{p}_3)}{\mathfrak{p}_1(\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3+\mathfrak{p}_4)}, & u_{1,2}&=1-\frac{(\mathfrak{p}_1+\mathfrak{p}_2)(\mathfrak{p}_2+\mathfrak{p}_4)}{\mathfrak{p}_2(\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3+\mathfrak{p}_4)},\\ u_{2,1}&=1-\frac{(\mathfrak{p}_1+\mathfrak{p}_3)(\mathfrak{p}_3+\mathfrak{p}_4)}{\mathfrak{p}_3(\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3+\mathfrak{p}_4)}, & u_{2,2}&=1-\frac{(\mathfrak{p}_2+\mathfrak{p}_4)(\mathfrak{p}_3+\mathfrak{p}_4)}{\mathfrak{p}_4(\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3+\mathfrak{p}_4)},\end{aligned}$$

$$\begin{aligned}p_1&=\frac{\mathfrak{p}_1\mathfrak{p}_2(\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3+\mathfrak{p}_4)}{(\mathfrak{p}_1+\mathfrak{p}_2)(\mathfrak{p}_1+\mathfrak{p}_3)(\mathfrak{p}_2+\mathfrak{p}_4)}, & p_2&=\frac{\mathfrak{p}_3\mathfrak{p}_4(\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3+\mathfrak{p}_4)}{(\mathfrak{p}_1+\mathfrak{p}_3)(\mathfrak{p}_2+\mathfrak{p}_4)(\mathfrak{p}_3+\mathfrak{p}_4)},\\ \tilde p_1&=\frac{\mathfrak{p}_1\mathfrak{p}_3(\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3+\mathfrak{p}_4)}{(\mathfrak{p}_1+\mathfrak{p}_2)(\mathfrak{p}_1+\mathfrak{p}_3)(\mathfrak{p}_3+\mathfrak{p}_4)}, & \tilde p_2&=\frac{\mathfrak{p}_2\mathfrak{p}_4(\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3+\mathfrak{p}_4)}{(\mathfrak{p}_1+\mathfrak{p}_2)(\mathfrak{p}_2+\mathfrak{p}_4)(\mathfrak{p}_3+\mathfrak{p}_4)}\end{aligned}$$

and

$$p_0=\tilde p_0=1-p_1-p_2,$$

where the numbers $\{\mathfrak{p}_j\}_{1\leqslant j\leqslant 4}$ are essentially arbitrary, although certain combinations are forbidden in order to avoid dividing by 0. A straightforward computation shows that (1.2) holds and we obtain the polynomials defined by Hoare and Rahman [HR08]. The constructions in the present paper correspond to the ones in [IT]. The bispectral involution $\mathfrak{b}$ in this case amounts to exchanging the parameters $\mathfrak{p}_2$ and $\mathfrak{p}_3$. We note also that independently of [IT], Grünbaum and Rahman [GR10] used different methods to derive an interesting recurrence relation for these polynomials which was suggested in [Gru07].

7.2 The multivariate Krawtchouk polynomials defined by Milch

Within the context of orthogonal polynomials, one is often interested in explicit formulas for the polynomials in terms of the parameters which appear in the measure. We construct one such example which gives the multivariate Krawtchouk polynomials discovered by Milch [Mil68].

Let us fix nonzero complex numbers $p_0, p_1, \ldots, p_d$ such that $p_0 + p_1 + \cdots + p_d = 1$. We define $\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_d$ in terms of $p_0, p_1, \ldots, p_d$ as follows:

$$\tilde{p}_0 = p_0, \tag{7.1a}$$

$$\tilde{p}_k = \frac{p_k p_0}{(1-\sum_{j=1}^{k} p_j)(1-\sum_{j=1}^{k-1} p_j)} \quad \text{for } k = 1, 2, \ldots, d. \tag{7.1b}$$

Next we define a $(d+1) \times (d+1)$ matrix $U = (u_{i,j})$ with entries

$$u_{i,j} = \delta_{0,i} \quad \text{when } i < j, \tag{7.2a}$$

$$u_{i,j} = 1 \quad \text{when } i > j, \tag{7.2b}$$

$$u_{0,0} = 1, \tag{7.2c}$$

$$u_{i,i} = -\frac{1-\sum_{k=1}^{i} p_k}{p_i} \quad \text{for } i = 1, 2, \ldots, d. \tag{7.2d}$$

Thus, U is a matrix of the form (1.1) where the remaining entries are 0's and 1's above and below the diagonal, respectively, and the diagonal entries are given in (7.2d). One can check that with the above notation (1.2) holds and therefore we obtain explicit formulas for all constructions in the paper. Up to a permutation of the variables and the parameters, this choice leads to the multivariate Krawtchouk polynomials considered in [Mil68] and the corresponding bispectral difference operators constructed in [GI10, § 5.4].

7.3 The multivariate Krawtchouk polynomials introduced by Dougherty and Skriganov

Fix now $q \notin \{0, 1\}$ and define $p_0, p_1, \ldots, p_d$ as follows:

$$p_0 = q^{-d}, \tag{7.3a}$$

$$p_k = q^{-d+k-1}(q-1) \quad \text{for } k = 1, 2, \ldots, d. \tag{7.3b}$$

We denote

$$P = \tilde{P} = \operatorname{diag}(p_0, p_1, \ldots, p_d) \tag{7.4}$$

and we consider also a $(d+1) \times (d+1)$ matrix $U = (u_{i,j})$ with entries

$$u_{i,j} = 1 \quad \text{when } i + j \leqslant d, \tag{7.5a}$$

$$u_{i,j} = \frac{1}{1-q} \quad \text{when } i + j = d + 1, \tag{7.5b}$$

$$u_{i,j} = 0 \quad \text{when } i + j > d + 1. \tag{7.5c}$$

With the above notation, we see that (1.2) holds, leading to the family of multivariate Krawtchouk polynomials introduced in [DS02].

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References

BI84 E. Bannai and T. Ito, *Algebraic combinatorics. I. Association schemes* (Benjamin/Cummings, Menlo Park, CA, 1984).

DS02 S. T. Dougherty and M. M. Skriganov, *MacWilliams duality and the Rosenbloom–Tsfasman metric*, Mosc. Math. J. **2** (2002), 81–97.

DG86 J. J. Duistermaat and F. A. Grünbaum, *Differential equations in the spectral parameter*, Comm. Math. Phys. **103** (1986), 177–240.

DX01 C. F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, Encyclopedia of Mathematics and its Applications, vol. 81 (Cambridge University Press, Cambridge, 2001).

GI10 J. Geronimo and P. Iliev, *Bispectrality of multivariable Racah–Wilson polynomials*, Constr. Approx. **31** (2010), 417–457.

Gri71 R. C. Griffiths, *Orthogonal polynomials on the multinomial distribution*, Austral. J. Statist. **13** (1971), 27–35.

Gru07 F. A. Grünbaum, *The Rahman polynomials are bispectral*, SIGMA Symmetry Integrability Geom. Methods Appl. **3** (2007) 11 (Paper 065).

GR10 F. A. Grünbaum and M. Rahman, *On a family of 2-variable orthogonal Krawtchouk polynomials*, SIGMA Symmetry Integrability Geom. Methods Appl. **6** (2010) 12 (Paper 090).

HR08 M. R. Hoare and M. Rahman, *A probabilistic origin for a new class of bivariate polynomials*, SIGMA Symmetry Integrability Geom. Methods Appl. **4** (2008) 18 (Paper 089).

IT P. Iliev and P. Terwilliger, *The Rahman polynomials and the Lie algebra* $\mathfrak{sl}_3(\mathbb{C})$, Trans. Amer. Math. Soc., to appear.

IX07 P. Iliev and Y. Xu, *Discrete orthogonal polynomials and difference equations of several variables*, Adv. Math. **212** (2007), 1–36.

Mil68 P. R. Milch, *A multi-dimensional linear growth birth and death process*, Ann. Math. Statist. **39** (1968), 727–754.

MT04 H. Mizukawa and H. Tanaka, $(n+1, m+1)$*-hypergeometric functions associated to character algebras*, Proc. Amer. Math. Soc. **132** (2004), 2613–2618.

Ter01 P. Terwilliger, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other*, Linear Algebra Appl. **330** (2001), 149–203.

Ter06 P. Terwilliger, *An algebraic approach to the Askey scheme of orthogonal polynomials*, in *Orthogonal polynomials and special functions*, Lecture Notes in Mathematics, vol. 1883 (Springer, Berlin, 2006), 255–330.

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