ASYMMETRIC MINIMA OF INDEFINITE TERNARY QUADRATIC FORMS

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1. Introduction and preliminaries

Let $f = f(\boldsymbol{x}) = f(x_1, x_2, \cdots, x_n)$ be an indefinite n-ary quadratic form of determinant det (f); that is, $f(\boldsymbol{x}) = \boldsymbol{x}'A\boldsymbol{x}$ where A is a real symmetric matrix with determinant det (f). Such a form is said to take the value v if there exists integral $\boldsymbol{x} \neq \boldsymbol{0}$ such that $f(\boldsymbol{x}) = v$.

The problem of asymmetric minima is to find, for a given signature s and for each $t \geqq 0$, the value $\phi_n^s(t)$, defined to be the infimum of the set of all positive α such that every normalised form f (that is, every form with $d(f) = |\det (f)| = 1$) takes a value in the closed interval $[-\alpha, t\alpha]$. The value $\phi_n^s(t)$ is thus a measure of the size of the least closed interval $I = [-a, b]$ containing the origin and with asymmetry $b/a = t$, such that every normalised form f takes a value in any open interval containing I.

By considering $-f$ in place of f it is easy to see that $\phi_n^s(t)$ need only be evaluated for $s \geqq 0$, except when $t = 0$, in which case it is necessary to consider separately the intervals $[-a, 0]$ and $[0, b]$.

For $t = 1$, $n = 3$, the value of s is irrelevant and the problem reduces to the symmetric minimum problem for indefinite ternary quadratic forms. In this case the solution is known—Markoff [5] has shown that

$$\phi_3^1(1) = (\tfrac{2}{3})^{\frac{1}{3}}$$

and Venkov [8] has determined the first eleven successive minima.

For $n = 2$, Segre [6] has shown that

$$\phi_2^0(t) \leqq 2(t^2+4t)^{-\frac{1}{2}} \qquad t \geqq 1$$

with equality if and only if t is integral. By considering the relation between $\phi_2^0(t)$ and $\phi_2^0(1/t)$ we can obtain from the above a bound for $\phi_2^0(t)$ for $0 < t < 1$.

Tornheim [7] has shown how to calculate $\phi_2^0(t)$ for any given $t > 0$ in terms of infinite chains $[p_i]$, $-\infty < i < \infty$, of positive integers, and

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simple continued fractions associated with these chains. However $\phi_2^0(t)$ appears to be an extremely complicated function.

It comes therefore as a surprise to find that $t\phi_3^1(t)$ is a continuous piecewise linear function of t. This property of $\phi_3^1(t)$ is a consequence of the following theorem, which is the main result of this paper. A further consequence is that every normalised form f takes a value in the closed interval

$$I = [-\phi_3^1(t), t\phi_3^1(t)],$$

not merely in any open interval containing I.

THEOREM A. *Every normalised indefinite ternary quadratic form of signature* 1 *takes a value in each of the following closed intervals*:

$$\begin{aligned}
&I_0 : [0, \sqrt[3]{\tfrac{4}{3}}]\\
&I_1 : [-\sqrt[3]{\tfrac{1}{48}}, \sqrt[3]{\tfrac{54}{49}}]\\
&I_2 : [-\sqrt[3]{\tfrac{2}{49}}, \sqrt[3]{\tfrac{8}{9}}]\\
&I_3 : [-\sqrt[3]{\tfrac{1}{9}}, \sqrt[3]{\tfrac{125}{144}}]\\
&I_4 : [-\sqrt[3]{\tfrac{3}{16}}, \sqrt[3]{\tfrac{2}{3}}]\\
&I_5 : [-\sqrt[3]{\tfrac{2}{3}}, \sqrt[3]{\tfrac{27}{112}}]\\
&I_6 : [-\sqrt[3]{\tfrac{125}{112}}, \sqrt[3]{\tfrac{2}{9}}]\\
&I_7 : [-\sqrt[3]{\tfrac{16}{9}}, \sqrt[3]{\tfrac{1}{24}}]\\
&I_8 : [-\sqrt[3]{\tfrac{8}{3}}, \sqrt[3]{\tfrac{2}{135}}]\\
&I_9 : [-\sqrt[3]{\tfrac{16}{5}}, 0].
\end{aligned}$$

Furthermore if we define:

$$\begin{aligned}
f_1 &= (x+\tfrac{1}{2}z)^2 && -\tfrac{1}{2}(z^2-2yz-2y^2)\\
f_2 &= (x+\tfrac{1}{6}y+\tfrac{1}{2}z)^2 && -\tfrac{7}{12}(z^2-2yz-\tfrac{5}{3}y^2)\\
f_3 &= (x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2 && -\tfrac{3}{4}(z^2-2yz-\ y^2)\\
f_4 &= (x+\tfrac{4}{5}y+\tfrac{2}{5}z)^2 && -\tfrac{24}{25}(z^2-\ yz-\ y^2)\\
f_5 &= (x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2 && -\tfrac{5}{4}(z^2-\tfrac{6}{5}yz-\tfrac{3}{5}y^2)\\
f_6 &= (x+\tfrac{1}{3}y)^2 && -\tfrac{8}{3}(z^2-\ yz-\tfrac{1}{3}y^2)\\
f_7 &= (x+\tfrac{1}{2}y)^2 && -3(z^2-\ yz-\tfrac{1}{4}y^2)\\
f_8 &= x^2 && -8(z^2-\ yz-\tfrac{1}{8}y^2)\\
f_9 &= (x+\tfrac{1}{2}y)^2 && -15(z^2-\ yz-\tfrac{1}{20}y^2),
\end{aligned}$$

and let F_i, $1 \leqq i \leqq 9$, denote that multiple of f_i which has determinant -1, then for $0 \leqq i \leqq 8$ closure is required on the left of interval I_{i+1} and on the right of interval I_i only for forms equivalent (under an integral unimodular transformation) to F_{i+1}.

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Clearly the closure conditions of this theorem imply that if I is any interval about the origin in which every normalised indefinite ternary quadratic form of signature 1 takes a value then I must contain an interval I_i for some i with $0 \leqq i \leqq 9$. Thus in particular for every $t \geqq 0$ the interval $[-\phi_3^1(t), t\phi_3^1(t)]$ must have an end-point in common with an interval I_i. From this it follows that as t increases from zero, $\phi_3^1(t)$ and $t\phi_3^1(t)$ remain fixed alternately, so that the graph of $t\phi_3^1(t)$ is piecewise linear and continuous. Thus if we let $I_i = [-\alpha_i, \beta_i]$, we have that

$$\phi_3^1(t) = \begin{cases} \min\limits_{0 \leq i \leq 9} \{\max(\alpha_i, \beta_i/t)\} : t > 0 \\ \alpha_9 \qquad\qquad\qquad\qquad : t = 0, \end{cases}$$

with a similar expression for $\phi_3^{-1}(t)$.

It is of interest to note that the forms f_i have rational coefficients. The following table gives $m_-(f_i)$ and $d(f_i)$, while $m_+(f_i) = 1$ for all i, where $m_+(f)$ and $m_-(f)$ denote the infimum of the positive values taken by f and $-f$ respectively.

TABLE 1.1

i	1	2	3	4	5	6	7	8	9
$m_-(f_i)$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$	1	$\frac{5}{3}$	2	4	6
$d(f_i)$	$\frac{3}{4}$	$\frac{49}{54}$	$\frac{9}{8}$	$\frac{144}{125}$	$\frac{3}{2}$	$\frac{112}{27}$	$\frac{9}{2}$	24	$\frac{135}{2}$

For the proof of this theorem we shall make extensive use of the properties of reduced binary quadratic forms and the continued fractions associated with them [2]. Given an indefinite binary quadratic form $q(x, y)$ not taking the value 0 and with $d(q) > 0$ the chain of reduced forms equivalent (under an integral unimodular transformation) to q is denoted by (q_i), where

$$q_i(x, y) = (-1)^i a_i x^2 + b_i xy + (-1)^{i+1} a_{i+1} y^2, \; -\infty < i < \infty,$$

and $a_i > 0$, $b_i > 0$ for all i. Associated with this chain there is a doubly infinite chain $[p_i]$ of positive integers p_i. Defining, for each i,

$$F_i = \frac{\Delta + b_i}{2a_{i+1}}, \quad S_i = \frac{\Delta - b_i}{2a_{i+1}}$$

(where $\Delta^2 = D = \frac{1}{4}d(q)$ is the discriminant of q), we then have, in the usual notation for simple continued fractions, that

[2] For this classical theory of reduced forms see G. Frobenius, Sitz.-Ber. Preuss. Akad. Wiss. Berlin (1913) 202–211.

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$$F_i = (p_i, p_{i+1}, p_{i+2}, \cdots), \quad S_i = (0, p_{i-1}, p_{i-2}, \cdots).$$

We also have that

$$\begin{aligned} q_i(x, y) &= (-1)^{i+1} a_{i+1}[y^2 + (-1)^{i+1}(F_i - S_i)xy - F_i S_i x^2] \\ &= (-1)^{i+1} a_{i+1}(y \pm F_i x)(y \mp S_i x). \end{aligned}$$

In addition, if we set $K_i = F_i + S_i$, then

$$a_{i+1} K_i = \Delta.$$

2. The forms F_i

In this section we consider the special forms F_i and show that the closure conditions of the intervals I_i are necessary. The forms F_i are considered in separate lemmas, each giving $m_+(F_i)$ and $m_-(F_i)$ for some i.

LEMMA 2.1. *$m_+(F_1) = \sqrt[3]{\frac{4}{3}}$ and $m_-(F_1) = \sqrt[3]{\frac{1}{48}}$.*

This follows from the work of Barnes [1].

LEMMA 2.2. *$m_+(F_2) = \sqrt[3]{\frac{54}{49}}$ and $m_-(F_2) = \sqrt[3]{\frac{2}{49}}$.*

PROOF. $F_2 = \sqrt[3]{\frac{54}{49}}\{(x+\frac{1}{6}y+\frac{1}{2}z)^2 - \frac{7}{12}(z^2-2yz-\frac{5}{3}y^2)\}$. For the proof we consider the integral form

$$\begin{aligned} G_2(x, y, z) &= 3\sqrt[3]{\tfrac{49}{54}}\, F_2(x-4z, y, z) \\ &= 3(x-y)^2 - 21xz + 35z^2 + 7xy. \end{aligned}$$

Then we must prove that $m_+(G_2) = 3$ and $m_-(G_2) = 1$. As G_2 takes the values 3 and -1, and as taking congruences mod 7 shows G_2 cannot take the values 1 or 2, we only need to show that G_2 cannot take the value 0.

Suppose to the contrary that $G_2\,(x, y, z) = 0$ has a non-trivial solution. Then there exist relatively prime X, Y, Z with

$$3(X-Y)^2 - 21XZ + 35Z^2 + 7XY = 0.$$

This implies that $X \equiv Y \pmod 7$. Setting $X = Y + 7t$ and taking congruences mod 49 yields that

$$(Y+2Z)^2 + Z^2 \equiv 0 \pmod 7.$$

This can have only the solution $Y+2Z \equiv Z \equiv 0 \pmod 7$, which implies that X, Y, Z are not relatively prime, contrary to our initial assumption. This contradiction shows that G_2 cannot take the value 0 and completes the proof of the lemma.

LEMMA 2.3. *$m_+(F_3) = \sqrt[3]{\frac{8}{9}}$ and $m_-(F_3) = \sqrt[3]{\frac{1}{9}}$.*

PROOF. $F_3 = \sqrt[3]{\frac{8}{9}}\{(x+\frac{1}{2}y+\frac{1}{2}z)^2 - \frac{3}{4}(z^2-2yz-y^2)\}$. For the proof we consider the integral form

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$$\begin{aligned} G_3(x, y, z) &= 2\sqrt[3]{\tfrac{9}{8}}\, F_3(x-y, y, x+y-z) \\ &= 3x^2+3y^2-z^2. \end{aligned}$$

Then we must prove that $m_+(G_3) = 2$ and $m_-(G_3) = 1$. As G_3 clearly takes the values 2 and -1, and as taking congruences mod 3 eliminates the value 1, we only need to show that G_3 does not take the value 0. This is trivial, as the equation

$$3X^2+3Y^2-Z^2 = 0$$

is insoluble in relatively prime integers mod 9.

LEMMA 2.4. $m_+(F_4) = \sqrt[3]{\tfrac{125}{144}}$ *and* $m_-(F_4) = \sqrt[3]{\tfrac{3}{16}}$.

PROOF. $F_4 = \sqrt[3]{\tfrac{125}{144}}\{(x+\tfrac{4}{5}y+\tfrac{2}{5}z)^2-\tfrac{24}{25}(z^2-yz-y^2)\}$. For the proof we consider the integral form

$$\begin{aligned} G_4(x, y, z) &= 5\sqrt[3]{\tfrac{144}{125}}\, F_4(x, y, z) \\ &= 5x^2+8xy+4xz+8yz+8y^2-4z^2. \end{aligned}$$

Then we must show that $m_+(G_4) = 5$ and $m_-(G_4) = 3$. As G_4 clearly takes the values 5 and -3, and as

$$G_4 \equiv 5(x+2z)^2 \equiv 0,5 \quad \text{or} \quad 4 \qquad (\text{mod } 8)$$

and

$$G_4 \equiv 2(x+2y-2z)^2 \equiv 0 \quad \text{or} \quad 2 \quad (\text{mod } 3)$$

it is clear that we only have to show that G_4 does not take the value 0.

Suppose to the contrary that there exist relatively prime X, Y, Z with $G_4(X, Y, Z) = 0$. Then taking congruences mod 2 shows that $X = 2t$ for some integer t, and so

$$G_4(X, Y, Z) \equiv 4(t+Z)^2+8(Y^2+YZ+Z^2) \quad (\text{mod } 16),$$

which is impossible, as at least one of Y and Z must be odd for X, Y, Z to be relatively prime.

LEMMA 2.5. $m_+(F_5) = m_-(F_5) = \sqrt[3]{\tfrac{2}{3}}$.
This follows from the work of Markoff [4].

LEMMA 2.6. $m_+(F_6) = \sqrt[3]{\tfrac{27}{112}}$ *and* $m_-(F_6) = \sqrt[3]{\tfrac{125}{112}}$.

PROOF. $F_6 = \sqrt[3]{\tfrac{27}{112}}\{(x+\tfrac{1}{3}y)^2-\tfrac{8}{3}(z^2-yz-\tfrac{1}{3}y^2)\}$. For the proof we consider the integral form

$$\begin{aligned} G_6(x, y, z) &= 3\sqrt[3]{\tfrac{112}{27}}\, F_6(x+y, y, z) \\ &= 3x^2+8xy+8y^2-8z^2+8yz. \end{aligned}$$

Then we must show that $m_+(G_6) = 3$ and $m_-(G_6) = 5$. As G_6 takes the values 3 and -5, and as taking congruences mod 8 shows that G_6 cannot

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take the values ± 1, ± 2 and -3, we only need to show that G_6 cannot take the values 0 or -4.

Suppose that $G_6(X, Y, Z) = -4$. Then taking congruences mod 8 shows that $X = 2t$ for some integer t. Hence

$$-1 = \tfrac{1}{4}G_6(X, Y, Z) \equiv 3+2(Y^2+YZ+Z^2) \pmod 4$$

which implies that $Y = 2s$ and $Z = 2r$ for some integers r and s. However this implies that $G_6(t, s, r) = -1$, which we know is impossible.

Suppose that $G_6(X, Y, Z) = 0$ for relatively prime X, Y, Z. Then taking congruences mod 8 shows that $X = 4t$ for some integer t. Hence

$$0 = \tfrac{1}{8}G_6(X, Y, Z) \equiv Y^2+YZ+Z^2 \pmod 2,$$

which is impossible as at least one of Y and Z must be odd for X, Y, Z to be relatively prime. This shows that G_6 cannot take the values 0 or -4, and completes the proof of the lemma.

LEMMA 2.7. *$m_+(F_7) = \sqrt[3]{\frac{2}{9}}$ and $m_-(F_7) = \sqrt[3]{\frac{16}{9}}$.*

PROOF. $F_7 = \sqrt[3]{\frac{2}{9}}\{(x+\frac{1}{2}y)^2-3(z^2-yz-\frac{1}{4}y^2)\}$. For the proof we consider the integral form

$$\begin{aligned} G_7(x, y, z) &= \sqrt[3]{\tfrac{9}{2}}\, F_7(x+y, y, z) \\ &= x^2+3xy+3y^2-3z^2+3yz. \end{aligned}$$

Then we must show that $m_+(G_7) = 1$ and $m_-(G_7) = 2$. As G_7 takes the values 1 and -2, and as taking congruences mod 3 shows that G_7 cannot take the value -1, we only need to show that G_7 cannot take the value 0.

Suppose to the contrary that $G_7(X, Y, Z) = 0$ for relatively prime X, Y, Z. Then taking congruences mod 3 shows that $X = 3t$ for some integer t, and so

$$0 = \tfrac{1}{3}G_7(X, Y, Z) \equiv Y^2+YZ-Z^2 \pmod 3.$$

However this is impossible as at least one of Y and Z must not be divisible by 3 for X, Y, Z to be relatively prime.

LEMMA 2.8. *$m_+(F_8) = \sqrt[3]{\frac{1}{24}}$ and $m_-(F_8) = \sqrt[3]{\frac{8}{3}}$.*

PROOF. $F_8 = \sqrt[3]{\frac{1}{24}}\{x^2-8(z^2-yz-\frac{1}{8}y^2)\}$. For the proof we consider the integral form

$$\begin{aligned} G_8(x, y, z) &= \sqrt[3]{24}\, F_8(x, y, z) \\ &= x^2+y^2+8yz-8z^2. \end{aligned}$$

Then we must show that $m_+(G_8) = 1$ and $m_-(G_8) = 4$. As G_8 takes the values 1 and -4, and as taking congruences mod 8 shows that G_8 cannot

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take the values -2 or -1, we only need to show that G_8 cannot take the values -3 or 0.

Suppose to the contrary that there exist relatively prime X, Y, Z with $G_8(X, Y, Z) = 0$ or -3. Then

$$X^2+(Y+4Z)^2 \equiv 0 \pmod 3$$

and so $X = 3t$, $Y+4Z = 3s$ for some integers t and s. Then

$$1 \text{ or } 0 = -\tfrac{1}{3}G_8(X, Y, Z) \equiv 8Z^2 \pmod 3,$$

which is impossible as Z cannot be divisible by 3 if X, Y, Z are to be relatively prime.

LEMMA 2.9. *$m_+(F_9) = \sqrt[3]{\tfrac{2}{135}}$ and $m_-(F_9) = \sqrt[3]{\tfrac{16}{5}}$.*

This follows from the work of Barnes and Oppenheim [2].

3. The method of proof of theorem A

We first break down the theorem into ten sub-theorems which when combined together are equivalent to theorem A. Each of these sub-theorems takes the following form for some i, $0 \leqq i \leqq 9$, where a_i, b_i, I_i and F_i are as in the statement of theorem A.

THEOREM A_i. *Every normalised indefinite ternary quadratic form of signature* 1 *takes a value in the closed interval*

$$I_i = [-\sqrt[3]{a_i}, \sqrt[3]{b_i}].$$

Furthermore (for $0 \leqq i \leqq 8$) closure is required on the right only for forms equivalent to F_{i+1}, and (for $1 \leqq i \leqq 9$) closure is required on the left only for forms equivalent to F_i.

We now take the theorems A_i and reduce them to a form in which they are more easily proven. Consider, for $1 \leqq i \leqq 8$, in place of theorem A_i the theorem B_i as follows.

THEOREM B_i. *If g is any indefinite ternary quadratic form of signature* 1 *with $d(g) = d$ where*

$$0 < d \leqq 1/b_i,$$

and if $m_+(g) = 1$ then either

$$m_-(g) < \sqrt[3]{a_i d}$$

or g is equivalent to a multiple of either F_i or F_{i+1}.

It is easily seen that theorem A_i follows from theorem B_i, for if f is any normalised form with $m_+(f) = m$, then

(a) If $0 \leqq m < \sqrt[3]{b_i}$, f clearly takes a value in the interior I_i^0 of I_i.

(b) If $m \geqq \sqrt[3]{b_i}$ consider the form $g(x, y, z) = f(x, y, z)/m$. This has

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$$d(g) = 1/m^3 \leqq 1/b_i,$$

and applying theorem B_i gives that either

(i) $m_-(g) < \sqrt[3]{a_i d}$, from which it follows that $m_-(f) < \sqrt[3]{a_i}$, and so f takes a value in I_i^0, or

(ii) g is equivalent to a multiple of either F_i or F_{i+1}, from which it follows, on comparing determinants, that f is equivalent to either F_i or F_{i+1}.

(c) The closure conditions follow automatically from the results of § 2.

Thus if we can establish theorems A_0 and A_9 and prove theorems $B_1, B_2, \cdots, B_8$ we will have proved theorem A. As

(i) Theorem A_0 follows from the results of Barnes [1] on observing that if $Q_1(x, y, z) = -x^2+8(y^2+yz+z^2)$ then

$$F_1(x, y, z) = \tfrac{1}{4}Q_1(z-2x-2y, x, y)\sqrt[3]{\tfrac{4}{3}},$$

and

(ii) Theorem A_9 follows from the results of Barnes and Oppenheim [2] on observing that if $Q_2(x, y, z) = -x^2-xy-y^2+90z^2$ then

$$F_9(x, y, z) = -Q_2(x-5z, y+10z, -z)\sqrt[3]{\tfrac{2}{135}},$$

it is sufficient to prove theorems $B_1, B_2, \cdots, B_8$.

The proof of theorem B_i is simplified by the use of the following result on the approximation of indefinite ternary quadratic forms.

THEOREM 3.1. *Let f be an indefinite ternary quadratic form of signature* 1 *such that both $m_+(f)$ and $m_-(f)$ are non-zero. Then if f does not attain the value $m_+(f)$ we can associate with f another indefinite ternary quadratic form f' with the following properties.*

(i) $\det(f') = \det(f)$.

(ii) $m_+(f') = m_+(f)$; $m_-(f') \geqq m_-(f)$.

(iii) *f' attains the value $m_+(f)$.*

(iv) *f' is not a multiple of a form with integral coefficients.*

PROOF. As $m_+(f)$ is not attained by f we can find, for each integer $n \geqq 2$, relatively prime x_n, y_n, z_n such that

$$m_+(f) < f(x_n, y_n, z_n) \leqq \left(1+\frac{1}{n}\right) m_+(f).$$

Let $f(x_n, y_n, z_n) = (1+\delta_n)m_+(f)$. Then there exists a form g_n equivalent to f such that

$$g_n = m_+(f)(1+\delta_n)[(x+\lambda_n y+\mu_n z)^2+q_n(y, z)].$$

Now q_n is an indefinite binary quadratic form which cannot take a value in the open interval

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$$(3.1) \qquad (-m_-(f)/2m_+(f), \tfrac{1}{4})$$

as otherwise by choosing x suitably we could obtain a value of g_n, and hence of f, contradicting the definition of either $m_+(f)$ or $m_-(f)$. Hence we can choose a reduced form, say $c_n y^2 + d_n yz + e_n z^2$, from the chain of reduced forms equivalent to q_n. Then by passing to an equivalent form, using the notation $f \sim g$ to denote that f is equivalent to g, we have

$$g_n \sim h_n = m_+(f)(1+\delta_n)[(x+\alpha_n y+\beta_n z)^2 + c_n y^2 + d_n yz + e_n z^2]$$

where we may assume without loss of generality that

$$|\alpha_n| \leqq \tfrac{1}{2}, \ |\beta_n| \leqq \tfrac{1}{2}.$$

Clearly as q_n cannot take a value in the open interval (3.1) both $|c_n|$ and $|e_n|$ must be bounded away from zero by $\min\{\frac{1}{4}, m_-(f)/2m_+(f)\}$. Then as

$$(3.2) \qquad 4d(f) = (1+\delta_n)^3(m_+(f))^3(d_n + 4|c_n e_n|)$$

it is clear that the sequences $\{c_n\}$, $\{d_n\}$ and $\{e_n\}$ are bounded. As $\{\alpha_n\}$ and $\{\beta_n\}$ are also bounded sequences we can choose a subsequence $\{\gamma_n\}$ of $\{1/n\}$ such that the corresponding subsequences of $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ converge to limits c, d, e, α and β respectively. We shall show that

$$f' = m_+(f)[(x+\alpha y+\beta z)^2 + cy^2 + dyz + ez^2]$$

has the desired properties.

By taking limits of the subsequences corresponding to $\{\gamma_n\}$ in (3.2) we have that

$$4d(f) = (m_+(f))^3(d^2 + 4|ce|).$$

Then property (i) follows as the right hand side of this equation is $-4 \det(f')$ and as f' must clearly have signature 1.

Property (iii) is trivial.

Property (ii) clearly follows on showing that f takes values arbitrarily close to any value taken by f'. If f' takes the value v at X, Y, Z let $B = \max(|X|, |Y|, |Z|)$. From the definition of c, d, e, α and β it is clear that for any $\sigma > 0$ we can choose m such that the coefficients of x^2, y^2, z^2, xy, xz and yz in h_m differ from the corresponding coefficients in f' by at most σ.

Then

$$|h_m(X, Y, Z) - f'(X, Y, Z)| = |h_m(X, Y, Z) - v| \leqq 6\sigma B^2,$$

and as $h_m \sim f$ it is clear that f takes values arbitrarily close to any value v taken by f'.

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Using the notation that f is in the ε-neighbourhood (abbreviated nhd) $N_g(\varepsilon)$ of g if the coefficients of x^2, y^2, z^2, xy, xz and yz in f differ by at most ε from the corresponding coefficients in g, then we have seen above that for any $\varepsilon > 0$ we can choose m such that h_m is in $N_{f'}(\varepsilon)$.

In order to show that f' cannot be a multiple of a form with integral coefficients we refer to the results of Cassels and Swinnerton-Dyer [**3**] concerning the isolation of indefinite ternary quadratic forms with integral coefficients. This result is that if g is such a form and (μ, η) is any open interval there exists a nhd $N_g(\varepsilon)$ such that any form lying in $N_g(\varepsilon)$, not a multiple of g, takes a value in (μ, η). If we assume that kf' is integral for some number k and take $(\mu, \eta) = (0, \frac{1}{2}km_+(f))$ then the above isolation theorem shows that there exists $N_{f'}(\varepsilon)$ such that every form g in $N_{f'}(\varepsilon)$ with $m_+(g) \geqq \frac{1}{2}m_+(f)$ is a multiple of f'. As there exists m such that h_m is in $N_{f'}(\varepsilon)$, h_m, and thus f, must be equivalent to a multiple of f'. However this implies, using properties (ii) and (iii), that f takes the value $m_+(f)$, in contradiction to the given. This shows property (iv) and completes the proof of the theorem.

We may now simplify the theorems B_i as follows. Suppose that theorem B_i is false. Then there exists a form g of signature 1, with $d(g) = d$ where $0 < d \leqq 1/b_i$, with $m_+(g) = 1$, such that g is not equivalent to a multiple of F_i or F_{i+1} and such that

$$m_-(g) \geqq \sqrt[3]{a_i d}.$$

If $m_+(g)$ is not attained by g then by the above theorem there exists g', not a multiple of an integral form (and hence not equivalent to a multiple of F_i or F_{i+1}), with $d(g') = d$, $m_+(g') = 1$ attained by g', and such that $m_-(g') \geqq m_-(g) \geqq \sqrt[3]{a_i d}$. Hence if theorem B_i is false it still remains false if we insert the extra condition that $m_+(g)$ is attained by g.

Let theorem C_i denote theorem B_i with this extra assumption. Then clearly theorem B_i will follow once we have established theorem C_i.

For the proofs of theorems C_i we use a chain of forms (g_i), $-\infty < i < \infty$, equivalent to and associated with a given indefinite ternary form f.

Let f be an indefinite ternary quadratic form of signature 1 taking the value $m_+(f) = 1$. Then we can find an equivalent form

$$g = (x+\lambda y+\mu z)^2+q(y, z).$$

Now q is an indefinite binary quadratic form with $d(q) = d(f) \neq 0$, and it cannot take a value in the open interval

$$(-m_-(g)-\tfrac{1}{4}, \tfrac{3}{4}) \tag{3.3}$$

as otherwise we could choose x suitably to obtain a contradiction to the

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definition of either $m_+(g) = 1$ or $m_-(g)$. Hence there exists a chain of reduced forms

$$(3.4) \qquad q_i = (-1)^i a_i y^2 + b_i yz + (-1)^{i+1} a_{i+1} z^2, \; -\infty < i < \infty,$$

each equivalent to q. By applying a suitable $y-z$ transformation we may replace $q(y, z)$ in g by any one of the $q_i(y, z)$ giving

$$g_i' = (x+\alpha_i y+\beta_i z)^2 + q_i(y, z)$$

equivalent to f. Then by changing the sign of y if necessary and by applying a suitable parallel transformation to x we obtain a chain of forms

$$g_i = (x+\lambda_i y+\mu_i z)^2 + (-1)^{i+1} a_{i+1}(z-F_i y)(z+S_i y)$$

with $|\lambda_i| \leqq \frac{1}{2}$ and $|\mu_i| \leqq \frac{1}{2}$ such that each form g_i of the chain is equivalent to f. We shall call such a chain an "equivalence chain" for f. It should be noted that there may be a number of distinct equivalence chains for a given f, depending on the initial choice of g.

4. The proof of theorem C_1

The proof makes use of the following results.

LEMMA 4.1. *Let $k \geqq 2$ be integral and let q be an indefinite binary quadratic form. Define*

$$\begin{aligned} A &= [k^2+k+(3k-1)\sqrt{k^2+4k}]/(4k-2), \\ B &= \min\,(4k^2, k^2+6k+1), \\ d &= \min\,\{A^2 m_-^2/4k^2,\; Bm_+^2/4,\; Bm_-^2/4k^2\} \end{aligned}$$

where $m_+ = m_+(q)$ and $m_- = m_-(q)$. Then either q is equivalent to a multiple of $x^2-kxy-ky^2$ or $d(q) \geqq d$.

PROOF. The proof of this result depends on the work of Tornheim [7]. Put

$$Q(x, y) = q(x, y)/2\sqrt{d(q)}$$

so that Q has discriminant $\Delta^2 = 1$ and let

$$(4.1) \qquad \begin{cases} M = m_+(Q) = m_+(q)/2\sqrt{d(q)} \\ N = m_-(Q) = m_-(q)/2\sqrt{d(q)} \\ P = \max\,(1/M, k/N). \end{cases}$$

Then Tornheim has shown that either

(a) $P \geqq 2k$, or

(b) $P = \sqrt{k^2+4k}$ and Q is equivalent to

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$$M(x^2-kxy-ky^2) = N(x^2-kxy-ky^2)/k,$$

or

(c) from the proof of lemma 7 of his paper,

$$N \leqq k/A,$$

or

(d) from his lemmas 8 and 10 the chain of p_i for Q contains at least one $p_i \geqq k+1$ and

$$P \geqq \sqrt{k^2+6k+1}.$$

Now (a) and (d) give that

$$1/M \text{ or } k/N \geqq \min(2k, \sqrt{k^2+6k+1}) = \sqrt{B}$$

from which, using (4.1), we have that either

$$d(q) \geqq m_+^2 B/4 \geqq d,$$

or

$$d(q) \geqq m_-^2 B/4k^2 \geqq d.$$

Similarly (c) gives that

$$d(q) \geqq m_-^2 A^2/4k^2 \geqq d.$$

The lemma now follows on observing that the alternative (b) implies that q is equivalent to a multiple of $x^2-kxy-ky^2$.

LEMMA 4.2. *Both* $h_1(x) = x^3-\frac{1}{16}(x+\frac{1}{4})^2$ *and* $h_2(x) = x^3-\frac{1}{18}(x+\frac{1}{4})^2$ *have only one real root.*

PROOF. Evaluation of the roots of the derivatives of h_1 and h_2 shows that these roots are at most $\frac{1}{8}$ in absolute value. Then h_1 and h_2 are negative at these points, and so their graphs have both turning points below the x-axis. This implies that $h_1(x)$ and $h_2(x)$ have only one real root.

We are now in a position to prove theorem C_1 which for reference is re-stated.

If g is any indefinite ternary quadratic form of signature 1, *with* $d(g) = d$ *where* $0 < d \leqq \frac{49}{54}$, *and if* $m_+(g) = m_+ = 1$ *is attained by g then either*

(a) $m_-(g) < \sqrt[3]{d/48}$, or

(b) *g is equivalent to a multiple of either F_1 or F_2.*

As indicated at the end of §3 we consider in place of g an equivalence chain (g_i) of forms equivalent to g. We have

$$g_i = (x+\lambda_i y+\mu_i z)^2+(-1)^{i+1}a_{i+1}(z-F_i y)(z+S_i y)$$

where as indicated at the end of § 1

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$$a_{i+1}K_i = \Delta; \quad \Delta^2 = 4d. \tag{4.2}$$

As $(-1)^{i+1}a_{i+1}(z-F_i y)(z+S_i y)$ cannot take any values in the interval (3.3) we have, assuming that

$$m_-(g) = m_- \geqq \sqrt[3]{d/48}, \tag{4.3}$$

the following:

$$a_i \geqq \tfrac{3}{4} \qquad (i \text{ even}), \tag{4.4}$$

$$a_i \geqq m_- + \tfrac{1}{4} \geqq \sqrt[3]{d/48} + \tfrac{1}{4} \quad (i \text{ odd}). \tag{4.5}$$

Using (4.2) and setting

$$d = 49\beta/54, \qquad 0 < \beta \leqq 1 \tag{4.6}$$

we obtain

$$K_i = 7\sqrt{6\beta}/9a_{i+1}.$$

Then using the bounds (4.4) and (4.5) we find that

$$K_i \leqq 28\sqrt{6\beta}/27 < 2.5403\sqrt{\beta} \qquad (i \text{ odd}), \tag{4.7}$$

$$K_i \leqq 7\sqrt{6\beta}[9\sqrt[3]{\beta}(\tfrac{1}{4}+\sqrt[3]{\tfrac{49}{2592}}]^{-1} < 3.6893\sqrt[6]{\beta} \quad (i \text{ even}). \tag{4.8}$$

As $p_i < F_i < K_i$ we conclude that

$$p_i \leqq 2 \quad (i \text{ odd}); \qquad p_i \leqq 3 \quad (i \text{ even}).$$

The proof is now presented as a series of lemmas, each eliminating various possibilities for combinations of p_i occurring in the chain $[p_i]$. In these lemmas the following property will be used.

If the sequence $(r, s, \cdots, t) = (p_i, p_{i+1}, \cdots, p_{i+j})$ cannot occur in the chain $[p_i]$ then neither can the sequence

$$(t, \cdots, s, r) = (p_{k-j}, \cdots, p_{k-1}, p_k)$$

where $k \equiv i \pmod 2$.

This follows from the fact that replacing y by $-y$ reverses the order of the chain $[p_i]$ without affecting the values taken by the form.

For simplicity, λ and μ will replace λ_i and μ_i in the local considerations of the chain $[p_i]$ in the following work.

LEMMA 4.3. *The chain cannot contain either $p_i = 3$ with i even or $p_i = 2$ with i odd.*

PROOF. Let $p_i = 2$ with i odd and suppose that one of p_{i-1}, p_{i+1} is not 3. Then

$$K_i > 2 + (0, 2, 1) + (0, 3, 1) = 2\tfrac{7}{12}$$

which contradicts (4.7). Thus if $p_i = 2$ with i odd then $p_{i-1} = p_{i+1} = 3$.

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Let $p_i = 3$ with i even and suppose that one of p_{i+1}, p_{i-1} is not 2. Then

$$K_i > 3+(0, 1, 1)+(0, 2, 1) = 3\tfrac{5}{6}$$

which contradicts (4.8). Thus we must have $p_{i-1} = p_{i+1} = 2$, and so $p_{i-2} = p_{i+2} = 3$. Then

$$K_i > 3+2(0, 2, 3, 3) = \tfrac{89}{23}$$

which again contradicts (4.8). As $p_i = 3$ (i even) leads to a contradiction and $p_i = 2$ (i odd) implies that $p_{i+1} = 3$ lemma 4.3 follows.

From this lemma we can conclude that

$$p_i = 1 \quad (i \text{ odd}); \qquad p_i \leqq 2 \quad (i \text{ even}).$$

LEMMA 4.4. *The chain cannot have* $p_{i-1} = p_{i+1} = 1$ *where* i *is odd.*

PROOF. Suppose that $p_{i-1} = p_{i+1} = 1$ with i odd. Then

$$F_i \geqq (1, 1, \overline{1, 2}) = 1+1/\sqrt{3} > 1.57735.\text{[3]} \tag{4.9}$$

Similarly $S_i > .57735$, and so $K_i > 2.1547$. Using (4.7) we can obtain that $\beta > .71944$ and combining this with (4.3) and (4.6) we find that

$$m_- > .2386. \tag{4.10}$$

Now

$$\begin{aligned} F_i &\leqq (1, 1, \overline{1}) = (\sqrt{5}+1)/2 \quad \text{and} \\ S_i &\leqq (0, 1, \overline{1}) = (\sqrt{5}-1)/2. \end{aligned} \tag{4.11}$$

Using these bounds together with the lower bounds (4.9) we obtain that

$$\begin{aligned} .91068 &< F_i S_i \leqq 1 \\ .91068 &< (F_i-1)(S_i+1) \leqq 1. \end{aligned} \tag{4.12}$$

In addition we have, with regard to (4.2), (4.4) and (4.6), that

$$.75 \leqq a_{i+1} = 7\sqrt{6\beta}/9K_i < .8844.$$

Suppose, contrary to what we wish to prove, that $a_{i+1} \leqq .81$. Then as $m_+ = 1$, choosing x so that $(x+\mu)^2 \leqq \frac{1}{4}$, it is clear that we must have the value $(x+\mu)^2+a_{i+1} \geqq 1$. Therefore

$$(x+\mu)^2 \geqq 1-a_{i+1} \geqq .19.$$

This implies that

$$||\mu-\tfrac{1}{2}|| < .0642, \tag{4.13}$$

[3] The symbol $(1, 1, \overline{1})$ is used to denote the continued fraction $(1, 1, 1, \cdots, 1, \cdots)$. This usage extends to the symbol $(p, \cdots, q, \overline{r, \cdots, s})$ in the obvious way.

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where $||t||$ denotes the distance from t to the nearest integer. Choosing x so that $\frac{1}{4} \leqq (x+\lambda)^2 \leqq 1$ gives g_i the value $(x+\lambda)^2 - a_{i+1}F_iS_i$ which is less than 1. Thus

$$(x+\lambda)^2 \leqq a_{i+1}F_iS_i - m_-.$$

Then using (4.10) and (4.12) gives that

$$(x+\lambda)^2 < .5714$$

and so

$$||\lambda-\tfrac{1}{2}|| < .256.$$

Combining this with (4.13) yields that

$$||\lambda-\mu|| < 3202,$$

so we can choose x such that

$$(x+\lambda-\mu)^2 < .103.$$

However using the bounds (4.9) and (4.11) we find that

$$a_{i+1}(1+F_i)(1-S_i) < .8963,$$

giving

$$(x+\lambda-\mu)^2 + a_{i+1}(1+F_i)(1-S_i) < .9993.$$

This is a value of g_i contradicting $m_+ = 1$, and shows that we cannot have $a_{i+1} \leqq .81$. Thus we have

$$.81 < a_{i+1} < .8844. \tag{4.14}$$

In the following values of g_i we choose x such that the square lies between 1 and 2.25 inclusive:

$$(x+\lambda)^2 - a_{i+1}F_iS_i,$$
$$(x+\lambda+\mu)^2 - a_{i+1}(F_i-1)(S_i+1).$$

Equations (4.12) show that these values are non-negative, so they must be at least 1. Thus

$$(x+\lambda)^2 \geqq 1 + a_{i+1}F_iS_i. \tag{4.15}$$

Then using (4.12) and (4.14) we have $(x+\lambda)^2 > 1.73856$, which yields that $||\lambda-\frac{1}{2}|| < .182$. Similarly $||\lambda+\mu-\frac{1}{2}|| < .182$. Thus $||\mu|| < .364$, so we can choose x such that

$$(x+\mu)^2 < (.364)^2 < .1325.$$

In order that the value $(x+\mu)^2 + a_{i+1}$ shall not contradict $m_+ = 1$, we must have $a_{i+1} > .8675$. Using this instead of (4.14) in (4.15) and repeating the argument gives that $||\mu|| < .326$, so we can choose x such that

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$$(x+\mu)^2+a_{i+1} < (.326)^2+.8844 < 1.$$

This contradicts $m_+ = 1$ and completes the proof of the lemma.

LEMMA 4.5. *The chain cannot have $p_{i-3} = p_{i-1} = 2$, $p_{i+1} = 1$ where i is odd.*

PROOF. Suppose to the contrary that such an i was in the chain. Then the previous lemma implies that $p_{i+3} = 2$, and so

$$F_{i-1} = (2, 1, 1, 1, 2, 1, \cdots) \geqq (\overline{2, 1, 1, 1}) > 2.6329,$$
$$S_{i-1} = (0, 1, 2, 1, \cdots) \geqq (0, \overline{1, 2, 1, 1}) > .7247.$$

Thus $K_{i-1} > 3.3576$. Using (4.2), (4.3) and (4.5) we find that

$$m_- \geqq \sqrt[3]{K_i^2(m_-+\tfrac{1}{4})^2/3}/4,$$

and inserting the above bound for K_i gives that

$$m_- > \sqrt[3]{3.7578(m_-+\tfrac{1}{4})^2}/4.$$

By iterating on this, commencing with $m_- \geqq 0$, we eventually obtain that $m_- > .242$.

The following bounds on F_i and S_i may be easily obtained.

$$1.57735 < (1, 1, \overline{1, 2}) \leqq F_i \leqq (\overline{1, 1, 1, 2}) < 1.580,$$
$$.366 < (0, 2, \overline{1, 2}) \leqq S_i \leqq (0, 2, \overline{1, 2, 1, 1}) < .36702.$$

Then $K_i > 1.9433$, and using (4.2) and (4.6) we can deduce that $a_{i+1} < .9804$. Combining this with the bounds for F_i and S_i yields that

$$\begin{aligned} &a_{i+1}F_iS_i < .5686, \\ &a_{i+1}(1+3F_i)(3S_i-1) < .5686. \end{aligned} \tag{4.16}$$

Choosing x with $\frac{1}{4} \leqq (x+\lambda)^2 \leqq 1$ gives, by the same method as in the previous lemma, that

$$(x+\lambda)^2 \leqq a_{i+1}F_iS_i-m_-.$$

Using the above bounds for m_- and $a_{i+1}F_iS_i$ gives that

$$(x+\lambda)^2 < .3266 < (.5716)^2,$$

and so $||\lambda-\frac{1}{2}|| < .0716$. Similarly we can prove that $||3\lambda-\mu-\frac{1}{2}|| < .0716$, and so $||4\lambda-\mu|| < .1432$. Now

$$a_{i+1}(7.309)(.464) < a_{i+1}(1+4F_i)(4S_i-1) < 3.36,$$

and we can choose x such that $3.4 < (x+4\lambda-\mu)^2 \leqq 4$. This gives a positive value

$$(x+4\lambda-\mu)^2-a_{i+1}(1+4F_i)(4S_i-1)$$

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of g_i, so in order not to contradict $m_+ = 1$ we must have

$$a_{i+1}(7.309)(.464) \leqq 3.$$

Thus $a_{i+1} < .8847$. This enables us to revise the bounds in (4.16), and repeating the analysis yields that $||\lambda-\frac{1}{2}|| < .021$ and that $||3\lambda-\mu-\frac{1}{2}|| < .021$. Then $||\mu|| < .084$, so we can choose x such that

$$0 < (x+\mu)^2+a_{i+1} < (.084)^2+.8847 < 1.$$

This contradiction to $m_+ = 1$ completes the proof of the lemma.

It follows from the above lemmas that the chain $[p_i]$ must be one of the following:

(a) $\infty(1, 2)\infty$, i.e. for all j, $p_{2j} = 2$, $p_{2j+1} = 1$.

(b) $\infty(1, 1, 1, 2)\infty$, i.e. for all j, $p_{4j-1} = p_{4j} = p_{4j+1} = 1$, $p_{4j+2} = 2$.

We now consider these special cases in turn.

Lemma 4.6. *If the chain* $[p_i]$ *is* $\infty(1, 2)\infty$ *then* $g \sim F_1\sqrt[3]{\frac{3}{4}}$.

Proof. If the chain is $\infty(1, 2)\infty$, we have for i even that

$$g_i = (x+\lambda_i y+\mu_i z)^2-a_{i+1}(z^2-2yz-2y^2).$$

Since $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = 3a^2 \leqq \tfrac{49}{54},$$

and so

$$a \leqq 7\sqrt{2}/18 < .55.$$

In addition, $d/48 = a^2/16$, and so (4.3) and (4.5) yield that

$$m_-^3 \geqq (m_-+\tfrac{1}{4})^2/16,$$

i.e.

$$h_1(m_-) \geqq 0.$$

By using lemma 4.2, noting that $h_1(\frac{1}{4}) = 0$, we have

$$m_- \geqq \tfrac{1}{4}; \quad a \geqq \tfrac{1}{2}.$$

Consider the binary quadratic form

$$t(x, z) = az^2-(x+\mu z)^2,$$

the negative of a section of g. This must have

$$m_+(t) \geqq \tfrac{1}{4}, \quad m_-(t) = 1.$$

Then taking $k = 4$ in lemma 4.1 we have that either

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(a) $t \sim (z^2-4xz-4x^2)/4$ and $a = d(t) = \frac{1}{2}$, or

(b) $a = d(t) > .5389$.

For the moment let us consider the second possibility. This gives

$$m_- \geqq \sqrt[3]{a^2/16} > .26.$$

Choosing, without loss of generality, $0 \leqq \mu \leqq \frac{1}{2}$, we have in the section $-t(x, z)$ with $x = -z = 1$ that

$$(1-\mu)^2-a < .5.$$

Then this value must be at most $-m_-$, and so

$$(1-\mu)^2 \leqq a-m_- < .29 < (.5386)^2,$$

from which we can deduce that $.4614 < \mu \leqq .5$. Then in the value $-t(1, 3)$ we have that

$$5.66 < (1+3\mu)^2 \leqq 6.25,$$
$$4.85 < 9a < 4.95.$$

In order not to contradict $m_+ = 1$ we must have $(1+3\mu)^2 > 5.85$, giving $.4728 < \mu \leqq .5$. In the value $-t(5, -4)$ we have that

$$9 \leqq (5-4\mu)^2 < 9.67,$$
$$8.622 < 16a < 8.8.$$

Then as $m_+ = 1$ we must have $16a < 8.67$. In the value $-t(1, 4)$ we have that

$$8.35 < (1+4\mu)^2 \leqq 9,$$
$$8.622 < 16a < 8.67.$$

Then as $m_+ = 1$, $m_- > .26$ we have that

$$8.35 < (1+4\mu)^2 < 8.67-.26 = 8.41 = (2.9)^2.$$

Hence we must have that

$$.4728 < \mu < .475. \tag{4.17}$$

By an identical treatment applied to the sections

$$-t_1 = (x+(\lambda-\mu)z_1)^2-az_1^2: \quad y = -z = z_1$$
$$-t_2 = (x+(\lambda+3\mu)z_2)^2-az_2^2: \quad z = 3z_2 = 3y$$

we can derive that

$$.4728 < \lambda-\mu < .475 \quad \text{or} \quad .525 < \lambda-\mu < .5272, \tag{4.18}$$

$$.4728 < \lambda+3\mu < .475 \quad \text{or} \quad .525 < \lambda+3\mu < .5272 \tag{4.19}$$

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(modulo 1). These inequalities (4.17), (4.18) and (4.19) can be shown to be inconsistent by adding 4 times (4.17) to (4.18).

This eliminates the possibility that $a > .5389$ and leaves $a = \frac{1}{2}$. In this case

$$t \sim \tfrac{1}{2}z^2 - (x+\tfrac{1}{2}z)^2,$$
$$t_1 \sim \tfrac{1}{2}z_1^2 - (x+\tfrac{1}{2}z_1)^2.$$

This yields on considering the types of forms equivalent to $\frac{1}{2}z^2-(x+\frac{1}{2}z)^2$ that $\mu \equiv \lambda - \mu \equiv \frac{1}{2}$ (mod 1), from which it follows that g is equivalent to

$$(x+\tfrac{1}{2}z)^2 - \tfrac{1}{2}(z^2-2yz-2y^2) = F_1\sqrt[3]{\tfrac{3}{4}}.$$

LEMMA 4.7. *If the chain* $[p_i]$ *is* $\infty(1, 1, 1, 2)\infty$ *then* $g \sim F_2\sqrt[3]{\frac{49}{54}}$.

PROOF. If the chain is $\infty(1, 1, 1, 2)\infty$ we have for even i with $p_i = 2$ that

$$g_i = (x+\lambda_i y+\mu_i z)^2 - a_{i+1}(z^2-2yz-\tfrac{5}{3}y^2).$$

As $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = 8a^2/3 \leqq \tfrac{49}{54},$$

and so

$$a \leqq \tfrac{7}{12}.$$

In addition $d/48 = a^2/18$ and so as in the previous lemma we obtain that $h_2(m_-) \geqq 0$. Since $h_2(.23) < 0$ we must have $m_- > .23$. By the same method as in the previous lemma it can be shown that either

(a) $az^2-(x+\mu z)^2 \sim \frac{1}{2}z^2-(x+\frac{1}{2}z)^2$, or

(b) $a > .5389$.

For the moment let us consider the first possibility. In this case $a = \frac{1}{2}$. If we set $y = 3z_3$, $z = -2z_3$ then we must have

$$az_3^2-(x+(3\lambda-2\mu)z_3)^2 \sim \tfrac{1}{2}z_3^2-(x+\tfrac{1}{2}z_3)^2,$$

which yields, taking (a) into consideration as well, that $\mu \equiv 3\lambda - 2\mu \equiv \frac{1}{2}$ (mod 1). From this we can deduce that $\lambda \equiv \frac{1}{2}$ or $\pm\frac{1}{6}$ (mod 1). However $\lambda \equiv \pm\frac{1}{6}$ gives the section $(x+\lambda)^2+\frac{5}{6}$ the value $\frac{31}{36}$, and $\lambda \equiv \frac{1}{2}$ gives the section $(x+\lambda-\mu)^2-\frac{1}{2}(1+2-\frac{5}{3})$ the value $\frac{1}{3}$, in each case contradicting $m_+ = 1$.

This eliminates the possibility that $a = \frac{1}{2}$, leaving $a > .5389$, from which we obtain that

$$m_- \geqq \sqrt[3]{a^2/18} > .252.$$

Choosing x with $\frac{1}{4} \leqq (x+\mu)^2 \leqq 1$ in the section $(x+\mu)^2-a$ gives a value less than 1, so this value is at most $-m_-$. Therefore

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$$(x+\mu)^2 < \tfrac{7}{12} - .252 < .3314 < (.5757)^2.$$

This yields that $||\mu - \tfrac{1}{2}|| < .0757$. Thus we can choose x such that

$$5.0 < (x+3\mu)^2 \leqq 6.25.$$

Then as $4.8501 < 9a \leqq 5.25$ this gives $(x+3\mu)^2 - 9a$ a value greater than $-m_-$, so this value is at least 1. This implies that

$$(x+3\mu)^2 > 5.8501 > (2.4178)^2,$$

from which it follows that

$$(4.20) \qquad ||\mu - \tfrac{1}{2}|| < .0274.$$

The value $(x+\lambda-\mu)^2 - 4a/3$ with x chosen such that $1 \leqq (x+\lambda-\mu)^2 \leqq \frac{9}{4}$ yields, as $.718 < 4a/3 \leqq \frac{7}{9}$, a positive value of g. This value must be at least 1, so

$$(x+\lambda-\mu)^2 > 1.718 > (1.31)^2,$$

which yields that

$$(4.21) \qquad ||\lambda - \mu - \tfrac{1}{2}|| < .19.$$

Since $5a/3 \leqq \frac{35}{36}$ it is clear from the sections

$$(x+\lambda)^2 + 5a/3; \quad (x+2\lambda-\mu)^2 + 5a/3;$$
$$(x+\lambda+2\mu)^2 + 5a/3; \quad (x+2\lambda+5\mu)^2 + 5a/3;$$

that we must have

$$(4.22) \qquad ||\lambda||,\ ||\lambda+2\mu||,\ ||2\lambda-\mu||,\ ||2\lambda+5\mu|| \text{ each at least } \tfrac{1}{6}.$$

It is easily verified that the only solutions to the congruence inequalities (4.20), (4.21) and (4.22) are $\lambda \equiv \pm\frac{1}{6}$, $\mu \equiv \frac{1}{2} \pmod 1$. Then in order that the section $(x+\lambda)^2 + 5a/3$ shall not take a value contradicting $m_+ = 1$ we must have $a = \frac{7}{12}$. Thus we must have

$$g \sim (x + \tfrac{1}{6}y + \tfrac{1}{2}z)^2 - \tfrac{7}{12}(z^2 - 2yz - \tfrac{5}{3}y^2) = F_2 \sqrt[3]{\tfrac{49}{54}}$$

as required.

Combining the lemmas proven we have shown that if $m_-(g) \geqq \sqrt[3]{d/48}$ then g is equivalent to a multiple of either F_1 or F_2. This is clearly equivalent to proving theorem C_1.

5. The proof of theorem C_2

For reference theorem C_2 is re-stated.

If g is any indefinite ternary quadratic form of signature 1, *with* $d(g) = d$ *where*

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$$0 < d \leqq \tfrac{9}{8},$$

and if $m_+(g) = m_+ = 1$ *is attained by* g *then either*

(a) $m_-(g) < \sqrt[3]{2d/49}$, *or*

(b) g *is equivalent to a multiple of either* F_2 *or* F_3.

As in § 4 we consider in place of g an equivalence chain (g_i) of forms equivalent to g. For simplicity we use the same notation as in § 4, renaming (4.2) as (5.1), i.e.

$$a_{i+1}K_i = \Delta; \quad \Delta^2 = 4d, \tag{5.1}$$

and replacing (4.3) by the assumption that

$$m_-(g) = m_- \geqq \sqrt{2d/49}. \tag{5.2}$$

Similarly (4.4) and (4.5) become

$$a_i \geqq \tfrac{3}{4} \qquad (i \text{ even}) \tag{5.3}$$

$$a_i \geqq m_- + \tfrac{1}{4} \geqq \sqrt[3]{2d/49} + \tfrac{1}{4} \quad (i \text{ odd}), \tag{5.4}$$

from which, using (5.1) and setting

$$d = 9\beta/8, \quad 0 < \beta \leqq 1, \tag{5.5}$$

we obtain that

$$K_i = 3\sqrt{2\beta}/2a_{i+1}. \tag{5.6}$$

Then using (5.3) and (5.4) yields that

$$K_i \leqq 2\sqrt{2\beta} < 2.82843\sqrt{\beta} \quad (i \text{ odd}), \tag{5.7}$$

$$K_i < 3.48859\sqrt[6]{\beta} \qquad (i \text{ even}). \tag{5.8}$$

Thus

$$p_i \leqq 2 \ (i \text{ odd}); \qquad p_i \leqq 3 \ (i \text{ even}).$$

The proof is now presented as a series of lemmas, with the use of λ, μ for λ_i, μ_i respectively for simplicity.

LEMMA 5.1. $p_i < 3$ *for all even* i.

PROOF. If $p_i = 3$ with i even then $F_i > (3, 2, 1) = \frac{10}{3}$, $S_i > (0, 2, 1) = \frac{1}{3}$, and so $K_i > \frac{11}{3}$ which contradicts (5.8).

LEMMA 5.2. *If* $p_i = 2$ *with* i *odd then* $p_{i-1} = p_{i+1} = 2$.

PROOF. Let $p_i = 2$ with i odd and let one of p_{i-1}, p_{i+1} be 1. Then

$$K_i > 2 + (0, 2, 1) + (0, 1, 1) = \tfrac{17}{6}$$

which contradicts (5.7).

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LEMMA 5.3. *If $p_j = 1$ for some odd j then either (a) $p_{j-2} = p_{j+2} = 1$ or (b) $p_{j-1} = p_{j+1} = 2$ and one of p_{j-2}, p_{j+2} is 1.*

PROOF. Suppose that $p_j = 1$ with j odd and that one of p_{j-2}, p_{j+2} is not 1. Then lemma 5.2 shows that there are in effect two possible cases where (b) does not hold, viz.

(i) $p_{j-2} = p_{j+2} = 2$; the chain is $\cdots, 2, 2, 2, 1_j, 2, 2, 2, \cdots$

(ii) $p_{j-2} = 1, p_{j+2} = 2$; the chain is $\cdots, 1, 1, 1_j, 2, 2, 2, \cdots$

It should be noticed that the reverse situation to (ii) i.e. $p_{j-2} = 2, p_{j+2} = 1$ is equivalent to (ii) — this was observed in § 4.

As the method of elimination of possibility (i) is similar to that of possibility (ii) it suffices to give the proof that (ii) cannot occur.

If (ii) occurred, we would have, taking $i = j+1$, that $F_i > 2.4142$, $S_i > \cdot 618$, and hence that $K_i > 3.0322$. Using this in (5.6) yields that $a_{i+1} < .69961$. In addition, $F_i < 2.42265$ and $S_i < .634$. Hence consideration of the value $g_i(x, 1, 2)$ with x chosen such that $(x+\lambda+2\mu)^2 \leqq \frac{1}{4}$ yields that $a_{i+1} > .67369$ and that

$$||\lambda+2\mu-\tfrac{1}{2}|| < .03. \tag{5.9}$$

As $m_-^3 \geqq a_{i+1}^2 K_i^2/98$ we have that $m_- > .349$. Then considering the value $g_i(x, 0, 1)$ with x chosen such that $\frac{1}{4} \leqq (x+\mu)^2 \leqq 1$ yields that $||\mu-\frac{1}{2}|| < .0925$, and so using (5.9) we have that $||\lambda-\mu|| < .31$. Hence by choosing x such that $1 \leqq (x+\lambda-\mu)^2 < 1.72$ we obtain a value $g_i(x, 1, -1)$ lying in the open interval $(.085, .879)$, contradicting $m_+ = 1$.

LEMMA 5.4. *If $p_j = 1$ for some odd j then $p_i = 1$ for all odd i.*

PROOF. Considering the above lemma, it suffices to show that if $p_j = 1$ with j odd then the situation $p_{j-1} = p_{j+1} = 2$, $p_{j-2} = 1$ cannot occur.

Suppose such a situation did occur. Then setting $i = j-1$ we have that $F_i > 2.7071$ and $S_i > .618$. Hence $K_i > 3.3251$, and so $a_{i+1} < .638$ follows from (5.6). Combining (5.1), (5.2) and (5.4) yields that

$$m_-^3 \geqq K_i^2(m_-+\tfrac{1}{4})^2/98,$$

and inserting the bound for K_i gives that

$$98m_-^3 > 11.05629(m_-+\tfrac{1}{4})^2.$$

Iterating on this, commencing from $m_- \geqq 0$, eventually gives that $m_- > .339$ and so $a_{i+1} > .589$.

Considering the value $g_i(x, 0, 1)$ where x is chosen such that $\frac{1}{4} \leqq (x+\mu)^2 \leqq 1$ yields that $||\mu-\frac{1}{2}|| < .04681$, and so in the section $g_i(x, 0, 3)$ we can choose x such that $5.567 < (x+3\mu)^2 \leqq 6.25$. However as

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$5.301 < 9a_{i+1} < 5.742$ this gives a value of g_i contradicting either $m_+ = 1$ or $m_- > .339$.

LEMMA 5.5. *If $p_j = 1$ and $p_{j+1} = 2$ with j odd then $p_{j-1} = p_{j+3} = 1$.*

PROOF. This is a direct consequence of the proof of the above lemma.

LEMMA 5.6. *If $p_j = 1$ and $p_{j+1} = 2$ with j odd then $p_{j-3} = 2$.*

PROOF. Suppose to the contrary that, for some odd j, $p_{j-3} = p_j = 1$, $p_{j+1} = 2$. Then setting $i = j+1$ we have that $2.618 < F_i < \frac{79}{30}$, $.618 < S_i < .62021$, and hence that $K_i > 3.236$. By using a method similar to that used in lemma 5.4 we obtain that $a_{i+1} < .6556$ and that $m_- > .33$.

Applying lemma 4.1, with $k = 3$, to the binary form

$$t(x, z) = -g_i(x, 0, z)$$

which has $m_-(t) = m_+(g_i) = 1$ and $m_+(t) \geqq m_-(g_i) > .33$, we find that either

(i) $d(t) > .6577$, which contradicts the previous bound $a_{i+1} < .6556$, or

(ii) $d(t) = a_{i+1} = \frac{7}{12}$. Then as $K_i < 3.2536$ we have that $d < \frac{49}{54}$ and the result follows from theorem C_1.

LEMMA 5.7. *$p_i = 2$ for at least one i.*

PROOF. If $p_i = 1$ for all i, then for i even

$$g_i = (x+\lambda_i y+\mu_i z)^2 - a_{i+1}(z^2-yz-y^2).$$

As $g_i \sim g$ there is no loss of generality in dropping the suffixes. Then $d = d(g) = 5a^2/4 \leqq \frac{9}{8}$, and so

$$a < .9487. \tag{5.10}$$

If $a \leqq .852$ we have that $d < \frac{49}{54}$ and the result follows from theorem C_1. Hence it is sufficient to assume that $a > .852$.

Considering the values $g(x_1, 1, -1)$ and $g(x_2, 0, 1)$ where

$$1 \leqq (x_2+\mu)^2 \leqq \tfrac{9}{4} \quad\text{and}\quad 1 \leqq (x_1+\lambda-\mu)^2 \leqq \tfrac{9}{4}$$

we find that $||\mu-\frac{1}{2}|| < .14$ and that $||\lambda-\mu-\frac{1}{2}|| < .14$ in order not to contradict $m_+ = 1$. Hence $||\lambda|| < .28$, and so, for suitable x,

$$O < g(x, 1, 0) < .0784+a.$$

Thus $a > .921$. Repeating this argument twice yields that $a > .95$, contradicting (5.10).

From the above lemmas it follows that either

(i) $p_j = 2$ for all j, i.e. the chain is $\infty(2)\infty$, or

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(ii) there exists an even i such that

$$p_j = 2 \quad (j \equiv i \bmod 4),$$
$$p_j = 1 \quad (\text{otherwise}),$$

i.e. the chain is $\infty(1, 1, 1, 2)\infty$. We shall now consider these remaining two possibilities.

LEMMA 5.8. *If the chain* $[p_i]$ *is* $\infty(1, 1, 1, 2)\infty$ *then* $g \sim F_2\sqrt[3]{\frac{49}{54}}$.

PROOF. If the chain is $\infty(1, 1, 1, 2)\infty$ then for any even i with $p_i = 2$ we have that

$$g_i = (x+\lambda_i y+\mu_i z)^2 - a_{i+1}(z^2-2yz-\tfrac{5}{3}y^2).$$

As $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then $d = d(g) = 8a^3/3 \leqq \frac{9}{8}$ and so $a < .65$. By the usual method we find that $m_-^3 \geqq 16(m_-+\frac{1}{4})^2/147$, and so

$$(3m_- - 1)(49m_-^2+11m_-+1) \geqq 0.$$

Hence $m_- \geqq \frac{1}{3}$ and $a \geqq \frac{7}{12}$.

Now considering the value $g(x, 0, 1)$ where $\frac{1}{4} \leqq (x+\mu)^2 \leqq 1$ we find that $\|\mu-\frac{1}{2}\| < .041$. Then considering the value $g(x, 0, 3)$ where $5.65 < (x+3\mu)^2 \leqq 6.25$ we find, if $a > \frac{7}{12}$, that g takes a value in the open interval $(-.2, 1)$, contradicting either $m_+ = 1$ or $m_- \geqq \frac{1}{3}$. Hence $a = \frac{7}{12}$, $d = \frac{49}{54}$, and the lemma follows from the results of theorem C_1.

LEMMA 5.9. *If the chain* $[p_i]$ *is* $\infty(2)\infty$ *then* $g \sim F_3\sqrt[3]{\frac{9}{8}}$.

PROOF. If the chain is $\infty(2)\infty$ we have for i even that

$$g_i = (x+\lambda_i y+\mu_i z)^2 - a_{i+1}(z^2-2yz-y^2).$$

As $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then $d = d(g) = 2a^3 \leqq \frac{9}{8}$ and so $a \leqq \frac{3}{4}$. As $m_+ = 1$, considering the sections $g(x, 1, 0)$ and $g(x, 1, 2)$ we find that $a = \frac{3}{4}$ and $\lambda \equiv \lambda+2\mu \equiv \frac{1}{2}$ (mod 1). Then as $\mu \equiv 0$ implies that $g(x, 0, 1)$ takes the value $\frac{1}{4}$, contradicting $m_+ = 1$, we must have $\mu \equiv \frac{1}{2}$. Hence

$$g \sim (x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2 - \tfrac{3}{4}(z^2-2yz-y^2) = F_3\sqrt[3]{\tfrac{9}{8}}$$

as required.

6. The proof of theorems C_3 and C_4

Consider the following result.

THEOREM D. *If g is any indefinite ternary quadratic form of signature* 1, *with $d(g) = d$ where*

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$$0 < d \leqq \tfrac{3}{2},$$

and if $m_+(g) = m_+ = 1$ *is attained by* g, *then either*

(*a*) $m_-(g) < \sqrt[3]{d/9}$, *or*

(*b*) g *is equivalent to a multiple of either* F_3, F_4 *or* F_5.

This theorem is stronger than either theorem C_3, which makes the stronger assumption that $d \leqq \frac{144}{125}$, or theorem C_4, which has the weaker conclusion that $m_-(g) < \sqrt[3]{3d/16}$. Thus theorems C_3 and C_4 will follow from the work of § 2 when we prove theorem D.

Applying theorem D to normalised forms, in the way that theorems A_i are deduced from theorems B_i, it can be seen that every normalised indefinite ternary quadratic form of signature 1, not equivalent to F_4, takes a value in the closed interval $[-\sqrt[3]{\frac{1}{9}}, \sqrt[3]{\frac{2}{3}}]$, the intersection of intervals I_3 and I_4.

To prove theorem D we consider, as usual, in place of g, an equivalence chain (g_i) of forms equivalent to g.

Assuming that $m_-(g) \geqq \sqrt[3]{d/9}$ and using the same notation as in the previous chapters we have that

(6.1) $$a_{i+1}K_i = \Delta; \quad \Delta^2 = 4d,$$

(6.2) $$m_-(g) = m_- \geqq \sqrt[3]{d/9},$$

(6.3) $$a_i \geqq \tfrac{3}{4} \qquad (i \text{ even}),$$

(6.4) $$a_i \geqq m_- + \tfrac{1}{4} \geqq \sqrt[3]{d/9} + \tfrac{1}{4} \qquad (i \text{ odd}),$$

(6.5) $$d = 3\beta/2, \quad 0 < \beta \leqq 1, \qquad \text{and}$$

(6.6) $$K_i = \sqrt{6\beta}/a_{i+1}.$$

Using the bounds (6.3) and (6.4) in (6.6) we obtain that

(6.7) $$K_i \leqq 4\sqrt{6\beta}/3 < 3.266\sqrt{\beta} \qquad (i \text{ odd}),$$

(6.8) $$K_i \leqq \sqrt{6\beta}[\sqrt[3]{\beta}(\sqrt[3]{\tfrac{1}{6}} + \tfrac{1}{4})]^{-1} < 3.062\sqrt[6]{\beta} \qquad (i \text{ even}).$$

Hence we must have $p_i \leqq 3$ for all i. If however $p_i = 3$ for some i we would have

$$K_i > (3, 4, 1) + (0, 4, 1) = 3.4$$

which contradicts the relevant one of (6.7) and (6.8). Thus we must have $p_i \leqq 2$ for all i.

We now present the proof as a series of lemmas.

LEMMA 6.1. *If* $p_i = 2$ *with* i *even then* $p_{i-1} = p_{i+1} = 2$.

PROOF. Let $p_i = 2$ where i is even and suppose that one of p_{i-1}, p_{i+1} is 1. Then

$$K_i \geqq 2 + (0, 1, \overline{1, 2}) + (0, 2, \overline{1, 2}) > 2.943,$$

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and comparing this with (6.8) yields that $\sqrt[6]{\beta} > .96113$, i.e. $\sqrt[3]{\beta} > .92377$. Hence as $m_- \geqq \sqrt[3]{\beta/6}$ we have that $m_- > .508$, and so $a_{i+1} > .758$. In addition, using the above bound for K_i in (6.6), we find that $a_{i+1} < .8324$. However applying lemma 4.1 with $k = 2$ to the form $t(x, z) = -g_i(x, 0, z)$, which has $m_-(t) = 1$ and $m_+(t) > .508$, yields that either

(a) $t \sim \frac{1}{2}(x^2 - 2xz - 2z^2)$, with $d(t) = a_{i+1} = \frac{3}{4}$, or

(b) $a_{i+1} = d(t) > .944$,

in either case contradicting $.758 < a_{i+1} < .8324$.

LEMMA 6.2. *If $p_i = 2$ with i even then $p_i = 2$ for all i and $g \sim F_3\sqrt[3]{\frac{9}{8}}$.*

PROOF. Let $p_i = 2$ where i is even. Then $K_i \geqq 2 + 2(0, \overline{2, 1}) = 1 + \sqrt{3}$. Combining (6.1), (6.2) and (6.4) we have that $m_-^3 \geqq K_i^2(m_- + \frac{1}{4})^2/36$, and inserting the above bound for K_i yields that

$$m_-^3 > 7.4641(m_- + \tfrac{1}{4})^2/36.$$

From this we obtain by iteration that $m_- > .478$, and hence $t(x, z) = -g_i(x, 0, z)$ has $m_-(t) = 1$ and $m_+(t) > .478$. Applying lemma 4.1 with $k = 2$ yields that either

(a) $a_{i+1} = \frac{3}{4}$, or

(b) $a_{i+1} > .913$.

However as (b) implies that $d = a_{i+1}^2 K_i^2/4 > 1.55$ which contradicts the given we must have $a_{i+1} = \frac{3}{4}$.

Now we have, using the previous lemma, that

$$F_i \leqq (\overline{2}); \quad S_i \leqq (0, \overline{2}) = 1/(\overline{2}),$$

and so $F_i S_i \leqq 1$ with equality if and only if $p_i = 2$ for all i. However as $F_i S_i < 1$ implies that $a_{i+1} F_i S_i < \frac{3}{4}$ which contradicts (6.3) we must have $p_i = 2$ for all i. Thus

$$g_i = (x + \lambda_i y + \mu_i z)^2 - \tfrac{3}{4}(z^2 - 2yz - y^2),$$

and so $d = d(g) = \frac{9}{8}$ and the lemma follows from theorem C_2.

For the remainder of the proof of theorem D we may assume that $p_i = 1$ for all even i.

LEMMA 6.3. *If $p_i = 2$ with i odd then $p_{i-2} = p_{i+2} = 1$.*

PROOF. Let $p_i = 2$ with i odd and suppose that one of p_{i-2}, p_{i+2} is 2. Then

$$K_i \geqq 2 + (0, 1, 2, \overline{1}) + (0, \overline{1}) > 3.31$$

which contradicts (7.7).

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LEMMA 6.4. *If* $p_i = 2$ *with* i *odd then* $p_{i-4} = p_{i+4} = 2$.

PROOF. Let $p_i = 2$ with i odd and suppose that one of p_{i-4}, p_{i+4} is 1. By considering the reverse chain if necessary we may assume that $p_{i-4} = 1$. Then $2.618 < F_i < 2.633$, $.618 < S_i < .62021$, and so $K_i > 3.236$. Using this in (6.6) gives that $a_{i+1} < .757$. In addition, from (6.3), $a_{i+1} \geqq .75$, so combining (6.1) and (6.2) with the above bound for K_i yields that $m_- > .545$.

Considering the section $g_i(x, 0, 1)$ yields that

$$||\mu - \tfrac{1}{2}|| < .0071. \tag{6.9}$$

Furthermore choosing x such that $1 \leqq (x+\lambda)^2 \leqq \frac{9}{4}$ in the section $g_i(x, 1, 0)$ yields that $||\lambda - \frac{1}{2}|| < .013$ in order not to contradict the bounds on m_+ and m_-. Combining this with (6.9) shows that $||5\lambda - 3\mu|| < .09$. Then the value $g_i(x, 5, -3)$ where $1 \leqq (x+5\lambda-3\mu)^2 < 1.19$ lies in the interval $(-{\cdot}24, .11)$, contradicting either $m_+ = 1$ or $m_- > .545$. This contradiction completes the proof of the lemma.

It follows from the above lemmas that if $g \nsim F_3\sqrt[3]{\frac{9}{8}}$ and if $p_i = 2$ for some i then $p_j = 2$ for $j \equiv i \pmod 4$ and $p_j = 1$ if $j \not\equiv i \pmod 4$, and so the chain is $\infty(1, 1, 2, 1)\infty$.

LEMMA 6.5. *If the chain* $[p_i]$ *is* $\infty(1, 1, 2, 1)\infty$ *then* $g \sim F_5\sqrt[3]{\frac{3}{2}}$.

PROOF. If the chain is $\infty(1, 1, 2, 1)\infty$ then for odd i with $p_i = 2$ we have that

$$g_i = (x+\lambda_i y+\mu_i z)^2 + a_{i+1}(z^2 - 2yz - \tfrac{5}{3}y^2).$$

As $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then $d = d(g) = 8a^2/3 \leqq \frac{3}{2}$ and so $a \leqq \frac{3}{4}$. Thus, using (6.3), $a = \frac{3}{4}$. Considering the sections $g(x, 0, 1)$ and $g(x, 1, 0)$ yields that $\mu \equiv \lambda \equiv \frac{1}{2} \pmod 1$. Hence

$$\begin{aligned} g &\sim (x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2 + \tfrac{3}{4}(z^2 - 2yz - \tfrac{5}{3}y^2) \\ &\sim (x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2 - \tfrac{5}{4}(z^2 - \tfrac{6}{5}yz - \tfrac{3}{5}y^2). \end{aligned}$$

i.e. $g \sim F_5\sqrt[3]{\frac{3}{2}}$.

There is only one further possibility left for the chain $[p_i]$, namely $p_i = 1$ for all i. We now consider this case.

LEMMA 6.6. *If the chain* $[p_i]$ *is* $\infty(1)\infty$ *then* $g \sim F_4\sqrt[3]{\frac{144}{125}}$.

PROOF. If the chain is $\infty(1)\infty$ then for even i

$$g_i = (x+\lambda_i y+\mu_i z)^2 - a_{i+1}(z^2 - yz - y^2).$$

As above we may drop the suffixes and take g_i to be g. Then $d = d(g) = 5a^2/4 \leqq \frac{3}{2}$ and so $a < 1.096$. Now considering the section $(x+\lambda)^2 + a$ it is clear that $a \geqq 1 - ||\lambda||^2$, and so

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(6.10) $$a \geqq 1-(||\mu-\tfrac{1}{2}||+||\lambda-\mu-\tfrac{1}{2}||)^2.$$

The value $g(x, 0, 1)$ where $1 \leqq (x+\mu)^2 \leqq \frac{9}{4}$ lies in the half-open interval $(-.096, 1.5]$. Hence, as $m_- \geqq \sqrt[3]{d/9} \geqq \sqrt[3]{\frac{5}{4}}$, this value of g must be at least 1, and so $(x+\mu)^2 \geqq 1+a$. Hence

$$||\mu-\tfrac{1}{2}|| \leqq \tfrac{3}{2}-\sqrt{1+a}.$$

Similarly it can be shown that $||\lambda-\mu-\frac{1}{2}|| \leqq \frac{3}{2}-\sqrt{1+a}$. Inserting these into (6.10) yields that

$$a \geqq 1-(3-2\sqrt{1+a})^2,$$

which on simplification gives, as $a > 0$, that $a \geqq \frac{24}{25}$.

Considering the sections $(x+\mu)^2-a$, $(x+\lambda-\mu)^2-a$ and $(x+\lambda+2\mu)^2-a$ with the squares chosen in the closed interval $[1, \frac{9}{4}]$ we find that

(6.11) $$||\mu-\tfrac{1}{2}|| \leqq .1,$$

(6.12) $$||\lambda-\mu-\tfrac{1}{2}|| \leqq .1 \quad \text{and}$$

(6.13) $$||\lambda+2\mu-\tfrac{1}{2}|| \leqq .1.$$

Taking $0 \leqq \mu \leqq \frac{1}{2}$ without loss of generality it is found that (6.11), (6.12) and (6.13) imply that $\mu = \cdot 4$ and that $\lambda \equiv \cdot 8 \pmod 1$.

As $3.84 \leqq 4a < 4.4$ the value $g(3, 0, -2)$ will contradict $m_+ = 1$ unless $a = \frac{24}{25}$. Hence

$$g \sim (x+\tfrac{4}{5}y+\tfrac{2}{5}z)^2-\tfrac{24}{25}(z^2-yz-y^2) = F_4\sqrt[3]{\tfrac{144}{125}}.$$

This completes the proof of theorem D.

7. The proof of theorems C_5 and C_6

Both of these theorems may be proved in a manner similar to the above. However it is simpler to deduce them from the following result of Venkov [8].

Let f be an indefinite ternary quadratic form and let

$$M(f) = \min\{m_+(f), m_-(f)\} \geqq \sqrt[3]{2d(f)/9}.$$

Then f must be equivalent to a multiple of one of the following forms.

$$\begin{aligned}
l_1 &= -x^2-xy-y^2+2z^2,\\
l_2 &= x^2+xy-y^2-2z^2,\\
l_3 &= -x^2-y^2+3z^2,\\
l_4 &= -x^2-xy+y^2-\tfrac{5}{2}z^2,\\
l_5 &= -x^2-\tfrac{6}{7}xy-y^2+\tfrac{2}{7}xz+\tfrac{18}{7}yz+\tfrac{17}{7}z^2,
\end{aligned}$$

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$$\begin{aligned}
l_6 &= -x^2-y^2-xz-yz+3z^2,\\
l_7 &= -x^2-xy-y^2+5z^2,\\
l_8 &= -\tfrac{7}{5}x^2+2xy-\tfrac{11}{5}y^2+\tfrac{9}{5}xz+\tfrac{1}{5}yz+z^2,\\
l_9 &= -x^2+\tfrac{2}{3}xy-y^2+\tfrac{8}{3}yz+\tfrac{8}{3}z^2,\\
l_{10} &= -x^2+xy-y^2+3yz+\tfrac{21}{8}z^2,\\
l_{11} &= -x^2+xy-y^2+2xz+2yz+2z^2.
\end{aligned}$$

Furthermore $M(l_i) = 1$ *for* $1 \leqq i \leqq 11$.

Deduction of theorem C_5. Let g be an indefinite ternary quadratic form of signature 1, with $d(g) = d$ where $0 < d \leqq \frac{112}{27}$, and let $m_+(g) = 1$ be attained by g. Furthermore let $m_-(g) \geqq \sqrt[3]{2d/3}$. Then $M(-g) > \sqrt[3]{2d/9}$, and so by Venkov's result it follows, on comparing signatures and minima m_+, that $g = -l_i$ for some i. Hence

$$m_+(l_i) = m_-(g) \geqq \sqrt[3]{2d/3}, \tag{7.1}$$

and

$$d(l_i) = d \leqq \tfrac{112}{27}. \tag{7.2}$$

As l_i takes the value 1 for $i = 2, 3, 4, 6, 7$ and 8, as l_5 takes the value $\frac{8}{7}$, and as $d(l_{11}) > d(l_{10}) > \frac{112}{27}$, it follows that the only l_i satisfying (7.1) and (7.2) are l_1 and l_9. Theorem C_5 now follows from the results of lemmas 2.5 and 2.6, on observing that

$$-l_1(x, y+z, z) = \sqrt[3]{\tfrac{3}{2}}\,F_5(x, y, z),$$

and

$$-l_9(x, -y, z) = \sqrt[3]{\tfrac{112}{27}}\,F_6(x, y, z). \tag{7.3}$$

Deduction of theorem C_6. Let g be an indefinite ternary quadratic form of signature 1, with $d(g) = d$ where $0 < d \leqq \frac{9}{2}$, and let $m_+(g) = 1$ be attained by g. Furthermore let $m_-(g) \geqq \sqrt[3]{125d/112}$. Then $M(-g) \geqq \sqrt[3]{2d/9}$ and so by Venkov's result it follows that $g = -l_i$ for some i. Since l_i takes the value 1 for $i = 1, 2, 3, 4, 6, 7$ and 8, since l_5 takes the value $\frac{8}{7}$, and since l_{10} takes the value $\frac{3}{2}$, it can be shown that $m_+(l_i) \leqq \sqrt[3]{4d/5}$ unless $i = 9$ or 11. Considering (7.3), and noting that

$$-l_{11}(x+z, -y, z) = \sqrt[3]{\tfrac{9}{2}}\,F_7(x, y, z),$$

theorem C_6 now follows from the results of lemmas 2.6 and 2.7.

8. The proof of theorem C_7

For reference theorem C_7 is restated.

Theorem C_7. *If g is any indefinite ternary quadratic form of signature* 1,

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with $d(g) = d$ *where* $0 < d \leqq 24$, *and if* $m_+(g) = m_+ = 1$ *is attained by* g *then either*

(*a*) $m_-(g) < \sqrt[3]{16d/9}$, *or*

(*b*) g *is equivalent to a multiple of either* F_7 *or* F_8.

PROOF. Let (g_i) be an equivalence chain of forms equivalent to g. Assuming that $m_-(g) \geqq \sqrt[3]{16d/9}$ and using the same notation as in previous sections we have

$$a_{i+1}K_i = \Delta; \quad \Delta^2 = 4d, \tag{8.1}$$

$$m_-(g) = m_- \geqq \sqrt[3]{16d/9}, \tag{8.2}$$

$$a_i \geqq \tfrac{3}{4} \qquad (i \text{ even}), \tag{8.3}$$

$$a_i \geqq m_- + \tfrac{1}{4} \geqq \sqrt[3]{16d/9} + \tfrac{1}{4} \quad (i \text{ odd}), \tag{8.4}$$

$$d = 24\beta, \quad 0 < \beta \leqq 1, \quad \text{and} \tag{8.5}$$

$$K_i = 4\sqrt{6\beta}/a_{i+1}. \tag{8.6}$$

Now if $\beta \leqq \frac{3}{16}$ then $d \leqq \frac{9}{2}$ and using the results of theorem C_6 and lemma 2.6 it follows that g is equivalent to a multiple of F_7. Hence we may assume from now on that $\beta > \frac{3}{16}$. Then

$$m_- > \sqrt[3]{\tfrac{16}{9} \cdot 24 \cdot \tfrac{3}{16}} = 2,$$

and so using the results of a previous paper [9] we find that either

(i) $g \sim (x+\frac{1}{2}z)^2 - 3(y^2 - yz - \frac{1}{4}z^2)$, or

(ii) $g \sim (x+\frac{1}{2}z)^2 - 3(y^2 - yz - \frac{1}{2}z^2)$, or

(iii) $g \sim x^2 - 3(y^2 - \frac{4}{3}yz - \frac{1}{3}z^2)$, or

(iv) $d \geqq 7.5$.

Now possibility (i) may be eliminated as $\beta \not> \frac{3}{16}$ for this form, and possibilities (ii) and (iii) may be eliminated as these have $m_- = 2 < \sqrt[3]{16d/9}$, contradicting equation (8.2). Hence $d \geqq 7.5$, $\beta \geqq \frac{5}{16}$, and using this in (8.2) yields that $m_- > 2.37$.

Applying the corollary to theorem 1 of the paper [9] to the sections $g_i(x, 0, z)$ where i is even yields that $a_{i+1} \geqq 4.62$ for all even i, and hence that $q_i(y, z)$, as defined in (3.4), can take no values in the open interval $(-4.62, .75)$. By a result of Segre [6] it follows that

$$d = d(q_i) \geqq \{(4.62)^2 + 3(4.62)\}/4 > 8.8.$$

We may now use this in (8.2) to obtain that $m_- > 2.5$. Repeating the above process yields that

$$m_- > 2.53; \quad a_{i+1} > 4.78 \quad (i \text{ even}). \tag{8.7}$$

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For the present we shall assume that

$$d \leqq \tfrac{243}{16}. \tag{8.8}$$

Then $\beta \leqq \frac{81}{128}$, and using this in (8.6) gives that $K_i < 1.631$ for all even i. Hence $p_i = 1$ for all even i. In addition, combining (8.3), (8.6) and (8.8) yields that $K_i < 10.393$ for odd i, and so $p_i \leqq 10$ for all odd i.

We now show that, for all even i,

$$4.78 < a_{i+1} \leqq 5.25. \tag{8.9}$$

For i even we have that $K_i > 1+2(0, 10, 1, 11) = \frac{155}{131}$, and using this in (8.6) gives that $a_{i+1} < 6.6$. Suppose, for some even i, that $5.25 < a_{i+1} < 6.6$. Then the value $g_i(x, 0, 1)$ where $4 \leqq (x+\mu)^2 \leqq 6.25$ contradicts either $m_+ = 1$ or $m_- > 2.53$ unless $a_{i+1} > 6.53$. However in this case, as $K_i > \frac{155}{131}$, combining (8.1) and (8.2) yields that $m_- > 2.8$, while $g_i(x, 0, 1)$ lies between -2.6 and $-.26$. This contradiction is sufficient to show (8.9).

The bound

$$K_i \leqq 6\sqrt{3}/7 < 1.485 \tag{8.10}$$

may be shown to hold for all even i as follows. From the corollary to theorem 1 of [9], as $m_- > 2$, we have that

$$a_{i+1} \geqq 4.25+(m_- - 2), \tag{8.11}$$

and combining this with (8.1) and (8.2) yields that

$$\phi_i(m_-) = m_-^3 - \tfrac{4}{9}K_i^2(2.25+m_-)^2 \geqq 0. \tag{8.12}$$

As (8.9) and (8.11) together imply that $m_- \leqq 3$, inequality (8.12) must be satisfied for some $m_- \leqq 3$. However using the known bounds on m_- and K_i it can easily be seen that the derivative

$$\phi_i'(m_-) = 3m_-^2 - \tfrac{8}{9}K_i^2(2.25+m_-) > 0,$$

and so (8.12) must be satisfied with $m_- = 3$. Hence $K_i \leqq 6\sqrt{3}/7$ as required.

The proof is now continued as a series of lemmas eliminating all possibilities for the chain $[p_i]$.

LEMMA 8.1. *$p_j \leqq 8$ for all odd j.*

PROOF. Let $p_i \geqq 9$ for some odd i. Then

$$\tfrac{12}{131} = (0, 10, 1, 11) < S_{i+1} < (0, 9, 2) = \tfrac{2}{19}.$$

Now using (11.3) we have that $a_{i+2}F_{i+1}S_{i+1} = a_{i+1} \geqq \frac{3}{4}$, and so, as $a_{i+2} \leqq \frac{21}{4}$ and $S_{i+1} < \frac{2}{19}$, it follows that

$$F_{i+1} > 1.357. \tag{8.13}$$

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Hence, taking (8.10) into account, $p_{i+2} = 2$. Thus $S_{i+3} > \frac{12}{35}$, and so, to satisfy (8.10), $F_{i+3} < 1.1422$. This implies that $p_{i+4} \geqq 7$, and so $F_{i+1} < (1, 2, 1, 7) < 1.35$, contradicting (8.13).

LEMMA 8.2. *$p_j \leqq 7$ for all odd j.*

PROOF. Let $p_{i-1} = 8$ with i even. Then

$$\tfrac{10}{89} = (0, 8, 1, 9) < S_i < (0, 8, 2) = \tfrac{2}{17},$$

and so (8.10) implies that $F_i < 1.373$. In addition $F_i > 1.214$ follows on considering the relation $a_{i+1}F_iS_i = a_i \geqq \frac{3}{4}$.

Now the value $g_i(x, 1, 0)$ where $(x+\lambda)^2 \leqq \frac{1}{4}$ contradicts $m_+ = 1$ unless

$$\|\lambda-\tfrac{1}{2}\| < .113. \tag{8.14}$$

In addition the value $g_i(x, 0, 1)$ where $4 \leqq (x+\mu)^2 \leqq 6.25$ contradicts either $m_+ = 1$ or $m_- > 2.53$ unless $\|\mu-\frac{1}{2}\| < .096$. Hence $\|\lambda-\mu\| < .209$. However this implies that the value $g_i(x, 1, -1)$ where $9 \leqq (x+\lambda-\mu)^2 < 10.3$ contradicts either $m_+ = 1$ or $m_- > 2.53$.

LEMMA 8.3. *$p_j \leqq 6$ for all odd j.*

PROOF. Let $p_{i-1} = 7$ with i even. Then

$$\tfrac{9}{71} = (0, 7, 1, 8) < S_i < (0, 7, 2) = \tfrac{2}{15},$$

and so (8.10) implies that $F_i < 1.359$. Now $F_i > \frac{80}{71}$, hence

$$8.81 < a_{i+1}(1+F_i)(1-S_i) < 10.82.$$

As one of the values $g_i(x_1, 1, -1)$, $g_i(x_2, 1, -1)$, where

$$6.25 \leqq (x_1+\lambda-\mu)^2 \leqq 9 \leqq (x_2+\lambda-\mu)^2 \leqq 12.25,$$

contradicts either $m_+ = 1$ or $m_- > 2.53$ if $\|\lambda-\mu-\frac{1}{2}\| \geqq .213$, it follows that

$$\|\lambda-\mu-\tfrac{1}{2}\| < .213. \tag{8.15}$$

In addition it can be shown, as in the proof of lemma 8.2, that

$$\|\mu-\tfrac{1}{2}\| < .096. \tag{8.16}$$

Combining this with (8.15) yields that $\|\lambda\| < .309$. Thus the value $g_i(x, 1, 0)$ where $0 \leqq (x+\lambda)^2 < .096$ contradicts $m_+ = 1$ unless $a_{i+1}F_iS_i > .904$. Using the known bounds on a_{i+1} and S_i this yields that

$$F_i > 1.291, \tag{8.17}$$

and so $K_i > 1.417$. Thus

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$$m_- > 2.85,\ a_{i+1} > 5.10, \tag{8.18}$$

for if $m_- \leqq 2.85$ inequality (8.12) yields, as $\phi_i'(m_-) > 0$, that $K_i < 1.416$, contradicting (8.17).

Using the bounds (8.17) and (8.18) it can be shown that $a_{i+1}(1+F_i)(1-S_i) > 10.12$, and so in order that the value $g_i(x_2, 1, -1)$ shall not contradict either $m_+ = 1$ or $m_- > 2.85$ we must have

$$||\lambda - \mu - \tfrac{1}{2}|| < .166. \tag{8.19}$$

In addition, (8.16) can be refined to $||\mu - \frac{1}{2}|| < .031$ by using (8.18), and combining this with (8.19) yields that $||\lambda|| < .197$. However this implies that the value $g_i(x, 1, 0)$ where $0 \leqq (x+\lambda)^2 < .04$ contradicts $m_+ = 1$, and completes the proof of the lemma.

LEMMA 8.4. *$p_j \leqq 5$ for all odd j.*

PROOF. Let $p_{i-1} = 6$ with i even. Then

$$\tfrac{8}{55} = (0, 6, 1, 7) < S_i < (0, 6, 2) = \tfrac{2}{13},$$

and so (8.10) implies that $F_i < 1.34$. Hence $p_{i+1} \geqq 3$. Similarly by considering the reverse chain, it can be shown that $p_{i-3} \geqq 3$, and so $S_i < \frac{4}{27}$. Thus as $F_i > \frac{63}{55}$ it follows that

$$8.73 < a_{i+1}(1+F_i)(1-S_i) < 10.5$$

and so, as in the proof of the previous lemma

$$||\lambda - \mu - \tfrac{1}{2}|| < .213. \tag{8.20}$$

In addition

$$||\mu - \tfrac{1}{2}|| < .06 \tag{8.21}$$

for otherwise one of the values $g_i(x_1, 0, 1), g_i(x_2, 0, 1)$ where $2.25 \leqq (x_1+\mu)^2 \leqq 4 \leqq (x_2+\mu)^2 \leqq 6.25$ would contradict either $m_+ = 1$ or $m_- > 2.53$. Furthermore as $a_{i-1} = a_{i+1}(1+6F_i)(1-6S_i)$ it can be shown similarly that $||6\lambda - \mu - \frac{1}{2}|| < .06$. Then combining this with (8.20) and (8.21) yields that $||\lambda|| < .187$. Thus if $p_{i+1} \geqq 4$ then $F_i < \frac{11}{9}$ and the value $g_i(x, 1, 0)$ where $(x+\lambda)^2 < .035$ contradicts $m_+ = 1$. Hence $p_{i+1} = 3$, $F_i > (1, 3, 1, 7) > 1.258$, and $K_i > 1.403$. We can use these bounds to find a lower bound on m_- and hence to obtain contradictory bounds on $||3\lambda + 6\mu||$ as follows.

As $\phi_i'(m_-) > 0$ inequality (8.12) yields, if $m_- \leqq 2.8$, that $K_i < 1.4$, contradicting the above bound. Hence $m_- > 2.8$ and $a_{k+1} > 5.05$ for all even k. Then the value $g_i(x, 0, 1)$ where $4 \leqq (x+\mu)^2 \leqq 6.25$ contradicts either $m_+ = 1$ or $m_- > 2.8$ unless $||\mu - \frac{1}{2}|| < .041$. Similarly, as

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$a_{i+3} = a_{i+1}(4-3F_i)(4+3S_i)$, it can be shown that $||3\lambda+4\mu-\frac{1}{2}|| < .041$, and so

$$(8.22) \qquad ||3\lambda+6\mu-\tfrac{1}{2}|| < .123.$$

However as $F_i < (1, 3, 2) = \frac{9}{7}$ the value $g_i(x, 1, 2)$ where

$$6.25 \leqq (x+\lambda+2\mu)^2 \leqq 9$$

contradicts either $m_+ = 1$ or $m_- > 2.8$ unless $||\lambda+2\mu|| < .051$. This contradiction to (8.22) completes the proof of the lemma.

Lemma 8.5. *$p_j \leqq 4$ for all odd j.*

Proof. Let $p_{i-1} = 5$ with i even. Then

$$\tfrac{7}{41} = (0, 5, 1, 6) < S_i < (0, 5, 2) = \tfrac{2}{11},$$

and so (8.10) implies that $F_i < 1.315$. Hence $p_{i+1} \geqq 3$. Considering the sections $g_i(x, 0, 1)$ and $g_i(x, 5, -1)$ in the same way as the sections $g_i(x, 0, 1)$ and $g_i(x, 6, -1)$ were considered in the proof of lemma 8.4 it can be shown that

$$(8.23) \qquad ||\mu-\tfrac{1}{2}|| < .06 \text{ and } ||5\lambda-\mu-\tfrac{1}{2}|| < .06.$$

Now

$$8.48 < a_{i+1}(1+F_i)(1-S_i) < 9.96$$

and so, by the same method as was used in the proof of lemma 8.3, it can be shown that

$$(8.24) \qquad ||\lambda-\mu-\tfrac{1}{2}|| < .213.$$

Combining (8.23) and (8.24) yields that $||\lambda+2\mu|| < .344$, and so the value $g_i(x, 1, 2)$ where $7.05 < (x+\lambda+2\mu)^2 \leqq 9$ contradicts either $m_+ = 1$ or $m_- > 2.53$ unless both

$$(8.25) \qquad a_{i+1}(2-F_i)(2+S_i) \leqq 8,$$

and

$$(8.26) \qquad ||\lambda+2\mu|| \leqq 0.1.$$

Using the known bounds on a_{i+1} and S_i, (8.25) yields that $F_i > 1.228$. Hence $p_{i+1} = 3$, and so $F_i > (1, 3, 1, 6) = \frac{34}{27}$. Thus $K_i > 1.403$, which yields, by the same method as was used in the proof of the previous lemma, that $||3\lambda+6\mu-\frac{1}{2}|| < .123$ which is incompatible with (8.26).

Lemma 8.6. *$p_j \leqq 3$ for all odd j.*

Proof. Let $p_{i-1} = 4$ with i even. Then

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$$\tfrac{6}{29} = (0, 4, 1, 5) < S_i < (0, 4, 2) = \tfrac{2}{9},$$

and so (8.10) implies that $F_i < 1.279$. Hence as in the proof of lemma 8.4 we find $p_{i+1} \geqq 3$, $p_{i-3} \geqq 3$ and so $S_i < (0, 4, 1, 3) = \frac{4}{19}$.

As $K_i > 1+2(0, 4, 1, 5) = \frac{41}{29}$ it can be shown, by the same method as was used in the proof of lemma 8.4, that $a_{i+1} > 5.05$, $m_- > 2.8$, and

$$(8.27) \qquad ||\mu - \tfrac{1}{2}|| < .041.$$

Now the values $g_i(x_1, 1, 2)$ and $g_i(x_2, 1, -1)$ where

$$6.25 \leqq (x_1+\lambda+2\mu)^2 \leqq 9 \text{ and } 6.25 \leqq (x_2+\lambda-\mu)^2 \leqq 9$$

contradict either $m_+ = 1$ or $m_- > 2.8$ unless both $||\lambda+2\mu-\frac{1}{2}|| < .1$ and $||\lambda-\mu-\frac{1}{2}|| < .1$. However by subtraction these yield $||3\mu|| < .2$ which is incompatible with (8.27).

As a consequence of the preceding lemmas p_{i-1} can only be 1, 2 or 3 for i even. Hence, for i even, we have that $K_i > 1+2(0, 3, 1, 4) > 1.485$ which contradicts (8.10). From this contradiction we can deduce that the assumption (8.8) was false. Thus we may assume from now on that

$$(8.28) \qquad d > \tfrac{243}{16}.$$

We now obtain bounds on m_-, a_{i+1} and μ_i for even i, and K_i and p_i for both odd and even i.

Firstly, inserting (8.28) into (8.2) yields that $m_- > 3$. Then by an obvious modification of the corollary to theorem 1 in [9] applied to the sections $g_i(x, 0, z)$ it follows that

$$(8.29) \qquad a_{i+1} \geqq 7+(m_- - 3) > 7$$

for all even i. Hence the binary form $q_i(y, z)$ can take no values in the open interval $(-7, \frac{3}{4})$, so by the result of Segre it follows that $d \geqq \frac{35}{2}$. Using this in place of (8.28) and repeating the above analysis yields that $m_- > 3.145$, $a_{i+1} > 7.145$ and $d > 18.12$. Repeating the iteration a few more times yields that $m_- > 3.19$ and that $a_{i+1} > 7.19$ for all even i.

Combining (8.2), (8.3), (8.6) and (8.29) yields that

$$(8.30) \qquad K_i \leqq 4\sqrt{6\beta}[4+4\sqrt[3]{2\beta/3}]^{-1} < 1.308\sqrt[6]{\beta}$$

and

$$(8.31) \qquad K_i \leqq 16\sqrt{6\beta}/3 < 13.07\sqrt{\beta}$$

for even and odd i respectively. Hence

$$p_i = 1 \quad (i \text{ even}); \quad p_i \leqq 13 \quad (i \text{ odd}).$$

Now if $p_i \leqq 3$ with i odd then $K_{i-1} > (1, 4)+(0, 14) > 1.32$ which contradicts (8.30). Hence $p_i \geqq 4$ for all odd i. In addition, if $p_i \geqq 12$ with i

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odd then $K_i > 12+2(0, 1, 4) = 13.6$ which contradicts (8.31). Hence

$$4 \leqq p_i \leqq 11$$

for all odd i.

To obtain an upper bound on a_{i+1} we first note that $a_{i+1} < 8.4$ for all even i follows from (8.6) on observing that $K_i > (1, 12)+(0, 12) = \frac{7}{6}$ for even i. However if $a_{i+1} > 8$ for some even i then the value $g_i(x, 0, 1)$ where $6.25 \leqq (x+\mu)^2 \leqq 9$ would contradict either $m_+ = 1$ or $m_- > 3.19$. Hence, for all even i, $7.19 < a_{i+1} \leqq 8$.

For i even the lower bound on a_{i+1} may be improved, and a bound on $||\mu_i||$ obtained, as follows. For a given even j it may be assumed without loss of generality that $0 \leqq \mu_j \leqq \frac{1}{2}$. Then we must have

$$(2+\mu_j)^2-a_{j+1} \leqq -m_- \tag{8.32}$$

and

$$(3-\mu_j)^2-a_{j+1} \geqq 1. \tag{8.33}$$

On subtraction these yield that $10\mu_j \leqq 4-m_-$ and hence that $\mu_j < .081$. Thus $g_j(5, 0, 2) < 0$, hence in order not to contradict the definition of m_- we must have $(5+2\mu_j)^2-4a_{j+1} \leqq -m_-$. Multiplying this inequality by $\frac{9}{4}$ and rearranging yields that

$$(8+3\mu_j)^2-9a_{j+1} \leqq 7.75+3\mu_j-9m_-/4 < .83.$$

Hence, as this is $g_j(8, 0, 3)$, we have that

$$(8+3\mu_j)^2-9a_{j+1} \leqq -m_-. \tag{8.34}$$

Subtracting this from 9 times (8.33) yields that $102\mu_j < 8-m_-$, and so

$$\mu_j < .048. \tag{8.35}$$

Inserting this into (8.34) yields that

$$a_{j+1} > 7.46. \tag{8.36}$$

Thus, as the even j was chosen arbitrarily, (8.35) and (8.36) imply that

$$||\mu_i|| < .048 \tag{8.37}$$

and

$$a_{i+1} > 7.46 \tag{8.38}$$

for all even i. Using this new bound on a_{i+1} in the argument immediately following (8.29) yields that $m_- > 3.26$, and this enables (8.38) to be refined to

$$a_{i+1} > 7.47 \qquad \text{(all even } i\text{)}. \tag{8.39}$$

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Having obtained these general bounds on a_{i+1}, $||\mu_i||$, m_-, K_i and p_i we now proceed to eliminate all possible chains $[p_i]$ except that which gives rise to F_8.

LEMMA 8.7. *$p_i < 11$ for some odd i.*

PROOF. Let $p_i = 11$ for all odd i. Then

$$g_j = (x+\lambda_j y+\mu_j z)^2 - a_{j+1}(z^2-yz-\tfrac{1}{11}y^2)$$

for even j. Thus the value $g_j(x, 1, 0)$ where $(x+\lambda_j)^2 \leqq \frac{1}{4}$ contradicts $m_+ = 1$.

LEMMA 8.8. *$p_j \leqq 10$ for all odd j.*

PROOF. Let $p_{i+1} = 11$ with i even and suppose that $p_{i-1} \leqq 10$. Then

$$\tfrac{168}{155} = (1, 11, 1, 12) < F_i < (1, 11, 1, 4) = \tfrac{64}{59}$$

and

$$\tfrac{13}{142} = (0, 10, 1, 12) < S_i < (0, 4, 1, 4) = \tfrac{5}{24}.$$

Hence in order that the value $g_i(x, 1, 1)$ where $(x+\lambda+\mu)^2 \leqq \frac{1}{4}$ shall not contradict $m_+ = 1$ we must have $||\lambda+\mu-\frac{1}{2}|| < .075$, which, combined with (8.37), yields that $||\lambda-\mu-\frac{1}{2}|| < .171$. However this implies that at least one of the values $g_i(x_1, 1, -1)$ and $g_i(x_2, 1, -1)$ where

$$11.07 < (x_1+\lambda-\mu)^2 \leqq 12.25$$

and

$$12.25 \leqq (x_2+\lambda-\mu)^2 < 13.48$$

contradicts either $m_+ = 1$ or $m_- > 3.26$.

Hence $p_{i+1} = 11$ implies that $p_{i-1} = 11$. Repeating this argument indefinitely to both the original and the reverse chains shows that $p_j = 11$ for all odd j, in contradiction to the result of lemma 8.7.

LEMMA 8.9. *If $p_{i-1} = 10$ with i even then $p_{i+1} \leqq 6$.*

PROOF. Let $p_{i-1} = 10$ with i even and suppose that $p_{i+1} \geqq 7$. Then $\frac{143}{131} < F_i < \frac{44}{39}$ and $\frac{12}{131} < S_i < \frac{5}{54}$. Hence the value $g_i(x, 1, -1)$ where $12.25 \leqq (x+\lambda-\mu)^2 \leqq 16$ contradicts either $m_+ = 1$ or $m_- > 3.26$ unless $||\lambda-\mu|| < .108$. Combining this with (8.37) yields that $||\lambda|| < .156$ and hence that the value $g_i(x, 1, 0)$ where $(x+\lambda)^2 < .025$ contradicts $m_+ = 1$. This contradiction is sufficient to prove the lemma.

LEMMA 8.10. *$p_j \leqq 9$ for all odd j.*

PROOF. Let $p_{i-1} = 10$ with i even. Then $\frac{95}{83} < F_i < \frac{29}{24}$ and $\frac{8}{87} < S_i < \frac{5}{54}$ as p_{i-3} and p_{i+1} can be at most 6. Hence $K_i > 1.235$, and so applying the steps

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$$(8.40)\qquad \begin{gathered} d = (a_{i+1}K_i)^2/4 \\ m_- \geqq \sqrt[3]{16d/9} \end{gathered}$$

yields that $m_- > 3.35$. Now the value $g_i(x, 1, 0)$ where $(x+\lambda)^2 \leqq \frac{1}{4}$ contradicts $m_+ = 1$ unless $||\lambda-\frac{1}{2}|| < .178$, and combining this with (8.37) yields that $||\lambda+2\mu-\frac{1}{2}|| < .274$. Then at least one of the values $g_i(x_1, 1, 2)$, $g_i(x_2, 1, 2)$ where $10.4 < (x_1+\lambda+2\mu)^2 \leqq 12.25$ and $12.25 \leqq (x_2+\lambda+2\mu)^2 < 14.3$ lies in the open interval $(-3.35, 1)$ contradicting either $m_+ = 1$ or $m_- > 3.35$.

LEMMA 8.11. *If $p_{i-1} = 9$ with i even then $p_{i+1} \leqq 5$.*

PROOF. Let $p_{i-1} = 9$ with i even and suppose that $p_{i+1} \geqq 6$. Then $\frac{120}{109} < F_i < \frac{39}{34}$ and $\frac{11}{109} < S_i < \frac{5}{49}$. Now the value $g_i(x, 1, 0)$ where $(x+\lambda)^2 \leqq \frac{1}{4}$ contradicts $m_+ = 1$ unless $||\lambda-\frac{1}{2}|| < .26$, and combining this with (8.37) yields that $||\lambda-\mu-\frac{1}{2}|| < .308$. Then the value $g_i(x, 1, -1)$ where $12.25 \leqq (x+\lambda-\mu)^2 < 14.51$ contradicts either $m_+ = 1$ or $m_- > 3.26$.

LEMMA 8.12. *If $p_{i-1} = 9$ with i even then $p_{i+1} = 4$.*

PROOF. Let $p_{i-1} = 9$ with i even and suppose that $p_{i+1} = 5$. Then $\frac{76}{65} < F_i < \frac{34}{29}$, $\frac{7}{69} < S_i < \frac{5}{49}$ and so $K_i > 1.27$. Using this in steps (8.40) yields that $m_- > 3.419$. Now the value $g_i(x, 1, -1)$ where

$$12.25 \leqq (x+\lambda-\mu)^2 \leqq 16$$

contradicts either $m_+ = 1$ or $m_- > 3.419$ unless both $||\lambda-\mu|| < .057$ and $a_{i+1}(1+F_i)(1-S_i) \leqq 15$. Using (8.37) and the known bounds on F_i and S_i these inequalities imply that $||\lambda|| < .105$ and $a_{i+1} < 7.71$. Hence the value $g_i(x, 1, 0)$ where $(x+\lambda)^2 \leqq \frac{1}{4}$ contradicts $m_+ = 1$.

LEMMA 8.13. *$p_j \leqq 8$ for all odd j.*

PROOF. Let $p_{i-1} = 9$ with i even. Then the above lemma shows that $p_{i-3} = p_{i+1} = 4$. Hence $\frac{65}{54} < F_i < \frac{29}{24}$, $\frac{6}{59} < S_i < \frac{5}{49}$, and so $K_i > 1.304$. Using this in steps (8.40) in conjunction with the bounds $a_{i+1} > 7.47$, $a_{i+1} \geqq 7.922$ and $a_{i+1} \geqq 7.992$ yields that $m_- > 3.48$, $m_- > 3.619$ and $m_- > 3.641$ respectively. Now

$$14.78 < a_{i+1}(1+F_i)(1-S_i) < 15.884,$$

where the upper bound may be reduced to 15.869 or 15.73 according as the upper bound on a_{i+1} is reduced to 7.992 or 7.922 respectively. Hence the value $g_i(x, 1, -1)$ where $12.25 \leqq (x+\lambda-\mu)^2 \leqq 16$ contradicts either $m_+ = 1$ or the relevant bound on m_- unless both $||\lambda-\mu|| < .03$ and $a_{i+1}(1+F_i)(1-S_i) \leqq 15$. Thus $||\lambda|| < .078$ and $a_{i+1} < 7.59$, and so the value $g_i(x, 1, 0)$ where $(x+\lambda)^2 \leqq \frac{1}{4}$ contradicts $m_+ = 1$.

LEMMA 8.14. *$p_j \geqq 5$ for all odd j.*

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PROOF. Let $p_j = 4$ with j odd. Then $K_{j+1} > (0, 5)+(1, 9) > 1.31$ in contradiction to (8.30).

LEMMA 8.15. *If $p_j = 8$ for all odd j then $g \sim F_8\sqrt[3]{}/24$.*

PROOF. Let $p_j = 8$ for all odd j. Then

$$g_i = (x+\lambda_i y+\mu_i z)^2 - a_{i+1}(z^2-yz-\tfrac{1}{8}y^2)$$

for i even. As $g_i \sim g$ the subscripts can be dropped without loss of generality. Then (8.37) and (8.38) imply that

$$||\mu|| < .048 \tag{8.41}$$

and $7.47 < a \leqq 8$. In addition $m_- \geqq \sqrt[3]{2a^2/3} > 3.33$.

Now the value $g(x, 1, -1)$ where $12.25 \leqq (x+\lambda-\mu)^2 \leqq 16$ contradicts either $m_+ = 1$ or $m_- > 3.33$ unless

$$||\lambda-\mu|| < .13. \tag{8.42}$$

Combining this with (8.41) yields that $||2\lambda-\mu|| < .31$. Then the value $g(x, 2, -1)$ where $16 \leqq (x+2\lambda-\mu)^2 < 18.6$ contradicts $m_- > 3.33$ unless $||2\lambda-\mu|| < .083$. Hence $||2\lambda|| < .131$, and combining this with (8.41) and (8.42) yields that

$$||\lambda|| < .066. \tag{8.43}$$

As the transformations $(y, z) \to (y+8z, -z)$ and $(y,z) \to (y+8z, y+9z)$ applied to g only replace (λ, μ) by $(\lambda, 8\lambda-\mu)$ and $(\lambda+\mu, 8\lambda+9\mu)$ respectively it is clear that any bound for $||\mu||$ must also hold for both $||8\lambda-\mu||$ and $||8\lambda+9\mu||$. Now let $||\mu|| \leqq r$ be the best such bound possible. Then $||8\lambda-\mu|| \leqq r$ and $||8\lambda+9\mu|| \leqq r$, and combining these three bounds yields that $||8\lambda|| \leqq 2r$ and $||10\mu|| \leqq 2r$. The second of these yields, as (8.41) implies that $r \leqq .048$, that $||\mu|| \leqq r/5$. Clearly, from the definition of r, this implies that $r = 0$. Hence $||\mu|| = 0$ and $||8\lambda|| = 0$. Thus as the second of these yields, considering (8.43), that $||\lambda|| = 0$, we must have $g \sim x^2-a(z^2-yz-\frac{1}{8}y^2)$. Hence as the value $a/8$ contradicts $m_+ = 1$ unless $a = 8$ we must have

$$g \sim x^2-8(z^2-yz-\tfrac{1}{8}y^2) = F_8\sqrt[3]{}/24$$

as required.

LEMMA 8.16. *If $p_{i-1} = 8$ with i even then $p_{i+1} \neq 7$.*

PROOF. Let $p_{i-1} = 8$ and $p_{i+1} = 7$ where i is even. Then $\frac{89}{79} < F_i < \frac{53}{47}$ and $\frac{10}{89} < S_i < \frac{6}{53}$, and so $K_i > 1.2389$. Using this in (8.1) and (8.5) yields that $a_{i+1} < 7.91$. This implies that the value $g_i(x, 1, -1)$ where $12.25 \leqq (x+\lambda-\mu)^2 \leqq 16$ contradicts either $m_+ = 1$ or $m_- > 3.26$ unless $||\lambda-\mu|| < .117$. Thus $||\lambda|| < .165$, and so the value $g_i(x, 1, 0)$ where

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$(x+\lambda)^2 \leqq \frac{1}{4}$ contradicts $m_+ = 1$ unless $a_{i+1} > 7.61$. Then repeating the above analysis with this new bound on a_{i+1} yields that $||\lambda-\mu|| < 0.83$, $||\lambda|| < .131$ and $a_{i+1} \geqq 7.69$. Hence $||2\lambda+3\mu|| < .406$, and so the value $g_i(x, 2, 3)$ where $16 \leqq (x+2\lambda+3\mu)^2 < 19.42$ contradicts either $m_+ = 1$ or $m_- > 3.26$.

Lemma 8.17. *If $p_{i-1} = 8$ with i even then $p_{i+1} \neq 6$.*

Proof. Let $p_{i-1} = 8$ and $p_{i+1} = 6$ with i even. Then $\frac{79}{69} < F_i < \frac{47}{41}$, $\frac{10}{89} < S_i < \frac{6}{53}$, and $K_i > 1.257$. Using this in (8.1), (8.5) and (8.40) yields that $m_- > 3.39$ and $a_{i+1} < 7.795$. Using the method of proof of the previous lemma yields that $||\lambda-\mu|| < .102$, and so $||2\lambda-\mu|| < .254$. Hence the value $g_i(x, 2, -1)$ where $16 \leqq (x+2\lambda-\mu)^2 < 18.1$ contradicts $m_- > 3.39$ unless both $||2\lambda-\mu|| < .064$ and $a_{i+1} > 7.59$. Hence $||2\lambda+3\mu|| < .16$, and so the value $g_i(x, 2, 3)$ where $16 \leqq (x+2\lambda+3\mu)^2 < 18.2$ contradicts either $m_+ = 1$ or $m_- > 3.39$.

Lemma 8.18. *If $p_{i-1} = 8$ with i even then $p_{i+1} \neq 5$.*

Proof. Let $p_{i-1} = 8$ and $p_{i+1} = 5$ with i even. Then

$$\tfrac{69}{59} < F_i < \tfrac{41}{35},\ \tfrac{10}{89} < S_i < \tfrac{6}{53}$$

and $K_i > 1.2818$. Using this in (8.1), (8.5) and (8.40) yields that $a_{i+1} < 7.644$ and $m_- > 3.44$. Using the method of proof of lemma 8.16 yields that $||\lambda-\mu|| < .08$. Hence $||2\lambda+3\mu|| < .4$, and so the value $g_i(x, 2, 3)$ where $12.96 < (x+2\lambda+3\mu)^2 \leqq 16$ contradicts either $m_+ = 1$ or $m_- > 3.44$.

From the above work it is clear that if $p_i = 8$ for some odd i then $p_i = 8$ for all odd i and $g \sim F_8\sqrt[3]{24}$. Hence for the rest of the proof we may assume that $p_j \leqq 7$ for all odd j.

Lemma 8.19. *If $p_{i-1} = 7$ with i even then $p_{i+1} = 7$.*

Proof. Let $p_{i-1} = 7$ with i even and let $p_{i+1} \leqq 6$. Then

$$\tfrac{71}{62} < F_i < \tfrac{41}{35},\quad \tfrac{9}{71} < S_i < \tfrac{6}{47},\quad \text{and}\quad K_i > 1.2719.$$

Using this in (8.1), (8.5) and (8.40) yields that $a_{i+1} < 7.704$ and $m_- > 3.42$. Then following the method of proof of lemma 8.16 yields that $||\lambda-\mu|| < .131$. Hence $||2\lambda-\mu|| < .310$, and so the value $g_i(x, 2, -1)$ where

$$16 \leqq (x+2\lambda-\mu)^2 < 18.6$$

contradicts either $m_+ = 1$ or $m_- > 3.42$.

Lemma 8.20. *$p_j = 6$ for all odd j.*

Proof. Let $p_{i-1} = 7$ with i even. Then lemma 8.19, applied to both the original and reverse chains shows that $p_j = 7$ for all odd j, and so

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$$g_i = (x+\lambda_i y+\mu_i z)^2 - a_{i+1}(z^2-yz-\tfrac{1}{7}y^2).$$

As usual the suffixes may be dropped without loss of generality. Then as $d = 11a^2/28$ and $a > 7.47$, equations (8.2) and (8.5) imply that $a < 7.82$ and $m_- > 3.39$. Following the method of proof of lemma 8.16 yields that $||\lambda-\mu|| < .144$. Hence $||2\lambda-\mu|| < .336$, and so the value $g_i(x, 2, -1)$ where $16 \leqq (x+2\lambda-\mu)^2 < 18.81$ contradicts either $m_+ = 1$ or $m_- > 3.39$. This contradiction implies that $p_j \leqq 6$ for all odd j. However if $p_i \leqq 5$ for some odd i then $K_{i-1} > (1, 5, 1, 7)+(0, 6, 1, 7) > 1.308$, contradicting (8.30). Hence $p_j = 6$ for all odd j.

LEMMA 8.21. *$p_i \neq 6$ for some odd i.*

PROOF. Let $p_j = 6$ for all odd j. Then

$$g_i = (x+\lambda_i y+\mu_i z)^2 - a_{i+1}(z^2-yz-\tfrac{1}{6}y^2)$$

for even i. As usual the suffixes may be dropped without loss of generality. Then as $d = 5a^2/12$ and $a > 7.47$, equations (8.2) and (8.5) yield that $a < 7.59$ and $m_- > 3.45$. Following the method of proof of lemma (8.16) yields that $||\lambda-\mu|| < .168$. Hence $||2\lambda-\mu|| < .384$, and so the value $g_i(x, 2, -1)$ where $16 \leqq (x+2\lambda-\mu)^2 < 19.22$ contradicts either $m_+ = 1$ or $m_- > 3.45$ unless $||2\lambda-\mu|| > .293$. Combining this with (8.37) yields that $.245 < ||2\lambda-\mu|| < .336$. Hence the value $g_i(x, 2, -2)$ where

$$52.49 < (x+2\lambda-2\mu)^2 < 53.82$$

contradicts $m_- > 3.45$.

The result of theorem C_7 now follows as we have shown that if $0 < d \leqq 24$ and if $m_-(g) \geqq \sqrt[3]{(16d/9)}$ then g is equivalent to a multiple of either F_7 or F_8.

9. The proof of theorem C_8

For reference theorem C_8 is restated.

THEOREM C_8. *If g is any indefinite ternary quadratic form of signature 1 with $d(g) = d$ where $0 < d \leqq 67.5$ and if $m_+(g) = m_+ = 1$ is attained by g then either*

(*a*) *$m_-(g) < \sqrt[3]{8d/3}$, or*

(*b*) *g is equivalent to a multiple of either F_8 or F_9.*

PROOF. Let (g_i) be an equivalence chain of forms equivalent to g. Assuming that $m_-(g) \geqq \sqrt[3]{8d/3}$ and using the usual notation we have that

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(9.1) $$a_{i+1}K_i = \Delta; \quad \Delta^2 = 4d,$$

(9.2) $$m_-(g) = m_- \geqq \sqrt[3]{8d/3},$$

(9.3) $$a_i \geqq m_- + \tfrac{1}{4} \geqq \sqrt[3]{8d/3} + \tfrac{1}{4} \quad (i \text{ odd}),$$

(9.4) $$a_i \geqq \tfrac{3}{4} \quad (i \text{ even}),$$

(9.5) $$d = 135\beta/2, \quad 0 < \beta \leqq 1, \quad \text{and}$$

(9.6) $$K_i = 3\sqrt{30\beta}/a_{i+1}.$$

Now if $d \leqq 24$ then theorem C_7 shows that $m_- < \sqrt[3]{8d/3}$ unless g is equivalent to a multiple of F_8. Thus it is only necessary to show, assuming that $d > 24$, that g is equivalent to a multiple of F_9.

Under this assumption we have that $m_- \geqq \sqrt[3]{8d/3} > 4$. Applying theorem 1 of [9] to the sections

(9.7) $$(x+\mu_i z)^2 - a_{i+1}z^2$$

of g_i (where i is even) yields that $a_{i+1} \geqq 10.25$ for all even i, and hence that $q_i(y, z)$ can take no values in the open interval $(-10.25, .75)$. Then applying Segre's result yields that $d = d(q_i) \geqq \frac{2173}{64}$, and using this in (9.2) gives $m_- > 4.49$. Thus, applying the corollary to theorem 1 of [9] to the sections (9.7) of g_i, it follows that $a_{i+1} > 10.74$ for all even i. Repetition of the above process yields, after a few iterations, that $d > 37.87$, $m_- > 4.65$ and

(9.8) $$a_{i+1} > 10.9 \quad (i \text{ even}).$$

For the present we shall assume that

(9.9) $$8d/3 \leqq 125.$$

Using this in (9.5) yields a bound on β which in conjunction with (9.4), (9.6) and (9.8) yields that

(9.10) $$K_i < 18.267 \quad (i \text{ odd})$$

and

(9.11) $$K_i < 1.257 \quad (i \text{ even}).$$

Hence $p_i = 1$ for all even i and $p_i \leqq 18$ for all odd i.

Now if $p_k \leqq 3$ for some odd k then $K_{k-1} > (1, 4)+(0, 19) > 1.3$ which contradicts (9.11). Hence $p_i \geqq 4$ for all odd i. Thus $p_i \leqq 16$ for all odd i, as otherwise, if $p_k \geqq 17$ with k odd, K_k would be greater than 18.6, contradicting (9.10). This in turn implies that $p_i \geqq 5$ for all odd i in order to satisfy (9.11). Thus we have shown that

(9.12) $$p_i = 1 \quad (i \text{ even}); \qquad 5 \leqq p_i \leqq 16 \quad (i \text{ odd}).$$

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Before commencing to eliminate various $[p_i]$ chains it is necessary to obtain upper bounds on a_{i+1} and $||\mu_i-\frac{1}{2}||$ for even i. From the bounds (9.12) it follows that $K_i > 1+2(0, 16, 1, 17) = \frac{341}{305}$ for even i, and inserting this into (9.1) and (9.9) gives that $a_{i+1} < 12.26$ for even i. For even i it is clear that $(3+||\mu_i||)^2-a_{i+1} \geqq 1$ and $(3-||\mu_i||)^2-a_{i+1} \leqq -m_-$ in order not to contradict either $m_+ = 1$ or the definition of m_-. The first of these yields that $a_{i+1} \leqq 11.25$, while subtracting the second from the first yields that

$$||\mu_i|| \geqq (1+m_-)/12 \tag{9.13}$$

for all even i. As $m_- > 4.65$ this implies that

$$||\mu_i-\tfrac{1}{2}|| < .03 \tag{9.14}$$

for all even i.

As $m_- > 4$, applying the corollary to theorem 1 of [9] to the sections (9.7) yields that

$$a_{i+1} \geqq 10.25+(m_--4) \quad (i \text{ even}). \tag{9.15}$$

Combining this with (9.1) and (9.2) yields that

$$\psi_i(m_-) = m_-^3-2K_i^2(6.25+m_-)^2/3 \geqq 0 \quad (i \text{ even}). \tag{9.16}$$

Now from (9.15), as $a_{i+1} \leqq 11.25$, it is clear that $m_- \leqq 5$ and so the inequality (9.16) must be satisfied for some $m_- \leqq 5$. However using the known bounds on m_- and K_i it can easily be shown that the derivative

$$\psi_i'(m_-) = 3m_-^2-4K_i^2(6.25+m_-)/3 > 0,$$

and so (9.16) must be satisfied with $m_- = 5$. Hence

$$K_i \leqq \sqrt{\tfrac{40}{27}} < 1.2172 \quad (i \text{ even}). \tag{9.17}$$

Furthermore as $\psi_i'(m_-) > 0$ it is clear that if $\psi_i(x) < 0$ for all allowable values of K_i then $m_- > x$.

The bounds (9.12) on p_i for i odd can now be improved to $6 \leqq p_i \leqq 16$, for if $p_k \leqq 5$ for some odd k then $K_{k-1} > (1, 6)+(0, 17) > 1.22$ contradicting (9.17).

The proof is now continued as a series of lemmas eliminating all possibilities for the chain $[p_i]$.

LEMMA 9.1. *$p_j \leqq 14$ for all odd j.*

PROOF. Let $15 \leqq p_{i-1} \leqq 16$ for some even i. Then $\frac{18}{17} < F_i < \frac{55}{48}$, $\frac{1}{17} < S_i < \frac{7}{111}$, and so $a_{i+1}F_iS_i < .813$. Hence $||\lambda-\frac{1}{2}|| < .07$ in order that the value $||\lambda||^2+a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$. Using (9.14) this

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implies that $||2\lambda-\mu-\frac{1}{2}|| < .17$, and so the value $g_i(x, 2, -1)$ where $28.4 < (x+2\lambda-\mu)^2 \leqq 30.25$ contradicts either $m_+ = 1$ or $m_- > 4.65$.

LEMMA 9.2. *$p_j \leqq 9$ for all odd j.*

PROOF. Let $10 \leqq p_{i-1} \leqq 14$ for some even i. Then $\frac{255}{239} < F_i < \frac{433}{378}$ and $\frac{16}{239} < S_i < \frac{7}{76}$. Thus $||\lambda-\mu|| < .23$ and $||\lambda+2\mu|| < .17$ in order that the values $g_i(x_1, 1, -1)$ and $g_i(x_2, 1, 2)$ where $16 \leqq (x_1+\lambda-\mu)^2 \leqq 20.25$ and $16 \leqq (x_2+\lambda+2\mu)^2 \leqq 20.25$ shall not contradict either $m_+ = 1$ or $m_- > 4.65$. However on subtraction these yield that $||3\mu|| < .4$, in contradiction with (9.14).

LEMMA 9.3. *$p_j > 9$ for at least one odd j.*

PROOF. Let $p_j \leqq 9$ for all odd j. Then

$$\frac{120}{109} < F_i < \frac{55}{48}, \quad \frac{11}{109} < S_i < \frac{7}{48}, \quad \text{and so} \quad K_i > 1.2018.$$

Hence as $\psi_i(4.93) < 0$ for $K_i > 1.2018$ we must have $m_- > 4.93$ and $a_{i+1} > 11.18$. Thus $||\lambda-\mu|| < .12$ and $||\lambda+2\mu|| < .12$ in order that the values $g_i(x_1, 1, -1)$ and $g_i(x_2, 1, 2)$, as in the proof of lemma 9.2, shall not contradict either $m_+ = 1$ or $m_- > 4.93$. However on subtraction these yield that $||3\mu|| < .24$, in contradiction with (9.14).

From the contradiction of lemmas 9.2 and 9.3 it is clear that the assumption (9.9) is false. Hence

$$8d/3 > 125 \tag{9.18}$$

and so, using (9.2), $m_- > 5$.

By an obvious modification of the corollary to lemma 1 of [9] applied to the sections (9.7) of g_i it follows that

$$a_{i+1} \geqq 14+(m_- -5) > 14 \tag{9.19}$$

for all even i. Hence the binary form $q_i(y, z)$ can take no values in the open interval $(-14, .75)$ and so by the result of Segre it follows that $d = d(q_i) \geqq 59.5$. This may now be used to obtain new bounds on m_- and a_{i+1}, and repeating this iterative process a number of times yields that $m_- > 5.538$ and $a_{i+1} > 14.538$ for all even i. Combining this with (9.4) and (9.6) gives that

$$K_i < 21.911 \quad (i \text{ odd}); \qquad K_i < 1.1304 \quad (i \text{ even}). \tag{9.20}$$

A tighter bound on K_i for i even may be obtained as follows. As $a_{i+1} \geqq 9+m_- \geqq 9+\sqrt[3]{2\Delta^2/3}$ we have that

$$K_i \leqq \Delta/[9+\sqrt[3]{2\Delta/3}].$$

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Now the right hand side of this inequality has positive derivative with respect to Δ (over the allowable range) and so as $\Delta \leqq \sqrt{270}$ we have that

$$K_i \leqq \sqrt{270}/(9+\sqrt[3]{180}) < 1.1222 \quad (i \text{ even}). \tag{9.21}$$

It is clear, from (9.20) and (9.21), that $p_i = 1$ for all even i and that $p_i \leqq 21$ for all odd i. Now suppose that $p_k = 21$ for some odd k. Then $K_k > (21, 2)+(0, 2) = 22$ contradicting (9.20). Hence $p_i \leqq 20$ for all odd i. In addition $p_i \geqq 13$ for all odd i, for if $p_k \leqq 12$ with k odd then $K_{k-1} > (1, 13)+(0, 21) > 1.124$ contradicting (9.21).

We now find upper bounds on a_{i+1} and $||\mu_i||$ for all even i. As $K_i > 1+2(0, 20, 1, 21) = \frac{505}{461}$, using (9.6) yields that $a_{i+1} < 15.01$. Then

$$(4-||\mu_i||)^2 - a_{i+1} \geqq 1$$

and

$$(3+||\mu_i||)^2 - a_{i+1} < -5.538$$

in order not to contradict either $m_+ = 1$ or $m_- > 5.538$. From the first of these it follows that $a_{i+1} \leqq 15$, while subtracting the first from the second yields that $||\mu_i|| < .033$. Hence

$$||\mu_i|| < .033 \quad (\text{all even } i). \tag{9.22}$$

We are now in a position to work on the $[p_i]$ chain, eliminating all possibilities except that which gives g as equivalent to a multiple of F_9.

LEMMA 9.4. *If $p_j = 20$ for all odd j then $g \sim F_9\sqrt[3]{\frac{135}{2}}$.*

PROOF. Let $p_j = 20$ for all odd j. Then

$$g_i = (x+\lambda_i y+\mu_i z)^2 - a_{i+1}(z^2 - yz - \tfrac{1}{20}y^2)$$

for any even i. Clearly, as $a_{i+1} \leqq 15$, the values $||\lambda_i||^2 + a_{i+1}/20$ and $||\lambda_i+\mu_i||^2 + a_{i+1}/20$ contradict $m_+ = 1$ unless $a_{i+1} = 15$, $||\lambda_i|| = \frac{1}{2}$ and $||\mu_i|| = 0$. Hence

$$g \sim g_i \sim (x+\tfrac{1}{2}y)^2 - 15(z^2 - yz - \tfrac{1}{20}y^2) = F_9\sqrt[3]{\tfrac{135}{2}}$$

as required.

In order to eliminate the other possibilities for the chain $[p_i]$ we shall suppose from now on that $p_i < 20$ for at least one odd i.

LEMMA 9.5. *If $p_{i-1} = 20$ with i even then $p_{i+1} \neq 19$.*

PROOF. Let $p_{i-1} = 20$ and $p_{i+1} = 19$ with i even. Then

$$\tfrac{461}{439} < F_i < \tfrac{293}{279}, \quad \tfrac{22}{461} < S_i < \tfrac{14}{293}$$

and so $a_{i+1}F_iS_i < .757$. Hence $||\lambda - \frac{1}{2}|| < .003$ and $a_{i+1} > 14.94$ in order that $g_i(x, 1, 0)$ where $(x+\lambda)^2 \leqq \frac{1}{4}$ shall not contradict $m_+ = 1$. This implies

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that $||\mu|| < .008$ as otherwise the value $(4-||\mu||)^2 - a_{i+1}$ will contradict either $m_+ = 1$ or $m_- > 5.538$.

Combining the above bounds on $||\lambda - \frac{1}{2}||$ and $||\mu||$ yields that $||8\lambda+9\mu|| < .096$, and so the value $g_i(x, 8, 9)$ where $81 \leqq (x+8\lambda+9\mu)^2 < 82.8$ contradicts $m_- > 5.538$.

LEMMA 9.6. $p_j \leqq 19$ *for all odd* j.

PROOF. Let $p_k = 20$ for some odd k. Then by the above lemma there must occur, either in the original or the reverse chain, an even i such that $p_{i-1} = 20$ and $13 \leqq p_{i+1} \leqq 18$. Then $\frac{441}{419} < F_i < \frac{209}{195}$, $\frac{22}{461} < S_i < \frac{14}{293}$ and $a_{i+1}F_iS_i < .7685$. Hence $||\lambda - \frac{1}{2}|| < .019$ and $a_{i+1} > 14.638$ in order that $g_i(x, 1, 0)$ where $(x+\lambda)^2 \leqq \frac{1}{4}$ shall not contradict $m_+ = 1$. Thus, using (9.22), $||2\lambda+3\mu|| < .137$, and so the value $g_i(x, 2, 3)$ where $36 \leqq (x+2\lambda+3\mu)^2 < 38$ contradicts $m_- > 5.538$.

LEMMA 9.7. $p_j \leqq 18$ *for all odd* j.

PROOF. Let $p_{i-1} = 19$ with i even. Then

$$\tfrac{440}{419} < F_i < \tfrac{209}{195},\ \tfrac{21}{419} < S_i < \tfrac{14}{279},\ \text{and } K_i > 1.1002.$$

Using this in (9.6) yields that $a_{i+1} < 14.936$. Hence $||\lambda - \frac{1}{2}|| < .06$ in order that the value $g_i(x, 1, 0)$ where $(x+\lambda)^2 \leqq \frac{1}{4}$ shall not contradict $m_+ = 1$. Combining this with (9.22) yields that $||2\lambda+3\mu|| < .22$, and so the value $g_i(x, 2, 3)$ where $36 \leqq (x+2\lambda+3\mu)^2 < 38.7$ contradicts either $m_+ = 1$ or $m_- > 5.538$ unless $a_{i+1} > 14.8$, $3-2F_i > .897$ and $||2\lambda+3\mu|| < .011$. Using (9.22) this yields that $||8\lambda+9\mu|| < .143$, and so one of the values $g_i(x_1, 8, 9)$, $g_i(x_2, 8, 9)$ where

$$78.4 < (x_1+8\lambda+9\mu)^2 \leqq 81 \text{ and } 81 \leqq (x_2+8\lambda+9\mu)^2 < 83.6$$

contradicts either $m_+ = 1$ or $m_- > 5.538$.

LEMMA 9.8. $p_j \geqq 14$ *for all odd* j.

PROOF. If $p_i \leqq 13$ for some odd i then

$$K_{i+1} > (1, 18, 1, 19) + (0, 13, 1, 19) > 1.1222,$$

contradicting (9.21).

LEMMA 9.9. $p_j \leqq 17$ *for all odd* j.

PROOF. Let $p_{i-1} = 18$ with i even. Then

$$\tfrac{399}{379} < F_i < \tfrac{239}{224},\ \tfrac{20}{379} < S_i < \tfrac{15}{284},\ \text{and } K_i > 1.1055.$$

Using the same method as in the proof of lemma 9.7 it can be shown that $a_{i+1} < 14.862$, $||\lambda - \frac{1}{2}|| < .1$ and $||2\lambda+3\mu|| < .3$. Hence the value $g_i(x, 2, 3)$ where $36 \leqq (x+2\lambda+3\mu)^2 < 39.7$ contradicts either $m_+ = 1$ or $m_- > 5.538$.

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LEMMA 9.10. *$p_j \geqq 15$ for all odd j.*

PROOF. If $p_i \leqq 14$ for some odd i then

$$K_{i+1} > (1, 17, 1, 18)+(0, 14, 1, 18) > 1.1222$$

contradicting (9.21).

LEMMA 9.11. *$p_j \leqq 16$ for all odd j.*

PROOF. Let $p_{i-1} = 17$ with i even. Then

$$\tfrac{360}{341} < F_i < \tfrac{271}{255},\ \tfrac{19}{341} < S_i < \tfrac{16}{287},\ \text{and } K_i > 1.1114.$$

Using the same method as in the proof of lemma 9.7 it can be shown that $a_{i+1} < 14.785$ and $||\lambda-\frac{1}{2}|| < .148$. Combining this with (9.22) yields that $||2\lambda-\mu|| < .33$ and so the value $g_i(x, 2, -1)$ where

$$36 \leqq (x+2\lambda-\mu)^2 < 40.1$$

contradicts $m_+ > 5.538$.

LEMMA 9.12. *$p_j \leqq 15$ for all odd j.*

PROOF. Let $p_{i-1} = 16$ with i even. Then

$$\tfrac{323}{305} < F_i < \tfrac{271}{255},\ \tfrac{18}{305} < S_i < \tfrac{16}{271},\ \text{and } K_i > 1.118.$$

Using this in (9.6) yields that $a_{i+1} < 14.7$. Hence $||2\lambda-\mu-\frac{1}{2}|| < .1$ in order that the value $g_i(x, 2, -1)$ where $36 \leqq (x+2\lambda-\mu)^2 \leqq 42.25$ shall not contradict either $m_+ = 1$ or $m_- > 5.538$. This implies that $||\lambda-\mu-\frac{1}{2}|| > .18$, and so the value $g_i(x, 1, -1)$ where $25 \leqq (x+\lambda-\mu)^2 < 28.4$ contradicts either $m_+ = 1$ or $m_- > 5.538$.

LEMMA 9.13. *$p_j > 15$ for at least one odd j.*

PROOF. If $p_j \leqq 15$ for all odd j then $K_i > 1+2(0, 15, 1, 16) > 1.1222$ for all even i, contradicting (9.21).

From the contradiction of lemmas 9.12 and 9.13 it is clear that we have eliminated all possible chains $[p_i]$ which have $p_j \leqq 19$ for at least one odd j. Hence $p_j = 20$ for all odd j and theorem C_8 follows from lemma 9.4.

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