INDUCED CONNECTIONS AND IMBEDDED RIEMANNIAN SPACES

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§ 1. Introduction

Let P be a principal fibre bundle over M with group G and with projection $\pi : P \rightarrow M$. By definition of a principal fibre bundle, G acts on P on the right. We shall denote this transformation law by ρ;

$$\rho(u, s) = u \cdot s \in P \qquad \text{for any } u \in P \text{ and } s \in G.$$

Given a continuous map h of a topological space M' into M, let $h^{-1}(P)$ be the set of points (x', u) of $M \times P$ such that $\pi(u) = h(x')$. Define the projection π' of $h^{-1}(P)$ onto M' and the right translations by G as follows;

$$\pi'(x', u) = x',$$
$$(x', u)s = (x', us).$$

The principal fibre bundle $h^{-1}(P)$, thus obtained, is said to be *induced by* h. The map $\tilde{h}$ of $h^{-1}(P)$ into P defined by

$$\tilde{h}(x', u) = u$$

is a bundle map in the sense that it commutes with the right translations by G.

A principal fibre bundle P is *universal* relative to a space M', if every principal fibre bundle over M' with group G can be induced by a map h of M' into M and if two such induced bundles are equivalent if and only if the maps are homotopic. It is well known that, if M' is a manifold and G is a compact Lie group, then there exists always a universal bundle P [7].

From now on, we assume that every bundle P is differentiable; P and M are differentiable manifolds and the projection π is differentiable and the structure group G is a Lie group (not necessarily connected).

Let P' be a principal fibre bundle over M' with group G and with projection π'. Let $\tilde{h}$ be a bundle map of P' into P. Assume that there is given

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an infinitesimal connection in P, which will be defined by a $\mathfrak{g}$-valued linear differential form ω on P with the following properties ($\mathfrak{g}$ is the Lie algebra of G) [2].[2)]

$$(\omega.\ 1) \qquad \omega(u \cdot \bar{s}) = s^{-1}\bar{s} \qquad \text{for any } \bar{s} \in T_s(G) \text{ and } u \in P;$$

$$(\omega.\ 2) \qquad \omega(\bar{u} \cdot s) = s^{-1}\omega(\bar{u})s \qquad \text{for any } s \in G \text{ and } \bar{u} \in T(P).$$

Let ω' be the differential form on P' induced from ω by $\tilde{h}$, i.e.,

$$\omega' = \omega \circ \delta\tilde{h},$$

where $\delta\tilde{h}$ is the differential of $\tilde{h}$.

It is easy to see that the form ω' satisfies the conditions (ω. 1, 2), hence defines an infinitesimal connection in P'. The connection in P' obtained in this way is said to be *induced* from the connection in P by $\tilde{h}$.

Naturally arises the following question. Let P be a universal principal fibre bundle relative to a manifold M'. Given any connection in any principal fibre bundle P' over M' with group G, does there exist a connection in P, from which the connection in P' is induced by a bundle map $\tilde{h}$ of P' into P? The purpose of the present paper is to study this question in the case where M' is an imbedded Riemannian space and P' is the bundle of orthogonal frames over M'. Suppose that M' is an n-dimensional Riemannian space imbedded in the $(n+k)$-dimensional Euclidean space. In the study of characteristic classes, Chern [1] and Pontrjagin [6] considered the natural map h (the generalization of Gaussian spherical map) of M' into $M_{n,k}$ (the Grassmann manifold) and the induced homomorphism h^* of $H^*(M_{n,k})$ into $H^*(M')$. Their results will be understood better if the problem is studied in the following two steps: (1) the relation between the canonical connection in the bundle of Grassmann $P_{n,k}$ and the Riemannian connection on M' and (2) the relation between the canonical connection in $P_{n,k}$ and the invariant Riemannian connection on $M_{n,k}$, This paper deals with part (1), and part (2) will be studied in another paper.

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2) If G is a Lie group whose multiplication law is given by $\varphi : G \times G \to G$, then $T(G)$ (the set of all tangent vectors to G) is also a Lie group whose multiplication law is the differential $\delta\varphi$ of φ. And G is considered as a subgroup of $T(G)$. The Lie algebra of G can be identified with $T_e(G)$ (the set of all tangent vectors to G at the unit e). The differential $\delta\rho$ of ρ gives the transformation law of $T(G)$ acting on $T(P)$. The notations in (ω. 1, 2) should be understood in this sense. For the detail, see [5].

§ 2. Universal bundles

Let R^{n+k} be the $(n+k)$-dimensional Euclidean space. Taking a point o in R^{n+k} as origin, we identify R^{n+k} with the $(n+k)$-dimensional vector space. A frame at o is a set of ordered vectors $e_1, \ldots, e_{n+k}$ at o which are orthonormal. Then there is a one-one correspondence between the set of all frames at o and the orthogonal group $O(n+k)$ in $n+k$ variables. If w_0 is a particular frame at o, the correspondence is given by

$$s(w_0) \longleftrightarrow s \qquad s \in O(n+k).$$

Let $M_{n,k}$ denote the set of all n-planes through the origin of R^{n+k}. If R^n is a fixed n-plane and R^k is its orthogonal complement, then we may identify

$$M_{n,k} = O(n+k)/O(n) \times O(k),$$

where $O(n)$ is the orthogonal subgroup leaving R^k pointwise fixed and $O(k)$ is the orthogonal subgroup leaving R^n pointwise fixed [7]. The manifold $M_{n,k}$ is called the *Grassmann manifold* of n-planes in $(n+k)$-space.

Remark. Our notation for the Grassmann manifold is slightly different from the one in Steenrod's book [7].

Let $SO(r)$ be the rotaion subgroup of $O(r)$ and define

$$\tilde{M}_{n,k} = SO(n+k)/SO(n) \times SO(k),$$

which will be called the Grassmann manifold of oriented n-planes in $(n+k)$-space.

Then $\tilde{M}_{n,k}$ is the simply connected two-fold covering of $M_{n,k}$.

Let

$$P_{n,k} = O(n+k)/\{1\} \times O(k), \qquad \tilde{P}_{n,k} = SO(n+k)/\{1\} \times SO(k).$$

The action of $O(n) \times \{1\}$ (resp. $SO(n) \times \{1\}$) on $O(n+k)$ (resp. $SO(n+k)$) on the right induces the action of $O(n) \times \{1\}$) (resp. $SO(n) \times \{1\}$) on $P_{n,k}$ (resp. $\tilde{P}_{n,k}$) on the right, hence $P_{n,k}$ (resp. $\tilde{P}_{n,k}$) is a principal fibre bundle over $M_{n,k}$ (resp. $\tilde{M}_{n,k}$) with group $O(n)$ (resp. $SO(n)$).

§ 3. Canonical connections in $P_{n,k}$ and $\tilde{P}_{n,k}$

Let $\mathfrak{o}(n+k)$, $\mathfrak{o}(n)$ and $\mathfrak{o}(k)$ be the Lie algebras of $O(n+k)$, $O(n)$ and $O(k)$ respectively. Since the algebra $\mathfrak{o}(n+k)$ is semi-simple and compact, the so-called Killing-Cartan bilinear form on $\mathfrak{o}(n+k)$ is definite.

Let $\mathfrak{m}_{n,k}$ be the orthogonal complement to $\mathfrak{o}(n) \dotplus \mathfrak{o}(k)$ with respect to the Killing-Cartan bilinear form. Then

$$\mathfrak{o}(n+k) = \mathfrak{o}(n) \dotplus \mathfrak{o}(k) \dotplus \mathfrak{m}_{n,k},$$
$$ad(s) \cdot \mathfrak{m}_{n,k} \subseteqq \mathfrak{m}_{n,k} \qquad \text{for any } s \in O(n) \times O(k).$$

Let θ be the left invariant $\mathfrak{o}(n+k)$-valued linear differential form on $O(n+k)$ defined by

$$\theta(\bar{s}) = s^{-1}\bar{s} \qquad \text{for any } \bar{s} \in T_s(O(n+k)).$$

Let ω be the $\mathfrak{o}(n)$-component of θ relative to the above decomposition of the Lie algebra $\mathfrak{o}(n+k)$. We shall show that this $\mathfrak{o}(n)$-valued differential form ω on $O(n+k)$ induces an $\mathfrak{o}(n)$-valued differential form on $P_{n,k}$ which defines a connection in $P_{n,k}$. Let $\bar{s}'$ be any element of $T_{s'}(O(k))$, where s' is an arbitrary point of $O(k)$. Then

$$\theta(\bar{s} \cdot \bar{s}') = (ss')^{-1}(\bar{s} \cdot \bar{s}') = s'^{-1}(s^{-1}\bar{s})s'(s'^{-1}\bar{s}')$$
$$= ad(s') \cdot (s^{-1}\bar{s}) + (s'^{-1}\bar{s}'),$$

because both $s^{-1}\bar{s}$ and $s'^{-1}\bar{s}'$ are considered to be in the Lie algebra $\mathfrak{o}(n+k)$ and the product of two elements in $T_e(O(n+k))$ (where e is the unit of the group) corresponds to the sum of corresponding two elements in the Lie algebra $\mathfrak{o}(n+k)$. Since $s'^{-1}\bar{s}'$ is in $\mathfrak{o}(k)$ and all $\mathfrak{o}(n)$, $\mathfrak{o}(k)$ and $\mathfrak{m}_{n,k}$ are stable by $ad(s')$ and furthermore the elements of $\mathfrak{o}(n)$ are pointwise fixed by $ad(s')$, we obtain

$$\omega(\bar{s} \cdot \bar{s}') = \omega(\bar{s}) \qquad \text{for any } \bar{s} \in T(O(n+k)) \text{ and } \bar{s}' \in T(O(k)).$$

Therefore ω induces an $\mathfrak{o}(n)$-valued linear differential form on $P_{n,k}$, which we shall denote by the same letter ω. Now we shall show that the form ω satisfies the conditions $(\omega.1, 2)$ in Section 1. Let $u \in P_{n,k}$ and $\bar{s} \in T_s(O(n))$. If $s' \in O(n+k)$ is a representative for u, then

$$\theta(s'\bar{s}) = (s's)^{-1}(s'\bar{s}) = s^{-1}\bar{s}.$$

This proves Condition $(\omega.1)$. Let $\bar{u} \in T(P_{n,k})$ and $s \in O(n)$. If $\bar{s}' \in T(O(n+k))$ is a representative for $\bar{u}$, then

$$\theta(\bar{s}'s) = (s's)^{-1}(\bar{s}'s) = s^{-1}s'^{-1}\bar{s}'s = s^{-1}\theta(\bar{s}')s,$$

hence

$$\omega(\bar{u}s) = s^{-1}\omega(\bar{u})s.$$

Ws call the *canonical connection* in $P_{n,k}$ the connection defined by the form ω. Now we shall find the structure equation for the canonical connection. Let η and ζ be the $\mathfrak{o}(k)$-component and the $\mathfrak{m}_{n,k}$-component of θ respectively;

$$\theta = \omega + \eta + \zeta$$

By a similar argument for ω, we can prove that the $\mathfrak{m}_{n,k}$-valued form on $O(n+k)$ induces naturally an $\mathfrak{m}_{n,k}$-valued form on $P_{n,k}$, which we shall denote by the same letter ζ. From the equation of Maurer-Cartan:[3)]

$$d\theta = -\frac{1}{2}[\theta, \theta]$$

it follows that

$$d\theta = -\frac{1}{2}[\omega, \omega] - \frac{1}{2}[\eta, \eta] - \frac{1}{2}[\omega+\eta, \zeta] - \frac{1}{2}[\zeta, \omega+\eta] - \frac{1}{2}[\zeta, \zeta],$$

because

$$[\eta, \omega] = [\omega, \eta] = 0.$$

If we compare the $\mathfrak{o}(n)$-component of both sides, then we obtain

$$d\omega = -\frac{1}{2}[\omega, \omega] - \frac{1}{2}[\zeta, \zeta]_1,$$

where $[\zeta, \zeta]_1$ is the $\mathfrak{o}(n)$-component of $[\zeta, \zeta]$ (we shall see later that $[\zeta, \zeta]$ has its values in $\mathfrak{o}(n)+\mathfrak{o}(k)$).

Hence the curvature form Ω of the canonical connection is given by

$$\Omega = -\frac{1}{2}[\zeta, \zeta]_1.$$

We can apply the same reasoning to $\tilde{P}_{n,k}$; starting from $\tilde{\theta}$, which is the restriction of θ on $SO(n+k)$, we define similarly the forms $\tilde{\omega}$, $\tilde{\eta}$ and $\tilde{\zeta}$. We have also the following structure equation of *E.* Cartan:

$$d\tilde{\omega} = \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] - \frac{1}{2}[\tilde{\zeta}, \tilde{\zeta}]_1,$$

§ 4. Natural coordinates in $P_{n,k}$ and $\tilde{P}_{n,k}$

We take an orthogonal basis for R^{n+k} in such a way that the elements of $O(n)$ and $O(k)$ can be expressed respectively as follows:

3) $[\theta, \theta]$ will be understood as follows:
$[\theta, \theta]\cdot(\bar{s}, \bar{s}') = [\theta(\bar{s}), \theta(\bar{s}')]$ for any $\bar{s}, \bar{s} \in T_s(O(n+k))$.

$$\begin{pmatrix} * & 0 \\ 0 & I_k \end{pmatrix}, \quad \begin{pmatrix} I_n & 0 \\ 0 & * \end{pmatrix},$$

where I_k and I_n are the identity matrices of degree k and n respectively. Then the elements in the Lie algebras $\mathfrak{o}(n)$ and $\mathfrak{o}(k)$ are expressed respectively as follows:

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix},$$

where A and B are skew-symmetric matrices of degree n and k respectively.

Let matrices (v^a_b) and (w^a_b), $(a, b = 1, \ldots, n+k)$, be elements in the Lie algebra $\mathfrak{o}(n+k)$. Then the Killing-Cartan bilinear form Φ on $\mathfrak{o}(n+k)$ is given by

$$\Phi(v, w) = \sum_{a,b=1}^{n+k} v^a_b w^b_a.$$

An easy calculation shows that the subspace $\mathfrak{m}_{n,k}$ of $\mathfrak{o}(n+k)$ consists of the matrices of the following form:

$$\begin{pmatrix} 0 & C \\ {}^tC & 0 \end{pmatrix}$$

where C is a matrix of $(k\text{-}n)$-type.

Now we shall prove the

Proposition 1. *There is a natural one-one correspondence between the points in $P_{n,k}$ and the matrices with the following properties:*[4]

$$\begin{pmatrix} y^1_1 & \cdots & y^1_n \\ \cdots & \cdots & \cdots \\ y^n_1 & \cdots & y^n_n \\ \cdots & \cdots & \cdots \\ y^{n+k}_1 & \cdots & y^{n+k}_n \end{pmatrix} \qquad \sum_{a=1}^{n+k} y^a_i y^a_j = \delta_{ij} \qquad i, j = 1, \ldots, n.$$

Proof. By adding k columns, a matrix of above type can be completed to an orthogonal matrix, which gives an element of $P_{n,k}$ by the natural projection map of $O(n+k)$ onto $P_{n,k}$. The element of $P_{n,k}$ obtained in this way depends only on the initial matrix (y^a_i) and is independent from the choice of k columns added to it. Because, if both

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } \begin{pmatrix} A & F \\ C & G \end{pmatrix}$$

[4] In this paper, the indices a, b run from 1 to $n+k$ and i, j run from 1 to n.

are orthogonal matrices, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}\begin{pmatrix} A & F \\ C & G \end{pmatrix}=\begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix}\begin{pmatrix} A & F \\ C & G \end{pmatrix}=\begin{pmatrix} I_n & O \\ O & {}^tBF+{}^tDG \end{pmatrix}$$

Now it is easy to see that the mapping thus defined gives a one-one correspondence between the matrices (y_i^a)'s and the points of $P_{n,k}$.

Therefore we shall take (y_i^a) as coordinate functions of $P_{n,k}$ (observe that they are not independent from each others, however they are valid throughout $P_{n,k}$). We shall express the canonical connection in $P_{n,k}$ in terms of these natural coordinate functions. The left invariant form on $O(n+k)$ is given by the following matrix

$$(\theta_b^a) = (y_b^a)^{-1}(dy_b^a),$$

where the y_b^a's are the natural coordinate functions on $O(n+k)$. Hence the differential form ω defining the canonical connection in $P_{n,k}$ is given by the matrix (ω_j^i) defined by

$$\omega_j^i = \sum_{a=1}^{n+k} y_i^a dy_j^a.$$

The same result holds for $\tilde{P}_{n,k}$; first of all, we note that Proposition 1 holds for $\tilde{P}_{n,k}$. (We complete (y_i^a) to a proper orthogonal matrix by adding k columns, which is possible for $k \geqq 1$.) Then the rest of argument is perfectly the same.

PROPOSITION 2. *The forms for the canonical connections in $P_{n,k}$ and $\tilde{P}_{n,k}$ are given by*

$$\omega_j^i = \sum y_i^a dy_j^a \qquad \tilde{\omega}_j^i = \sum y_i^a dy_j^a$$

where the y_i^a's are the natural coordinate functions on $P_{n,k}$ and $\tilde{P}_{n,k}$.

§5. Riemannian connections

Let M' be an n-dimensional Riemannian space and P' the bundle of orthogonal frames over M'. If M' is non-orientable, then P' is connected; and if M' is orientable, P' has two connected components and to choose one of them is to choose an orientation for M'. Now we shall define an R^n-valued linear differential form θ' on P'. Let $\bar{u}$ be any vector tangent to P' at u and $\bar{x}$ its

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projection on M', i.e., if π' is the projection of P' onto M', then $\delta\pi'(\bar{u}) = \bar{x}$. Since u is an orthogonal transformation of R^n onto $T_x(M')$, $u^{-1}(\bar{x})$ is an element of R^n. We define

$$\theta'(\bar{u}) = u^{-1}(\bar{x}).$$

Remark 1. The form θ' gives the structure of *soudure* in the tangent bundle $T(M')$ and is called the *form of soudure* [2], [5]. The definition of θ' in terms of local coordinates is given in [3].

If we choose an orthogonal basis for R^n, then θ' is a set of n real valued linear differential forms θ'^i, $i = 1, \ldots, n$. Then the Riemannian connection in P' (or on M') is a connection in P' defined by an $\mathfrak{o}(n)$-valued linear differential form $\omega' = (\omega'^i_j)$ such that

$$d\theta'^i = -\sum \omega'^i_j.$$

Remark 2. The Riemannian metric is parallel with respect to any connection in P'. The above condition implies the so-called torsionfreeness. It is well known that there is a unique connection with above property.

§ 6. Imbedded Riemannian spaces

Let M' be an n-dimensional Riemannian manifold imbedded isometrically in the $(n+k)$-dimensional Euclidean space R^{n+k}. Let u be any element of P'; it is an orthogonal frame at a point x of M' and can be considered as an orthogonal transformation of R^n onto $T_x(M')$ sending the origin of R^n into x. Let V_x be the n-plane in R^{n+k} which is parallel to $T_x(M')$ and passes through the origin o of R^{n+k} and let u' be the orthogonal transformation of R^n onto V_x corresponding to u. Considering R^n as a fixed subspace of R^{n+k} passing through the origin o, we extend u' to an orthogonal transformation u^* of R^{n+k} onto itself. Let v be the image of u^* under the natural projection of $O(n+k)$ onto $P_{n,k}$. Then it can be proved, by a similar method as in Proposition 1, that v depends only upon u and is independent from the choice of u^*. We shall denote by $\tilde{h}$ the mapping of P' into $P_{n,k}$ sending u to v. From the definition of $\tilde{h}$, it follows immediately that $\tilde{h}$ is a bundle map of P' into $P_{n,k}$.

If M' is orientable, we take the connected component of the bundle of orthogonal frames over M' corresponding to the orientation and denote it by P'. Then P' is a principal fibre bundle over M' with group $SO(n)$, which may be called the bundle of oriented orthogonal frames over M'. In the same way as above, we define a bundle map $\tilde{h}$ of P' into $P_{n,k}$.

We shall now introduce a coordinate system in P' as follows. Let $x^1, \dots, x^n, x^{n+1}, \dots, x^{n+k}$ be a Cartesian coordinate system in R^{n+k} such that $x^1, \dots, x^n$ form a coordinate system for the fixed subspace R^n. Let

$$e_i = (\partial/\partial x^i)_0 \qquad i = 1, \dots, n.$$

Then the e_i's form an orthogonal frame in R^n at the origin o. If u is an element of P', then

$$u(e_i) = \sum_{a=1}^{n+k} x_i^a (\partial/\partial x^a)_x \qquad i = 1, \dots, n,$$

where $x = \pi'(u)$ and the x_i^a's have the following property:

$$\sum x_i^a x_j^a = \delta_{ij} \qquad i, j = 1, \dots, n.$$

We shall take $(x^a; x_i^b)$, where $a, b = 1, \dots, n+k$ and $i = 1, \dots, n$, as a coordinate system in P', even though these functions are not independent on P'. With respect to this coordinate system, the form of soudure θ' can be expressed as follows:

$$\theta'^i = \sum x_i^a dx^a.$$

To prove this, we shall show first the following

PROPOSITION 3. *We have*

$$\sum_{b,j} x_j^a x_j^b dx^b = dx^a \qquad \text{on } P'.$$

Proof. Let $\bar{u}$ be any vector tangent to P' at u. Set

$$\lambda^a = dx^a(\delta\pi'(\bar{u})).$$

Then

$$\delta\pi'(\bar{u}) = \sum \lambda^a (\partial/\partial x^a)_x.$$

Since $\delta\pi'(\bar{u})$ is tangent to M' at $x = \pi(u)$, it is a linear combination of $u(e_1), \dots, u(e_n)$. Hence, if $u_j^a = x_j^a(u)$, then

$$\sum \lambda^a (\partial/\partial x^a)_x = \sum \mu^i u_i^a (\partial/\partial x^a)_x$$

or

$$\lambda^a = \sum \mu^i u_i^a \qquad \text{for some real numbers } \mu^i.$$

Then

$$(\sum x_j^a x_j^b dx^b)(\bar{u}) = \sum u_j^a u_j^b \cdot dx^b(\delta\pi'(\bar{u})) = \sum u_j^a u_j^b \lambda^b$$
$$= \sum u_j^a u_j^b \mu^i u_i^b = \sum u_i^a \mu^i = \lambda^a = dx^a(\bar{u}).$$

This completes the proof of the proposition.

We can now prove that the above defined form $\theta' = (\theta'^i)$ is the form of soudure; that is, we shall show that

$$u(\theta'(\bar{u})) = \delta\pi'(\bar{u}).$$

Using the same notations as in the proof of the proposition 3, we have

$$u(\theta'(\bar{u})) = u(\sum \theta'^i(\bar{u}) \cdot e_i) = \sum (x_i^b dx^b x_i^a)(\bar{u}) \cdot (\partial/\partial x^a)_x$$
$$= \sum dx^a(\bar{u}) \cdot (\partial/\partial x^a)_x = \delta\pi'(\bar{u}).$$

Let $\omega = (\omega_j^i)$ be the form defining the canonical connection in $P_{n,k}$. Then the linear differential form $\omega' = (\omega'^i_j)$ defining the connection induced from the canonical connection by $\tilde{h}$ is given as follows in terms of the coordinate system:

$$\omega'^i_j = \sum x_i^a dx_j^a.$$

This follows immediately from Proposition 2 and from the fact that

$$x_i^a(u) = y_i^a(\tilde{h}(u)) \qquad \text{for any } u \in P'.$$

We claim that the connection defined by ω' is the Riemannian connection on M'. In fact

$$d\theta'^i + \sum \omega'^i_j \wedge \theta'^j = \sum dx_i^a \wedge dx^a + \sum (x_i^a dx_j^a) \wedge (x_j^b dx^b)$$
$$= \sum dx_i^a \wedge dx^a - \sum (dx_i^a x_j^a) \wedge (x_j^b dx^b)$$
$$= \sum dx_i^a \wedge dx^a - \sum dx_i^a \wedge dx^a = 0 \qquad \text{(Prop. 3)}.$$

A similar argument holds for $\tilde{P}_{n,k}$ if M' is oriented.

THEOREM I. *Let M' be an n-dimensional Riemannian space imbedded in the $(n+k)$-dimensional Euclidean space and let $\tilde{h}$ be the natural bundle map of P' (the bundle of orthogonal frames over M') into $P_{n,k}$. Then the connection in P' induced from the canonical connection in $P_{n,k}$ by $\tilde{h}$ is nothing but the Riemannian connection on M'.*

If M' is oriented, let $\tilde{h}$ be the natural bundle map of P' (the bundle of oriented orthogonal frames over M') into $\tilde{P}_{n,k}$. Then a statement similar to the above one is true.

§ 7. Hypersurfaces

Consider the case where $k=1$ and M' is oriented. We shall identify $\tilde{P}_{n,1} = SO(n+1)$ with the bundle of oriented orthogonal frames Q over the n-dimensional unit sphere S^n in the following manner. Let R^n be a fixed n-plane in R^{n+1} and R its orthogonal complement. Let z be a unit vector in R (or a point in R with unit distance from the origin). Then for each $s \in P_{n,1} = SO(n+1)$, sz is a point on the unit sphere S^n, and $s(R^n)$ is an n-plane in R^{n+1} parallel to the tangent space $T_{sz}(S^n)$. Hence s defines naturally an orthogonal transformation of R^n onto $T_{sz}(S^n)$ and s can be considered as an orthogonal frame (oriented) over S^n at sz. It is easy to see that this correspondence is a bundle isomorphism between $P_{n,1}$ and Q and that it is nothing but the inverse of the bundle map $\tilde{h}$, applied to a particular case where $M' = S^n$. Hence the canonical connection in $\tilde{P}_{n,1}$ corresponds to the Riemannian connection in Q (or on S^n) (See Th. 1.) From this fact and from Theorem I, follows the

THEOREM II. *Let M' be an n-dimensional Riemannian manifold imbedded in the $(n+1)$-dimensional Euclidean space and let $\tilde{h}$ be the natural bundle map of P' (the bundle of oriented orthogonal frames over M') into Q (the bundle of oriented orthogonal frames over the unit sphere S^n). Then the connection induced from the Riemannian connection on S^n by $\tilde{h}$ is the Riemannian connection on M'.*

Remark. For the geometrical interpretation of Theorem II, see [4].

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