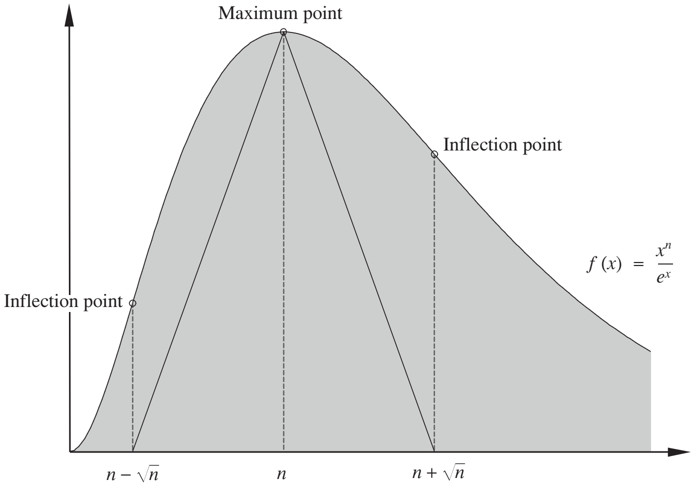
108.12 Proof without words: a lower bound for *n*!



$$n! = \Gamma(n+1) = \int_0^\infty f(x)\, dx > \frac{1}{2}\left(2\sqrt{n}\right) f(n) = \left(\frac{n}{e}\right)^n \sqrt{n}$$

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108.13 Indeterminate exponentials without tears

Every calculus student learns how to solve indeterminate limits of the form $f(n)^{g(n)}$ where $f(n) \to 1$ and $g(n) \to \infty$; most quickly learn to hate and fear this process. It is error-prone, full of tedious algebra, and requires careful attention to L'Hôpital's rule. Here is a typical "fairly simple" example.

$$\begin{aligned}
\lim_{n\to\infty} \ln\left(\frac{n+4}{n}\right)^{3n+1} &= \lim_{n\to\infty} \frac{\ln\left(\frac{n+4}{n}\right)}{\frac{1}{3n+1}} \\
&= \lim_{n\to\infty} \frac{\left(\frac{n}{n+4}\right)\left(\frac{-4}{n^2}\right)}{-\frac{3}{(3n+1)^2}} \quad \text{using L'Hôpital's rule} \\
&= \lim_{n\to\infty} \left(\frac{-4}{n(n+4)} \times \frac{-(3n+1)^2}{3}\right) \\
&= \lim_{n\to\infty} \frac{4(3n+1)^2}{3n(n+4)} = 12
\end{aligned}$$

and so

$$\lim_{n\to\infty} \left(\frac{n+4}{n}\right)^{3n+1} = e^{12}.$$