Note on The Envelope-Investigation.

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Obscurity in the direct discussion of The Envelope, as given in works on Differential Equations, has led writers on The Calculus to define the envelope of a family by a property which all know that it shares with any locus of multiple-points belonging to the family. The following presentation is an attempt by use of systematic notation to make clear the details of the direct process :—

Starting from the definition that

A curve is an envelope of a given family, if at each of its points it touches a member of the family :

let us suppose that a family is specified by the equation

$$\psi(x, y, u) = 0$$

in which ψ is a continuous function of *the three variables* x, y, u; continuous variation of u corresponds to continuous motion and deformation of a variable curve in the xy-plane, which takes in succession the curves of the family as positions.

If there is an envelope, it is the locus of a variable point P_u which has for a given value (u_0) of u, a *determinate* position (P_{u_0}) on the u_0-curve of the family : the coordinates (x_u, y_u) of P_u are therefore unknown functions of u such that

$$\psi(x_u, y_u, u) \equiv 0, \qquad - \quad - \quad - \quad - \quad (1)$$

and we seek to determine x_u, y_u from the fact that the locus of P_u touches the u_0-curve at the point P_{u_0}. This property gives the identical relation

$$\frac{dy_u}{du}\bigg/\frac{dx_u}{du} \equiv -\psi_x'(x_u, y_u, u)\big/\psi_y'(x_u, y_u, u)$$

i.e.,
$$\psi_x'(x_u, y_u, u) \cdot \frac{dx_u}{du} + \psi_y'(x_u, y_u, u) \cdot \frac{dy_u}{du} \equiv 0. \qquad - \quad (2)$$

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But from the identity (1),

$$\psi_x'(x_u,\ y_u,\ u)\,.\,\frac{dx_u}{du}+\psi_y'(x_u,\ y_u,\ u)\frac{dy_u}{du}+\psi_u'(x_u,\ y_u,\ u)\equiv 0\,;$$

hence (1) and (2) are equivalent to

$$\psi(x_u,\ y_u,\ u)\equiv 0 \text{ and } \psi_u'(x_u,\ y_u,\ u)\equiv 0.$$

Any envelope-locus is, therefore, represented in the relation between x and y which is the eliminant of u from the equations

$$\psi(x,\ y,\ u)=0,\ \ \psi_u'(x,\ y,\ u)=0.$$

The full locus of this eliminant-equation may be geometrically described as

(i) the locus of ultimate points of intersection of "consecutive" curves of the family; and therefore as

(ii) the locus of points in the xy-plane at which the equation $\psi(x,\ y,\ u)=0$ has two roots in u equal;

it obviously includes, on the description (i), the envelopes and multiple-point loci; and the description (ii) of this locus leads to the conclusion that the envelopes are also included in the "p-discriminant" of the differential equation that represents the family.

On the teaching to beginners of such transformations as $-(-a)=+a$.

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