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ON THE INTERCHANGEABILITY AND STOCHASTIC ORDERING OF $\cdot/M/1$ QUEUES IN TANDEM

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Abstract

A probabilistic proof is given of the fact that the departure process from two initially empty $\cdot/M/1$ queues in tandem is unaffected when the service rates are interchanged. As a consequence of this, we show that when the sum of the service rates at the two queues is held constant the departure process stochastically increases as the service rates become equal. The proofs are based on coupling of reflected random walks.

TANDEM QUEUES; RANDOM WALKS; COUPLING; NETWORKS; SERVER ALLOCATION

1. Introduction

Consider two initially empty $\cdot/M/1$ queues in tandem. Service rates are μ_1, μ_2 in nodes 1 and 2 respectively. For a fixed, but arbitrary, sequence of arrival times at node 1 denote by (D_t) the departure process from node 2. Denote by $(\underline{D}_t)$ the same process when μ_1 and μ_2 are interchanged. In Weber (1979) it is proved that $(D_t) \stackrel{d}{=} (\underline{D}_t)$. This is done by computing the joint characteristic function of the interarrival times of the processes. The same result is proved in Anantharam (1985) using the filtering equations for the $\cdot/M/1$ queue. Lehtonen (1986) uses involved pathwise arguments to obtain this result. Furthermore, he shows that, subject to $\mu_1 + \mu_2 = \mu$, (D_t) is stochastically increasing as $\mu_1 \to \mu/2$ monotonically.

In this note we provide a simple probabilistic proof of interchangeability and show that monotonicity is a consequence of it. The proofs are based on elementary fact about random walks. These are stated and proved in Section 2. They are used in Section 3 to prove the results.

2. Results on random walks

2.1. *Definitions.* Let $(\xi_i)_{i=0}^{\infty}$ be i.i.d. random variables such that

$$\xi_i = \begin{cases} +1, & \text{w.p. } q \\ -1, & \text{w.p. } p = 1 - q, \end{cases}$$

and take $q > p$.

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Define $(\bar{\xi}_i)_i$ as above with p and q interchanged. Define the unstable reflected random walk $X_0=0$, $X_{n+1}=(X_n+\xi_{n+1})^+((x)^+=\max(0,x))$. Also define times $(T_k)_{k=0}^\infty$, $(S_k)_{k=1}^\infty$ as follows (see Figure 1):

$$\begin{aligned} &T_0=0,\\ &S_{k+1}=\min\{n>T_k: X_{n+1}>X_n,\ X_l=X_n \quad \text{for some} \quad l>n\}, && k\geqq 0,\\ &T_{k+1}=\min\{n>S_{k+1}: X_n=X_{S_{n+1}}\}, && k\geqq 1. \end{aligned}$$

Call

$$B_k \stackrel{\text{def}}{=} \{X_n - X_{S_k},\ S_k\leqq n\leqq T_k\}, \qquad k\geqq 1,$$

the *busy cycles* of (X_n).

In words, when (X_n) is way from 0, a busy cycle starts at a point of increase of its path if (X_n) returns to the same level and never falls below that level in the future. The same holds trivially when (X_n) is at 0.

The random quantities $\bar{X}_n$, $\bar{T}_n$, $\bar{S}_n$, $\bar{B}_k$ are defined similarly in terms of the $\bar{\xi}_i$'s. Note that $\bar{X}_{\bar{S}_k}=\bar{X}_{\bar{T}_k}=0$ with probability 1 and thus the $(\bar{B}_k)$ correspond to the usual busy cycles of the stable reflected random walk $(\bar{X}_n)$.

2.2. *Lemma.*

(a) $B_k \stackrel{\text{d}}{=} \bar{B}_k$, $k\geqq 1$.

(b) $S_{k+1}-T_k \stackrel{\text{d}}{=} \bar{S}_{k+1}-\bar{T}_k \stackrel{\text{d}}{=} Y-1$ where Y is geometric with parameter p.($V \stackrel{\text{d}}{=} W$ means that the random quantities V and W have the same distribution.)

Furthermore, the elements of the sequences (B_k) and $(S_{k+1}-T_k)$ are independent.

Proof. (a) We first state a basic fact that will be used repeatedly. For $l>0$, define $r_l \stackrel{\text{def}}{=} P\{\min_{k>0}\sum_{i=1}^k \xi_i \leqq -l\}$. The r_l's uniquely satisfy the relation

$$r_l = pr_{l-1}+qr_{l+1}, \quad l>0, \quad \text{with} \quad r_0=1,$$

and one verifies that $r_l=(p/q)^l$ (see e.g. Shiryayev (1984). As a consequence of this fact,

$$\begin{aligned} P_l\Big\{\min_{k\geqq 1} X_k = 1\Big\} &= P_l\Big\{\min_{k\geqq 1} X_k>0\Big\} - P_l\Big\{\min_{k\geqq 1} X_k>1\Big\}\\ &=\Big(\frac{p}{q}\Big)^{l-1}\Big(1-\frac{p}{q}\Big), \end{aligned}$$

where $P_l\{\cdot\}$ denotes the probability of an event given that $X_0=l$.

Given $X_{S_k}=m$ and that $X_n=m+l$ for some $S_k<n<T_k$,

$$\begin{aligned} \Pr\{X_{n+1}=m+l+1\} &= P_{m+l}\Big\{X_1=m+l+1 \;\Big|\; \min_{k\geqq 1} X_k=m\Big\}\\ &=\frac{P_{m+l}\{X_1=m+l+1\}P_{m+l+1}\Big\{\min_{k\geqq 1} X_k=m\Big\}}{P_{m+l}\Big\{\min_{k\geqq 1} X_k=m\Big\}}=p. \end{aligned}$$

Thus, during B_k, (X_n) behaves as if p and q have been interchanged, i.e., $B_k \stackrel{\text{d}}{=} \bar{B}_k$.

(b) For $X_{T_k}=0$, $P_0\{X_1=1 \mid \min_{k\geq 1} X_k=0\}=q(p/q)=p$, and for $X_{T_k}=l>0$,

$$P_l\left\{\min_{k\geq 1} X_k=l \;\middle|\; \min_{k\geq 1} X_k>l-1\right\}=P_{l+1}\left\{\min_{k\geq 1} X_k=l \;\middle|\; \min_{k\geq 1} X_k>l-1\right\}$$

$$=\frac{P_{l+1}\left\{\min_{k\geq 1} x_k=l\right\}}{P_{l+1}\left\{\min_{k\geq 1} X_k>l-1\right\}}=\frac{\frac{p}{q}\left(1-\frac{p}{q}\right)}{1-\left(\frac{p}{q}\right)^2}=p.$$

This shows that $P\{S_{k+1}-T_k\geq n \mid S_{k+1}-T_k\geq n-1\}=q$. In words, when (X_n) is not in a busy cycle, the probability that one begins at the next step is p and the trials are independent at each step. This is easily seen to be true for the busy cycles of $(\bar{X}_n)$ also and the result follows.

To prove independence of the (B_k)'s we argue as follows. Let $F=\sigma\{X_s, s\leq m\}$ (or $G=\sigma\{X_s, s>m\}$) be the σ-field generated by $(X_k)_k$ up to and including (respectively after) time m. Then $B=\{\min_{k>m} X_k>l\}\in G$ for all $l\geq 0$. For events $A\in G$ and $C\in F$,

$$P(A, C \mid B, X_m)=\frac{P(A, B, C \mid X_m)}{P(B \mid X_m)}$$

$$=\frac{P(A, B \mid X_m)P(C \mid X_m)}{P(B \mid X_m)}=P(A \mid B, X_m)P(C \mid B, X_m).$$

This also proves independence of the $(S_{k+1}-T_k)$'s.

2.3. *Remark.* As a consequence of Lemma 2.2, processes (X_n) and $(\bar{X}_n)$ can be constructed on the same probability space such that $B_k=\bar{B}_k$ and $S_{k+1}-T_k=\bar{S}_{k+1}-\bar{T}_k$ a.s. for all k. As indicated in Figure 1, one obtains a sample path of $(\bar{X}_n)$ from a simple path of (X_n) by setting $X_n=0$, for $T_k\leq n\leq S_{k+1}$ and $\bar{B}_k=B_k$, $k=0, 1, \dots$. Lemma 2.2 then ensures that the marginals of the joint process $(X_n, \bar{X}_n)$ are identical. Note that the points of strict decrease for the two processes coincide because they are contained in the busy cycles.

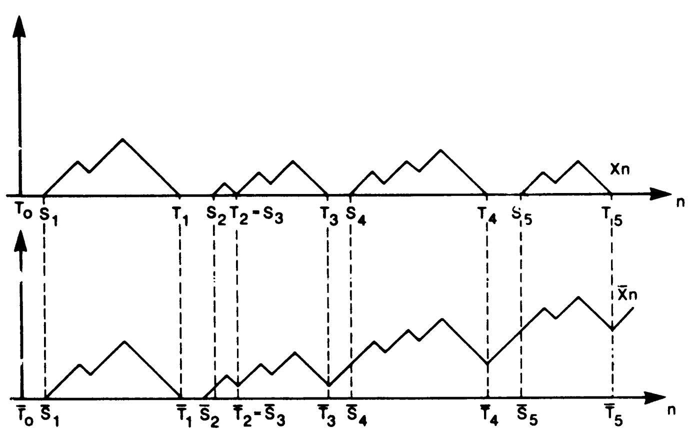

Figure 1.

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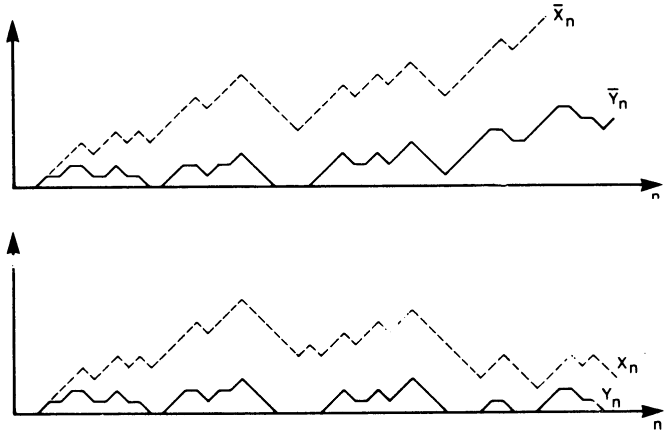

Figure 2.

We next prove an ordering result on the points of strict decrease of reflected random walks.

2.4. *Lemma.* Take $p' < p \leqq 1/2$ and denote by (d_i) and $d_i')$ the strictly decreasing points of the corresponding random walks denoted by (X_n) and (X_n'). Then $(d_i) \lneqq_{\text{st}} (d_i')$, i.e., $d_i \lneqq_{\text{st}} d_i'$ for all i.

Proof. Let $(\eta_i)_{i=0}^{\infty}$ be i.i.d. random variables such that

$$\eta_i = \begin{cases} 1, & \text{w.p. } p' \\ 0, & \text{w.p. } \alpha \\ -1, & \text{w.p. } p, \end{cases}$$

where $\alpha = 1 - p - p'$. Define $(\bar{\eta}_i)$ likewise with p and p' interchanged. We shall make use of the auxiliary reflected random walks (Y_n) defined as

$$Y_0 = \bar{Y}_0 = 0, \qquad Y_{n+1} = (Y_n + \eta_{n+1})^+,$$
$$\bar{Y}_{n+1} = (\bar{Y}_n + \bar{\eta}_{n+1})^+, \qquad n = 0, 1, \dots .$$

Denote the points of strict decrease of (Y_n), $(\bar{Y}_n)$ by (d_i^Y), $(\bar{d}_i^Y)$ respectively. Define $(m_i) = \{l \mid \eta_l = 0\}$ and $(\bar{m}_i)$ likewise. We can construct (η_i) and $(\bar{\eta}_i)$ such that $(m_i) = (\bar{m}_i)$. By Lemma 2.2 we can further arrange the construction so that $(d_i^Y) = (\bar{d}_i^Y)$.

We now obtain versions of (X_n) and (X_n') from (Y_n) and $(\bar{Y}_n)$ respectively by defining (see Figure 2)

$$X_0 = 0, \qquad X_{n+1} \begin{cases} X_n + 1, & \text{if } n + 1 \in (m_i) \\ (X_n + \eta_{n+1})^+, & \text{otherwise.} \end{cases}$$

Similarly, (X_n') is constructed from $(\bar{\eta}_i)$. It is easy to check that $p\{X_{n+1} = X_n + 1\} = 1 - p$ and $P\{X_{n+1}' = X_n' + 1\} = 1 - p'$. It remains to show that $(d_i') \subset (d_i)$ in this construction.

From the fact that $(d_i) \supset (d_i^Y) = (\bar{d}_i^Y) \subset (d_i')$, (see Figure 2) it suffices to consider $l \in (d_i') - (\bar{d}_i^Y)$. For such l, $Y_{l-1} = \bar{Y}_{l-1} = 0$, $\eta_l = \bar{\eta}_l = -1$ and $X_{l-1}' > 0$. But $(m_i) = (\bar{m}_i)$ and hence $X_{l-1} > 0$ which implies $l \in (d_i)$. Finally, note that $(d_i') - (\bar{d}_i^Y)$ is finite w.p. 1 because $(\bar{Y}_n)$ is an unstable reflected random walk whereas $(d_i) - (d_i^Y)$ is infinite w.p. 1 because (Y_n) is stable. Thus, the inclusion is strict with probability 1.

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3. Proof of the queueing results

3.1. *Definitions.* Denote by (A_t^1), (A_t^2) right-continuous versions of the virtual service processes in nodes 1 and 2 respectively. By virtual service process at a node with an exponential server we mean a Poisson process the points of which correspond to service completions whenever the node is not empty. As before, we consider a fixed, but arbitrary, sequence of arrival times at node 1. We discretize time by only considering points of $(A_t^1) \cup (A_t^2)$. We denote the process of customers in nodes 1 and 2 at these times by (z_n^1, z_n^2).

Assume $\mu_1 > \mu_2$ and define (X_n) as in the previous section with $\xi_i = \Delta A_i^1 - \Delta A_i^2$ (hence $q = \mu_1/(\mu_1 + \mu_2)$). Construct $(\bar{X}_n)$ pathwise from (X_n) as before. Note that in continuous time this corresponds to a construction of processes $(\bar{A}_t^1)$, $(\bar{A}_t^2)$ with rates μ_2, μ_1 respectively from (A_t^1), (A_t^2). In this construction, $A_t^1 \geqq \bar{A}_t^1$ for all $t \geqq 0$.

Write $\boldsymbol{C}$ (or $\boldsymbol{I}$) for the set of points of strict decrease (respectively increase) of (X_n) and observe that $\boldsymbol{C} = \bar{\boldsymbol{C}}$ (or $\bar{\boldsymbol{I}} \subset \boldsymbol{I}$). Let $(n_i)_i$ be the arrival points with the convention that each arrival point gets identified with the nearest embedding point to its left. Finally, let $(d_i)_i \subset \boldsymbol{C}$ be departure points $(z_{d_i}^2 = z_{d_i-1}^2 - 1)$ and let $(t_i)_i \subset \boldsymbol{I}$ be the points of intermediate transitions $(z_{t_i}^2 = z_{t_i-1}^2 + 1)$.

3.2. *Remark.* Suppose there is an infinite supply of customers in node 1. The departure process from node 1 is then Poisson and $X_n = z_n^2$ (respectively $\bar{X}_n^2 = \bar{z}_n^2$). By Remark 2.3, this implies $(d_i) = (\bar{d}_i) = \boldsymbol{C} = \bar{\boldsymbol{C}}$, i.e., starting with an empty $\cdot/M/1$ queue we can interchange the arrival and service rates without affecting the departure process.

We proceed with the following result.

3.3. *Lemma.*

(a) $\bar{z}_m^1 \geqq z_m^1$, *all* m.
(b) $z_m^2 \geqq \bar{z}_m^2$, *all* m.

Proof. Part (a) holds because arrivals are the same for both processes and $A_t^1 \geqq \bar{A}_t^1$. For (b), the fact that $\boldsymbol{C} = \bar{\boldsymbol{C}}$ must be used in addition to the above.

We next prove the result in Weber (1979).

3.4. *Theorem.*

$$(d_i)_i = (\bar{d}_i)_i.$$

Proof. Consider $m \in \boldsymbol{C} = \bar{\boldsymbol{C}}$ and assume $m \in (\bar{d}_i)$. Then $\bar{z}_{m-1}^2 > 0$ and from Lemma 3.3(b) $z_{m-1}^2 > 0$. Hence $m \in (d_i)$ and $(\bar{d}_i) \subset (d_i)$.

It remains to show that $m \in (d_i)$ implies $m \in (\bar{d}_i)$. We argue by induction on i. The fact is easily seen to be true for $i = 1$. For $i = k$ assume that $d_j = \bar{d}_j$ for all j such that $d_j \leqq n_k$. Let $l = z_{n_k}^1 + z_{n_k}^2 = \bar{z}_{n_k}^1 + \bar{z}_{n_k}^2$ be the number of customers that customer k finds ahead of him in the system. Then, $n_k \leqq d_j \leqq d_k$, and $d_j \in \bar{\boldsymbol{C}}$ for $j = k - l, \dots, k - 1$. Each point in $\bar{\boldsymbol{C}}$ (respectively $\boldsymbol{C}$) must be preceded (not necessarily immediately) by exactly one (respectively at least one) point in $\bar{\boldsymbol{I}}$ (respectively $\boldsymbol{I}$). Therefore, there are l distinct points s_j in $\bar{\boldsymbol{I}}$ such that

$$n_{k-j} \leqq t_{k-j} \leqq s_j < \bar{d}_{k-j}, \quad \text{and} \quad s_j \notin (\bar{t}_i)_{i=1}^{k-l-1} \quad \text{for} \quad j = 1, \dots, l.$$

This implies that the l customers ahead of the kth customer can depart during instants s_j and thus $\bar{d}_k \leqq d_k$. However, $(\bar{d}_i) \subset (d_i)$ implies that $\bar{d}_k = d_k$.

3.5. *Remark.* If an arbitrary queueing node is introduced between nodes 1 and 2, then from $A_t^1 \geqq \bar{A}_t^1$ it follows that when the network starts empty, $\bar{D}_t \underset{\text{st}}{\leqq} D_t$ (where $\underset{\text{st}}{\leqq}$ means stochastically less than or equal to).

The following theorem (Lehtonen (1986)) is a direct consequence of Lemma 2.4. Its proof is similar to the one of Theorem 3.4.

3.6. *Theorem.* Let $\mu_1 + \mu_2 = \mu_1 + \mu_2 = \mu$ with $\mu_1' > \mu_1 \geqq \mu/2$ and write (D_t) and (D_t') for the corresponding departure processes. Then, $(D_t) \underset{\text{st}}{\geq} (D_t')$.

3.7. *Remark.* As indicated in Lehtonen (1986), this result implies that if $N \cdot/M/1$ queues in series are such that $\mu_1 + \cdots + \mu_N = \mu$, then the departure process is stochastically maximized when $\mu_1 = \cdots = \mu_N = \mu/N$.

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