A THEOREM CONCERNING THREE FIELDS

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Several authors (**1**; **2**; **3**; **4**; **5**; **6**) have recently studied the existence and non-existence of certain types of extensions of a given field. In this note we prove a theorem closely related to these results which, in a sense, contains essential portions of each of these. We prove the

THEOREM. *Let $F \subset K \subset L$ be three fields (where we assume these inclusions all to be proper). Suppose that for every element x in L there exists a nontrivial polynomial $f_x(t)$ in the variable t with coefficients in F (and which depend on x) such that the element $f_x(x)$ is in K. Then either*

(a) *L is purely inseparable over K, or*

(b) *L, and so K, is algebraic over F.*

Proof. Suppose that L is not purely inseparable over K. Then there exists an element in L which is not in K which is separable over K. The set of all elements in L which are separable over K form a subfield L' of L. K is of course contained in L'; by supposing that L was not purely inseparable over K we have that $L' \neq K$. If this subfield L' were algebraic over F, then K would also be algebraic over F. This, combined with the fact that L is algebraic over K, would then lead to the desired conclusion that L is algebraic over F. So we suppose, to the contrary, that there is some element $a \in L'$, $a \notin K$ which is transcendental over F. (Being in L', a is of course separable over K.) We shall show that this leads to a contradiction.

Let $\tilde{L} = F(a)$, the set of all rational functions in a over the field F. Let $\tilde{K} = \tilde{L} \cap K$. Consider the three fields $F \subset \tilde{K} \subset \tilde{L}$. These inclusions are all proper since $a \in \tilde{L}$, $a \notin \tilde{K}$, and since a is algebraic over $\tilde{K}$ but not over F. Also, if $x \in \tilde{L}$ then there is a polynomial $f_x(t)$ with coefficients in F so that $f_x(x) \in K$; since $f_x(x) \in \tilde{L}$ it follows that $f_x(x) \in \tilde{K}$. Thus the conditions on the three fields F, K, L carry over to the three fields $F, \tilde{K}, \tilde{L}$.

By Lüroth's theorem $\tilde{K}$ is a rational function field over F in some s, $\tilde{K} = F(s)$. $\tilde{L} = \tilde{K}(a)$ is of finite degree and separable over $\tilde{K}$. Now Nagata, Nakayama and Tuzuku (**5**) have proved for this situation that there exist two distinct logarithmic valuations V_1 and V_2 on $\tilde{L}$ which coincide on $\tilde{K}$; a simple modification of their argument yields that we can find such V_1 and V_2 which, in addition, are trivial on the field F. Thus for these two valuations we have the following properties:

(1) There exists a $u \in \tilde{L}$, $u \notin \tilde{K}$ so that $V_1(u) \neq V_2(u)$;

(2) $V_1(k) = V_2(k)$ for all $k \in \tilde{K}$;

(3) $V_1(\alpha) = V_2(\alpha) = 0$ for all $\alpha \neq 0 \in F$.

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Without loss of generality we may assume that $V_1(u) > 0$. By hypothesis, $k = u^n + \alpha_{n-1}u^{n-1} + \ldots + \alpha_r u^r \in \tilde{K}$ for some $\alpha_i \in F$ $\alpha_r \neq 0$, $n \geqslant r \geqslant 1$. Thus $V_1(k) = V_2(k)$.

Since $V_1(\alpha_i) = 0$ (we only consider the non-zero multipliers that occur in the expression for k) and since $\alpha_r \neq 0$, $V_1(\alpha_r u^r) = rV_1(u) < V_1(\alpha_m u^m) = mV_1(u)$ for $m > r$ occurring in the expression for k with non-zero multiplier. Thus, since V_1 is a non-Archimedean valuation, it follows that $V_1(k) = rV_1(u)$. Since $0 < V_1(k) = V_2(k)$, it follows that $V_2(u) > 0$. Thus the argument used above for V_1 can be repeated and it follows that $V_2(k) = rV_2(u)$. But $V_1(k) = V_2(k)$; therefore we are led to $rV_1(u) = rV_2(u)$, which, since $r \neq 0$ implies that $V_1(u) = V_2(u)$. This is contrary to the assumption that $V_1(u) \neq V_2(u)$. The theorem is thereby established.

References

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