Nagoya Math. J., **223**(1) (2016), 66–86
DOI 10.1017/nmj.2016.24

NOTES ON AUTOMORPHISMS OF SURFACES OF GENERAL TYPE WITH $p_g = 0$ AND $K^2 = 7$

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Abstract. Let S be a smooth minimal complex surface of general type with $p_g = 0$ and $K^2 = 7$. We prove that any involution on S is in the center of the automorphism group of S. As an application, we show that the automorphism group of an Inoue surface with $K^2 = 7$ is isomorphic to $\mathbb{Z}_2^2$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$. We construct a 2-dimensional family of Inoue surfaces with automorphism groups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

§1. Introduction

The birational automorphism groups of projective varieties are extensively studied. Nowadays we know that, for a projective variety of general type X over an algebraically closed field of characteristic zero, the number of birational automorphisms of X is bounded by $c_d \cdot \mathrm{vol}(X, K_X)$, where c_d is a constant which only depends on the dimension d of X, and $\mathrm{vol}(X, K_X)$ is the volume of the canonical divisor K_X (cf. [13]). Furthermore, we know that $c_1 = 42$ and $c_2 = 42^2$ according to the classical Hurwitz theorem and to Xiao's theorem (cf. [25] and [26]). However, even in low dimensions, it is usually nontrivial to calculate the automorphism groups of explicit varieties of general type (for example, see [15], [23], [8] and [17]).

We focus on automorphisms of minimal smooth complex surfaces of general type with $p_g = 0$ and $K^2 = 7$. Involutions on such surfaces have been studied in [16] and [24]. All the possibilities of the quotient surfaces and the fixed loci of the involutions are listed. In order to find new examples, we have tried to classify such surfaces with commuting involutions in [10] and succeeded in constructing a new family of surfaces in [9]. We briefly recall the main results of [10]. Throughout the article, S denotes a minimal smooth surface of general type with $p_g = 0$ and $K^2 = 7$ over $\mathbb{C}$.

Received April 9, 2015. Revised June 16, 2015. Accepted June 26, 2015.
2010 Mathematics subject classification. 14J29, 14J50.
This work is partially supported by the China Postdoctoral Science Foundation (grant no. 2013M541062).

THEOREM 1.1. [10, Theorem 1.1] *Assume that the automorphism group* $\mathrm{Aut}(S)$ *contains a subgroup* $G = \{1, g_1, g_2, g_3\}$*, which is isomorphic to* $\mathbb{Z}_2^2$*. Let* R_{g_i} *be the divisorial part of the fixed locus of the involution* g_i *for* $i = 1, 2, 3$*. Then the canonical divisor* K_S *is ample and* $R_{g_i}^2 = -1$ *for* $i = 1, 2, 3$*. Moreover, there are only three numerical possibilities for the intersection numbers* $(K_S R_{g_1}, K_S R_{g_2}, K_S R_{g_3})$*: (a)* $(7, 5, 5)$*, (b)* $(5, 5, 3)$ *and (c)* $(5, 3, 1)$*. The intersection numbers* $(R_{g_1} R_{g_2}, R_{g_1} R_{g_3}, R_{g_2} R_{g_3})$ *have the following values: (a)* $(5, 9, 7)$*, (b)* $(7, 5, 1)$ *and (c)* $(1, 3, 1)$*, respectively.*

In the above theorem, we adopt the convention that $K_S R_{g_1} \geqslant K_S R_{g_2} \geqslant K_S R_{g_3}$, $R_{g_1} R_{g_2} \leqslant R_{g_1} R_{g_3}$ in case (a) and $R_{g_1} R_{g_3} \geqslant R_{g_2} R_{g_3}$ in case (b). Actually, we have completely classified the surfaces in case (a) and case (b) in [10]. But we do not know any example of the surfaces in case (c). One may ask whether there are noncommutative involutions on S. Here we give a negative answer.

THEOREM 1.2. *If* α *is an involution of* S*, then* α *is contained in the center of* $\mathrm{Aut}(S)$*.*

We prove the above theorem in Section 3. The key step is Theorem 3.1 which shows that any two involutions on S commute. Theorem 3.1 also has the following corollary.

COROLLARY 1.3. *Assume that* (S, G) *is a pair satisfying the assumption of Theorem 1.1. Then* g_1, g_2 *and* g_3 *are exactly all the involutions of* $\mathrm{Aut}(S)$*.*

The corollary immediately implies that if $\mathrm{Aut}(S)$ contains a nontrivial subgroup which is isomorphic to $\mathbb{Z}_2^r$, then $r = 1$ or $r = 2$. We remark that there are surfaces of general type with $p_g = 0$, $K^2 = 8$ and their automorphism groups contain subgroups which are isomorphic to $\mathbb{Z}_2^3$ (cf. [19, Example 4.2–4.4]).

As an application, we calculate the automorphism groups of the surfaces in the case (a) of Theorem 1.1. These surfaces are those constructed by Inoue in [14] who found the first examples of surfaces of general type with $p_g = 0$ and $K^2 = 7$. They can be described as finite Galois $\mathbb{Z}_2^2$-covers of the 4-nodal cubic surface (see Example 4.1, which is from [19, Example 4.1]).

THEOREM 1.4. *Let* S *be an Inoue surface. Then* $\mathrm{Aut}(S) \cong \mathbb{Z}_2^2$ *or* $\mathrm{Aut}(S) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$*. If* S *is a general Inoue surface, then* $\mathrm{Aut}(S) \cong \mathbb{Z}_2^2$*.*

Inoue surfaces form a 4-dimensional irreducible connected component in the Gieseker moduli space of canonical models of surfaces of general

https://doi.org/10.1017/nmj.2016.24 Published online by Cambridge University Press

type (cf. [3]). The proof of Theorem 1.4 actually shows that $\mathrm{Aut}(S)\cong\mathbb{Z}_2^2$ for S outside a 2-dimensional irreducible closed subset of this connected component (see Remark 4.3). We also exhibit a 2-dimensional family of Inoue surfaces with $\mathrm{Aut}(S)\cong\mathbb{Z}_2\times\mathbb{Z}_4$ (see Section 5). They are finite Galois $\mathbb{Z}_2\times\mathbb{Z}_4$-covers of a 5-nodal weak Del Pezzo surface of degree two, which is the minimal resolution of one node of the 6-nodal Del Pezzo surface of degree two.

§2. Preliminaries

2.1 Fixed point formulas and rational curves

Let X be a smooth projective surface over the complex number field. We only consider surfaces with $p_g(X)=q(X)=0$. In this case, X has Picard number $\rho(X)=10-K_X^2$ by Noether's formula and the Hodge decomposition. Also the expotential cohomology sequence gives $\mathrm{Pic}(X)\cong H^2(X,\mathbb{Z})$. Poincaré duality implies that the intersection form on $\mathrm{Num}(X)=\mathrm{Pic}(X)/\mathrm{Pic}(X)_{\mathrm{Tors}}$ is unimodular.

Assume that X has a nontrivial automorphism τ. Denote by $\mathrm{Fix}(\tau)$ the fixed locus of τ. Let k_τ be the number of isolated fixed points of τ and let R_τ be the divisorial part of $\mathrm{Fix}(\tau)$. Then R_τ is a disjoint union of irreducible smooth curves. We denote by $\tau^*: H^2(X,\mathbb{C})\to H^2(X,\mathbb{C})$ the induced linear map on the second singular cohomology group (note that $H^k(X,\mathbb{C})=0$ for $k=1,3$). The following proposition follows directly from the topological and holomorphic Lefschetz fixed point formulas (cf. [1, p. 567]; see also [11, Lemma 4.2]). The automorphism τ is called an involution if it is of order 2.

PROPOSITION 2.1. *If τ is an involution, then $k_\tau=K_XR_\tau+4$ and $\mathrm{tr}(\tau^*)=2-R_\tau^2$. If τ is of order 3, then $k_\tau=r_1+r_2=\mathrm{tr}(\tau^*)+2+K_XR_\tau+R_\tau^2$ and $r_1+2r_2=6+\frac{3}{2}K_XR_\tau-\frac{R_\tau^2}{2}$, where r_j is the number of isolated fixed points of τ of type $\frac{1}{3}(1,j)$ for $j=1,2$.*

Throughout this article, we denote by S a smooth minimal complex surface of general type with $p_g=0$ and $K_S^2=7$. Then $\rho(S)=3$ and S contains at most one (-2)-curve (this follows from Poincaré duality; cf. [10, Lemma 2.5]). Here an m-curve (for $m\leqslant 0$) on a smooth surface stands for an irreducible smooth rational curve with self-intersection number m. We have a similar result for (-3)-curves.

LEMMA 2.2. *The surface S contains at most one (-3)-curve.*

Proof. Assume by contradiction that S contains two distinct (-3)-curves C_1 and C_2. Then the intersection matrix of K_S, C_1 and C_2 has determinant $-7(C_1C_2)^2 + 2C_1C_2 + 69$. Since $K_SC_1 = K_SC_2 = 1$, the algebraic index theorem yields $C_1C_2 \leqslant 3$. Because the intersection form on $\mathrm{Num}(S)$ is unimodular and $\rho(S) = 3$, $-7(C_1C_2)^2 + 2C_1C_2 + 69$ is a square number. Then we have $C_1C_2 = 1$ since $C_1C_2 \geqslant 0$.

The divisor $K_S + C_1 + C_2$ has Zariski decomposition (see [21, Section 3]) $P + N$ with $P := K_S + \frac{1}{2}(C_1 + C_2)$ the positive part and $N := \frac{1}{2}(C_1 + C_2)$ the negative part. According to [21, Theorem 1.1], we have $c_2(S) - e(C_1 + C_2) - \frac{1}{3}P^2 - \frac{1}{4}N^2 \geqslant 0$. Here $e(C_1 + C_2)$ denotes the topological Euler number of $C_1 + C_2$ and $c_2(S)$ denotes the second Chern number of S. It is clear that $e(C_1 + C_2) = 3$ and the Noether formula gives $c_2(S) = 5$. The inequality above yields $-\frac{5}{12} \geqslant 0$, a contradiction. $\square$

LEMMA 2.3. (See also the table in [16]) *Let τ be an involution on S. Then $K_SR_\tau \in \{1, 3, 5, 7\}$ and $R_\tau^2 = \pm 1$. If $R_\tau^2 = 1$, then K_S is ample and R_τ is irreducible with $K_SR_\tau = 3$.*

Proof. For $R_\tau^2 = \pm 1$, see the proof of [4, Proposition 3.6]. According to [2, Lemma 3.2 and Proposition 3.3(v)], k_τ is an odd integer and $k_\tau \leqslant 11$. So $K_SR_\tau \in \{1, 3, 5, 7\}$ by Proposition 2.1.

Assume that $R_\tau^2 = 1$. If K_S is not ample, then S has a unique (-2)-curve C. The intersection number matrix of K_S, R_τ and C has determinant $-14 + 2(K_SR_\tau)^2 - 7(R_\tau C)^2$. The determinant equals 0, for, otherwise, the Chern classes of K_S, R_τ and C form a basis of $H^2(S, \mathbb{C})$ and they are τ^*-invariant, a contradiction to $\mathrm{tr}(\tau^*) = 2 - R_\tau^2 = 1$ by Proposition 2.1. It follows that $K_SR_\tau = 7$ and $(R_\tau C)^2 = 12$. This is impossible. So K_S is ample.

The algebraic index theorem gives $(K_SR_\tau)^2 \geqslant K_S^2R_\tau^2 = 7$ and thus $K_SR_\tau \in \{3, 5, 7\}$. Let $\pi_\tau : S \to \Sigma_\tau := S/\langle\tau\rangle$ be the quotient morphism. We have $K_S = \pi_\tau^*(K_{\Sigma_\tau}) + R_\tau$.

If $K_SR_\tau = 5$, then $k_\tau = K_SR_\tau + 4 = 9$ and $K_{\Sigma_\tau}^2 = \frac{1}{2}(K_S - R_\tau)^2 = -1$. So Σ has 9 nodes. The minimal resolution W_τ of Σ_τ has Picard number 11 and it contains 9 disjoint (-2)-curves. If Σ_τ has Kodaira dimension $\kappa(\Sigma_\tau) \geqslant 0$, by [11, Proposition 4.1], W_τ is minimal. This contradicts $K_{W_\tau}^2 = -1$. So $\kappa(\Sigma_\tau) = -\infty$ and W_τ is a rational surface. This contradicts [11, Theorem 3.3]. Hence $K_SR_\tau \neq 5$.

In the same manner we see that $K_SR_\tau \neq 7$ (see also [20]). So $K_SR_\tau = 3$. Because K_S is ample and R_τ is a disjoint union of smooth irreducible curves, the algebraic index theorem shows that R_τ is irreducible. $\square$

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2.2 Abelian covers

We briefly recall some facts from the theory of abelian covers from [22]. Assume that $\pi: X \to Y$ is a finite abelian cover between projective varieties with X normal and Y smooth. Let $\mathfrak{G}$ be the Galois group of π and let $\mathfrak{G}^*$ be the group of characters of $\mathfrak{G}$. Then the action of $\mathfrak{G}$ induces a splitting: $\pi_*(\mathcal{O}_X) = \bigoplus_{\chi\in\mathfrak{G}^*} \mathcal{L}_\chi^{-1}$, where $\mathcal{L}_\chi \in \operatorname{Pic}(Y)$ and $\mathcal{L}_1 = \mathcal{O}_X$. For each nontrivial cyclic subgroup $\mathfrak{C}$ of $\mathfrak{G}$ and each generator $\psi \in \mathfrak{C}^*$, there is a unique effective divisor $D_{\mathfrak{C},\psi}$ of Y associated to the pair $(\mathfrak{C}, \psi)$. The cover π is determined by $\mathcal{L}_\chi$ and $D_{\mathfrak{C},\psi}$ with some specified relations (cf. [22, Theorem 2.1]). We mainly apply this theory when $\mathfrak{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathfrak{G} \cong \mathbb{Z}_2^2$.

We set up some notation and conventions. Denote by $H = \langle g_1\rangle \times \langle g\rangle$ a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, where g_1, g are generators of H, g_1 is of order 2 and g is of order 4. Denote by $H^* = \langle\chi\rangle \times \langle\rho\rangle$ the group of characters of H, where $\chi(g_1) = -1$, $\rho(g) = \mathrm{i}$ and $\chi(g) = \rho(g_1) = 1$. The group H contains a unique subgroup $G = \{1, g_1, g_2, g_3\}$ which is isomorphic to $\mathbb{Z}_2^2$, where $g_2 = g^2$ and $g_3 = g_1g_2$. Denote by $\chi_i \in G^*$ the nontrivial character orthogonal to g_i for $i = 1, 2, 3$.

When $\mathfrak{G} = G$, we simply set $\mathcal{L}_i := \mathcal{L}_{\chi_i}$ and $\Delta_i := D_{\langle g_i\rangle,\psi}$, where ψ is the unique nontrivial character of $\langle g_i\rangle$. Similarly, when $\mathfrak{G} = H$, we set $D_i := D_{\langle g_i\rangle,\psi}$ for $1 \neq \psi \in \langle g_i\rangle^*$. For the cyclic group $\langle g\rangle \cong \mathbb{Z}_4$, we set $D_{g,\pm\mathrm{i}} := D_{\langle g\rangle,\psi}$ for $\psi \in \langle g\rangle^*$ with $\psi(g) = \pm\mathrm{i}$. We adopt a similar convention for the cyclic group $\langle g_1g\rangle \cong \mathbb{Z}_4$.

In what follows, the indices $i \in \{1, 2, 3\}$ should be understood as residue classes modulo 3. Also linear equivalence and numerical equivalence between divisors are denoted by $\equiv$ and $\overset{num}{\sim}$, respectively.

PROPOSITION 2.4. (cf. [6], [22, Theorem 2.1 and Corollary 3.1]) *Let $\pi: X \to Y$ be a finite abelian cover between projective varieties. Assume that X is normal and Y is smooth.*

(a) *If the Galois group of π is G, then π is determined by the following data: Divisors Δ_i and $\mathcal{L}_i$ $(i = 1, 2, 3)$ such that*

$$2\mathcal{L}_i \equiv \Delta_{i+1} + \Delta_{i+2}, \qquad \mathcal{L}_i + \Delta_i \equiv \mathcal{L}_{i+1} + \mathcal{L}_{i+2} \quad \textit{for } i = 1, 2, 3, \tag{2.1}$$

and moreover the divisors Δ_i are effective and $\Delta := \Delta_1 + \Delta_2 + \Delta_3$ is reduced.

(b) *If the Galois group of π is H, then π is determined by the following reduced data (see [22, Proposition 2.1]): Divisors $\mathcal{L}_\chi, \mathcal{L}_\rho, D_1, D_2, D_3,$*

$D_{g,\mathrm{i}}$, $D_{g,-\mathrm{i}}$, $D_{g_1g,\mathrm{i}}$ and $D_{g_1g,-\mathrm{i}}$ such that

$$2\mathcal{L}_\chi \equiv D_1 + D_3 + D_{g_1g,\mathrm{i}} + D_{g_1g,-\mathrm{i}},$$
$$4\mathcal{L}_\rho \equiv 2D_2 + 2D_3 + D_{g,\mathrm{i}} + 3D_{g,-\mathrm{i}} + D_{g_1g,\mathrm{i}} + 3D_{g_1g,-\mathrm{i}}, \tag{2.2}$$

and moreover the divisors D_1, D_2, D_3, $D_{g,\mathrm{i}}$, $D_{g,-\mathrm{i}}$, $D_{g_1g,\mathrm{i}}$ and $D_{g_1g,-\mathrm{i}}$ are effective and

$$D := D_1 + D_2 + D_3 + D_{g,\mathrm{i}} + D_{g,-\mathrm{i}} + D_{g_1g,\mathrm{i}} + D_{g_1g,-\mathrm{i}}$$

is reduced.

§3. Two involutions commute

We first deduce Corollary 1.3 and Theorem 1.2 from the following theorem.

THEOREM 3.1. *Let S be a smooth minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 7$. Assume that $\mathrm{Aut}(S)$ contains two distinct involutions α and β. Then $\alpha\beta = \beta\alpha$.*

Proof of Corollary 1.3. On the contrary, suppose that α is an involution of $\mathrm{Aut}(S)$ other than g_1, g_2, g_3. Theorem 3.1 implies $\langle \alpha, g_1, g_2\rangle \cong \mathbb{Z}_2^3$. Therefore, there are seven involutions θ in $\langle \alpha, g_1, g_2\rangle$ and seven numbers $K_SR_\theta \in \{1, 3, 5, 7\}$ by Lemma 2.3. Since any two involutions generate a subgroup of type $\mathbb{Z}_2^2$, by Theorem 1.1, we conclude that each of the three numbers $1, 3$ and 7 occurs at most once. Furthermore, Theorem 1.1(a) also implies that if 7 occurs, then the other six numbers are all equal to 5. Hence there are at least five involutions θ for which $K_SR_\theta = 5$. There must be a subgroup of type $\mathbb{Z}_2^2$ containing three of these five, a contradiction to Theorem 1.1. ∎

Proof of Theorem 1.2. We may assume that $\mathrm{Aut}(S)$ contains at least two involutions. These two involutions generate a subgroup $G \cong \mathbb{Z}_2^2$ by Theorem 3.1. We still denote by g_1, g_2 and g_3 the involutions of G. Let τ be any automorphism of S. Corollary 1.3 gives $\tau G\tau^{-1} = G$. Since $\tau(R_{g_i}) = R_{\tau g_i\tau^{-1}}$, we have $K_SR_{g_i} = K_SR_{\tau g_i\tau^{-1}}$ and $R_{g_i}R_{g_{i+1}} = R_{\tau g_i\tau^{-1}}R_{\tau g_{i+1}\tau^{-1}}$ for $i = 1, 2, 3$. From this observation and Theorem 1.1, we conclude that $\tau g_i\tau^{-1} = g_i$ for $i = 1, 2, 3$ and complete the proof. ∎

The remaining of this section is devoted to prove Theorem 3.1. *We assume by contradiction that $\alpha\beta \neq \beta\alpha$.* We will deduce a contradiction

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through a sequence of lemmas and propositions. We use the same notation as Section 2. Recall that $\mathrm{tr}(\alpha^*) = 2 - R_\alpha^2$, $R_\alpha^2 = \pm 1$ and $k_\alpha = K_S R_\alpha + 4$ (see Proposition 2.1 and Lemma 2.3).

LEMMA 3.2. *The order of $\alpha\beta$ is an odd integer.*

Proof. Assume by contradiction that the order of $\alpha\beta$ is $2k$ and $k \geqslant 2$. Let $\gamma := (\alpha\beta)^k = (\beta\alpha)^k$. Then γ is an involution and $\gamma\alpha = (\alpha\beta)^k\alpha = \alpha(\beta\alpha)^k = \alpha\gamma$. Therefore, $\langle\gamma, \alpha\rangle \cong \mathbb{Z}_2^2$. Then $R_\alpha^2 = R_\gamma^2 = -1$ by Theorem 1.1. Similarly, $\gamma\beta = \beta\gamma$ and $R_\beta^2 = R_\gamma^2 = -1$. So $\mathrm{tr}(\alpha^*) = \mathrm{tr}(\beta^*) = 3$.

Let $\iota := \alpha\beta\alpha$. Note that α, β and ι are three distinct involutions in $\mathrm{Aut}(S)$ and

$$\alpha(R_\iota) = R_\beta, \qquad \alpha(R_\beta) = R_\iota, \qquad \alpha(R_\alpha) = R_\alpha. \tag{3.1}$$

Recall that $\dim H^2(S, \mathbb{C}) = \rho(S) = 3$. Now $c_1(R_\alpha), c_1(R_\beta)$ and $c_1(R_\iota)$ are not a basis of $H^2(S, \mathbb{C})$, for otherwise, (3.1) implies $\mathrm{tr}(\alpha^*) = 1$, which is a contradiction to $\mathrm{tr}(\alpha^*) = 3$.

So the intersection number matrix of R_α, R_β and R_ι has determinant zero. That is $2x^2y + 2x^2 + y^2 - 1 = 0$, where $x := R_\alpha R_\iota = R_\alpha R_\beta$ (see (3.1)) and $y := R_\beta R_\iota$. Observe that $y \geqslant 0$; otherwise, $R_\beta = R_\iota$ and $\beta = \iota$, a contradiction to $\alpha\beta \neq \beta\alpha$. It follows that $x = 0$, $y = 1$ and the nontrivial linear relation among $c_1(R_\alpha), c_1(R_\beta)$ and $c_1(R_\iota)$ is $c_1(R_\beta) + c_1(R_\iota) = 0$. This contradicts the fact that the divisor $R_\beta + R_\iota$ is strictly effective. Hence the order of $\alpha\beta$ is an odd integer. ∎

Recall that our aim is to deduce a contradiction from the assumption $\alpha\beta \neq \beta\alpha$. According to the previous lemma, from now on, *we may assume that the order r of $\alpha\beta$ is an odd prime.* In fact, if $r = p(2t+1)$ for some prime $p \geqslant 3$ and some integer $t > 0$, then $\alpha' := (\alpha\beta)^t\alpha$ and $\beta' := (\beta\alpha)^t\beta$ are involutions and the order of $\alpha'\beta'$ is p. In particular, we have $\alpha'\beta' \neq \beta'\alpha'$. We may replace α, β by α', β' and continue our discussion.

The subgroup $\langle\alpha, \beta\rangle$ of $\mathrm{Aut}(S)$ is isomorphic to the dihedral group of order $2r$. Let D_r denote this subgroup. Since r is a prime, all the involutions in D_r are pairwise conjugate and D_r has exactly one nontrivial normal subgroup $\langle\alpha\beta\rangle$, which is the commutator subgroup. Any irreducible linear representation of D_r has dimension at most two, and any irreducible 2-dimensional representation of D_r is isomorphic to the matrix representation given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \quad \text{for some } c \neq 1 \text{ and } c^r = 1.$$

LEMMA 3.3. *With the same assumption as above, we have*

(a) *the canonical K_S is ample;*
(b) *the curves R_α and R_β generate a pencil $|F|$ of curves with $F^2 = 1$ and $K_S F = 3$, and $|F|$ has a simple base point p;*
(c) *the group D_r acts faithfully on $|F|$.*

Proof. Set $k := \frac{1}{2}(r-1)$ and set $\gamma := \alpha(\beta\alpha)^k = \beta(\alpha\beta)^k$. Then α, β and γ are three distinct involutions and they are pairwise conjugate. Therefore, $K_S R_\alpha = K_S R_\beta = K_S R_\gamma$ and $R_\alpha^2 = R_\beta^2 = R_\gamma^2$. Since $\gamma\alpha = \beta\gamma$ and $\gamma\beta = \alpha\gamma$,

$$\gamma(R_\alpha) = R_\beta, \qquad \gamma(R_\beta) = R_\alpha, \qquad \gamma(R_\gamma) = R_\gamma. \tag{3.2}$$

We claim that $R_\gamma^2 = R_\alpha^2 = R_\beta^2 = 1$. Otherwise, as in the proof of Lemma 3.2, we could deduce a contradiction by calculating the determinant of the intersection number matrix of R_α, R_β and R_γ and by calculating $\mathrm{tr}(\gamma^*)$.

Then (a) follows from Lemma 2.3. Lemma 2.3 also gives $K_S R_\alpha = K_S R_\beta = 3$. The algebraic index theorem implies $(R_\alpha + R_\beta)^2 \leqslant \frac{6^2}{7}$ and thus $R_\alpha R_\beta \leqslant 1$. Since $R_\alpha^2 = R_\beta^2 = 1$, the equality holds and $R_\alpha \overset{num}{\sim} R_\beta$. Similarly, we have $R_\gamma \overset{num}{\sim} R_\alpha$.

Let p be the unique intersection point of R_α and R_γ. Then (3.2) implies that R_α, R_β and R_γ pairwise intersect transversely at the point p. Recall that $\mathrm{Pic}(S) \cong H^2(S, \mathbb{Z})$ and $\mathrm{Num}(S) = \mathrm{Pic}(S)/\mathrm{Pic}(S)_{\mathrm{Tors}}$. Let m be the smallest positive integer such that $mR_\alpha \equiv mR_\gamma \equiv mR_\beta$. Let $\varepsilon : \tilde{S} \to S$ be the blowup at p, let E be the exceptional curve and let $\tilde{R}_\alpha$ be the strict transform of R_α, and so forth. Then $|m\tilde{R}_\alpha|$ induces a fibration $f : \tilde{S} \to \mathbb{P}^1$ and $m\tilde{R}_\gamma, m\tilde{R}_\alpha$ and $m\tilde{R}_\beta$ are fibers of f.

The fibration f has E as a m-section. If $m \geqslant 2$, we easily obtain a contradiction by applying the Hurwitz formula for $f|_E : E \to \mathbb{P}^1$. Therefore, $m = 1$, $R_\alpha \equiv R_\gamma \equiv R_\beta$ and $h^0(S, \mathcal{O}_S(R_\alpha)) = 2$. And (b) is proved.

For (c), first note that p is a fixed point of D_r. So D_r acts faithfully on the tangent space $\mathrm{T}_p S$ of S to the point p. According to the discussion before the lemma, this action is irreducible. Because r is an odd number, the corresponding action of D_r on $\mathbb{P}(\mathrm{T}_p S)$ is faithful. Since $F^2 = 1$, p is a smooth point of F and thus $\mathrm{T}_p F$ is a 1-dimensional linear subspace of $\mathrm{T}_p S$ for any $F \in |F|$. From this, we conclude that D_r acts faithfully on $|F|$. ∎

Because D_r acts faithfully on $|F|\cong\mathbb{P}^1$, every automorphism has exactly two invariant curves in $|F|$. For every involution $\theta\in D_r$, one of the two θ-invariant curves in $|F|$ is R_θ. Denote the other one by F_θ. Then F_θ contains the seven isolated fixed points of θ. Denote by F_0 one of the two $\alpha\beta$-invariant curves in $|F|$. Then the other one is $\alpha(F_0)(=\beta(F_0))$ and $\mathrm{Fix}(\alpha\beta)\subseteq F_0\cup\alpha(F_0)$. We shall show that F_0 is not 2-connected. But first we need the following lemma about the action of D_r on the singular cohomology group.

LEMMA 3.4. *The automorphism $\alpha\beta$ acts trivially on $H^2(S,\mathbb{C})$. In particular, the quotient surface S/D_r has Picard number* 2.

Proof. We have seen that α,β and thus D_r act trivially on the 2-dimensional linear subspace generated by $c_1(K_S)$ and $c_1(F)$. Because $H^2(S,\mathbb{C})$ is 3-dimensional and $\alpha\beta$ is contained in the kernel of any 1-dimensional representation of D_r, $\alpha\beta$ acts trivially on $H^2(S,\mathbb{C})$. Hence the invariant subspace of $H^2(S,\mathbb{C})$ for the D_r-action is 2-dimensional and S/D_r has Picard number 2. ∎

We analyze the members of the pencil $|F|$ which are not 2-connected. This will help us to determine the base locus of the linear system $|K_S+F|$ in the proof of Proposition 3.6 and to find a basis of $\mathrm{Num}(S)$. We continue to use the fact that S has Picard number 3.

LEMMA 3.5. *Assume that $|F|$ contains a curve which is not 2-connected. Then*

(a) *the curves in $|F|$ which are not 2-connected are exactly the $\alpha\beta$-invariant curves F_0 and $\alpha(F_0)$;*
(b) *$F_0=A+B$, where A and B are irreducible curves, and $K_SA=2$, $K_SB=1$, $A^2=0$, $B^2=-1$ and $AB=1$. Moreover, A contains the base point p of $|F|$.*

Proof. Assume that $A+B\in|F|$, $A>0$, $B>0$ and $AB\leqslant 1$. Because K_S is ample and $K_SF=3$, we may assume $K_SA=2$ and $K_SB=1$. Then B is irreducible. The algebraic index theorem and the adjunction formula imply $A^2\leqslant 0$ and $B^2\leqslant -1$. In particular, by Lemma 3.4, $\alpha\beta(B).B=B^2<0$ and thus $\alpha\beta(B)=B$. Hence $A+B$ is one of the $\alpha\beta$-invariant curves F_0 and $\alpha(F_0)$.

Because $A^2+B^2=F^2-2AB\geqslant -1$, the argument above yields $A^2=0$, $B^2=-1$ and $AB=1$. Then $FA=1$ and $FB=0$. So the simple base point p of $|F|$ belongs to A. It remains to show that A is irreducible.

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Assume $A = A_1 + A_2$ with $A_1 > 0$ and $A_2 > 0$. Because $K_S A = 2$ and K_S is ample, $K_S A_1 = K_S A_2 = 1$ and both A_1 and A_2 are irreducible. We now calculate the intersection numbers A_1^2, A_2^2 and $A_1 A_2$. For $i = 1, 2$, the algebraic index theorem and the adjunction formula imply $A_i^2 \in \{-1, -3\}$. Note that $A_i^2 = -3$ if and only if A_i is a (-3)-curve. Since $\alpha(F_0) \cap F_0$ consists of one point p, $\alpha(A_i) \neq A_i$ for $i = 1, 2$. We conclude that $A_1^2 = A_2^2 = -1$ by Lemma 2.2. Then $A_1 A_2 = \frac{1}{2}(A^2 - A_1^2 - A_2^2) = 1$.

Because $FA = 1$, we may assume $FA_1 = 1$ and $FA_2 = 0$. It follows that $p \in A_1$, $p \notin A_2$ and $A_2 B = A_2(F - A_1 - A - 2) = 0$. Since $p \notin A_2 \cup B$, S contains four disjoint curves $B, \alpha(B), A_2$ and $\alpha(A_2)$, all with self-intersection number (-1), a contradiction to $\rho(S) = 3$. Therefore A is irreducible. ∎

The following proposition determines the order of the automorphism $\alpha\beta$.

PROPOSITION 3.6. *The pencil $|F|$ contains a curve which is not* 2-*connected and the automorphism $\alpha\beta$ is of order* 3.

Proof. Let F be any curve in $|F|$. The long exact sequence of cohomology groups associated to the exact sequence $0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S + F) \to \mathcal{O}_F(K_F) \to 0$ shows that $h^0(S, \mathcal{O}_S(K_S + F)) = h^0(F, \mathcal{O}_F(K_F)) = p_a(F) = 3$ and the trace of $|K_S + F|$ on F is complete. Thus $|K_S + F|$ defines a rational map $h : S \dashrightarrow \mathbb{P}^2$ and h is defined on F whenever $|K_F|$ is base point free. In particular, h is defined on the smooth curve R_α $(\in |F|)$ and $h(R_\alpha)$ is the canonical image of R_α. The same statement holds by replacing α by β.

Because there is a D_r-linearization on $\mathcal{O}_S(K_S + F)$, the rational map h is D_r-equivariant. Therefore, $h(R_\alpha)$ is contained in the fixed locus of the action of α on $\mathbb{P}^2$. Note that an involution on $\mathbb{P}^2$ has a line and a point as the fixed locus. It follows that α acts trivially on $\mathbb{P}^2$ because $h(R_\alpha)$ is a conic curve or a quartic curve. Similarly, β and thus D_r act trivially on $\mathbb{P}^2$. Therefore, $h : S \dashrightarrow \mathbb{P}^2$ factors through the quotient morphism $S \to S/D_r$.

Note that K_S is ample, F is nef and $(K_S + F)^2 = 14$. First assume that h is a morphism. Then it is finite and it has degree 14. We thus get $|D_r| = \deg h$ and $r = 7$. It follows that the induced morphism $h' : S/D_r \to \mathbb{P}^2$ is an isomorphism. So the invariant linear subspace of $H^2(S, \mathbb{C})$ for the D_r-action is isomorphic to $H^2(\mathbb{P}^2, \mathbb{C})$, which is 1-dimensional. This contradicts Lemma 3.4 and thus h is not a morphism.

We now analyze the base locus of h. If F is 2-connected, $|K_F|$ is base point free by [5, Theorem 3.3] and h is defined on F. Hence the base locus of $|K_S + F|$ is contained in the curves of $|F|$, which are not 2-connected.

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According to Lemma 3.5, $|F|$ contains exactly two such curves $F_0 = A + B$ and $\alpha(F_0)$. Similar arguments as above show that the trace of $|K_S + A|$ (respectively $|K_S + B|$) on A (respectively B) is complete. Since $p_a(A) = 2$ and $p_a(B) = 1$, $|K_A|$ and $|K_B|$ are base point free by [5, Theorem 3.3]. Because $|K_S + F| \supseteq |K_S + A| + B, |K_S + B| + A$, $|K_S + F|$ has exactly two base points $q := A \cap B$ and $\alpha(q) = \alpha(A) \cap \alpha(B)$.

Therefore, h is a finite morphism outside the base locus and $\deg h = (K_S + F)^2 - 2 = 12$. Since h factors through S/D_r, we have $|D_r| = 6$ and $r = 3$. ∎

According to Lemma 3.5 and Proposition 3.6, the $\alpha\beta$-invariant curve F_0 has A and B as irreducible components with $AB = 1$. It is easy to check that F, A and $\alpha(A)$ generate $\mathrm{Num}(S)$ and $K_S \overset{num}{\sim} F + A + \alpha(A)$. We shall show that K_S is indeed linearly equivalent to $F + A + \alpha(A)$, hence contradict $p_g(S) = 0$ and complete the proof of Theorem 3.1. For this purpose, we turn to the quotient surface S/D_3 and analyze $\mathrm{Fix}(\alpha\beta)$.

Proposition 3.7. *The automorphism $\alpha\beta$ has $B \cup \alpha(B)$ as the divisorial part of the fixed locus and it has five isolated fixed points $p, q_1, q_2, \alpha(q_1)$ and $\alpha(q_2)$, where q_1 and q_2 are contained in A. Each isolated fixed point of $\alpha\beta$ is of type $\frac{1}{3}(1, 2)$.*

Proof. We have seen that F_0 and $\alpha(F_0)$ are $\alpha\beta$-invariant and $\mathrm{Fix}(\alpha\beta) \subseteq F_0 \cup \alpha(F_0)$. Moreover, the curves $A, \alpha(A), B$ and $\alpha(B)$ are $\alpha\beta$-invariant. Also note that a point q is a fixed point (respectively an isolated fixed point) of $\alpha\beta$ if and only if so is the point $\alpha(q)(= \beta(q))$.

We claim that neither A nor $\alpha(A)$ is contained in $\mathrm{Fix}(\alpha\beta)$. Otherwise, both A and $\alpha(A)$ are contained in $\mathrm{Fix}(\alpha\beta)$. Since $A \cap \alpha(A) = p$, this contradicts the fact that the divisorial part of $\alpha\beta$ is a disjoint union of smooth curves. The claim is proved.

Now assume by contradiction that B is not contained in $\mathrm{Fix}(\alpha\beta)$. Then nor is $\alpha(B)$ and $\mathrm{Fix}(\alpha\beta)$ consists of isolated fixed points. Then $\mathrm{Fix}(\alpha\beta)$ has five fixed points by Proposition 2.1 and Lemma 3.4. Three of these points are $p, q := A \cap B$ and $\alpha(q)$. Denote the other two by $p_1 (\in F_0)$ and by $\alpha(p_1)$. We must have $p_1 \in B$. Otherwise, the nontrivial automorphism $\alpha\beta|_B$ has exactly one fixed point q, which is a smooth point of B since $AB = 1$. This is impossible because $p_a(B) = 1$. Therefore, $p_1 \in B$. It follows that $\alpha\beta|_A$ has exactly two fixed points p and q, which are smooth points of A. Note that A has at most two singular points since $p_a(A) = 2$. Because the singular locus of A is $\alpha\beta$-invariant and $\alpha\beta|_A$ has order 3, we conclude that A is indeed smooth. However, the Hurwitz formula shows that $\alpha\beta|_A$ has either one or four fixed points, a contradiction.

So B and $\alpha(B)$ are contained in $\mathrm{Fix}(\alpha\beta)$. In particular, B and $\alpha(B)$ are smooth curves. Then $\mathrm{Fix}(\alpha\beta) \setminus \{B \cup \alpha(B)\}$ consists of five isolated fixed points and each fixed point is of type $\frac{1}{3}(1,2)$ by Proposition 2.1 and Lemma 3.4. These points must be contained in $A \cup \alpha(A)$. ∎

Now we are able to describe the quotient map $\pi : S \to Y := S/D_3$, where $D_3 = \{1, \alpha, \beta, \gamma, \alpha\beta, \beta\alpha\}$ and $\gamma := \alpha\beta\alpha = \beta\alpha\beta$. The divisorial parts and isolated fixed points of cyclic subgroups of D_3 are as follows (see the discussion before Lemma 3.4 and Proposition 3.7):

cyclic subgroups	divisorial part	isolated fixed points
$\langle\alpha\rangle$ (resp. $\langle\beta\rangle, \langle\gamma\rangle$) $\cong \mathbb{Z}_2$	R_α (resp. R_β, R_γ)	7 points on F_α (resp. F_β, F_γ)
$\langle\alpha\beta\rangle \cong \mathbb{Z}_3$	$B, \alpha(B)$	$p, q_1, q_2, \alpha(q_1), \alpha(q_2)$

Note that p is the unique point with the stabilizer D_3. From the action of D_3 on the tangent space T_pS (see the proof of Lemma 3.3(c)), it is easily seen that $\pi(p)$ is a smooth point of Y. We conclude that Y has seven nodes and two A_2-singularities $\pi(q_1)$ and $\pi(q_2)$. In particular, Y is Gorenstein. The ramification formula gives

$$K_S = \pi^*K_Y + R_\alpha + R_\beta + R_\gamma + 2B + 2\alpha(B) \equiv \pi^*K_Y + 3F + 2B + 2\alpha(B) \tag{3.3}$$

and thus $K_Y^2 = \frac{1}{6}(K_S - 3F - 2B - 2\alpha(B))^2 = -3$.

Let $B' = \pi(B)$. Then B' is contained in the smooth locus of Y. Note that B' is a smooth elliptic curve and $\pi^*B' = 3B + 3\alpha(B)$. So $B'^2 = -3$ and $K_YB' = 3$. Since $-K_YB' = K_Y^2 = B'^2$, $-K_Y \overset{num}{\sim} B'$ by Lemma 3.4. This implies that $H^0(mK_Y) = 0$ for $m \geqslant 1$. As the quotient of S, Y has irregularity $q(Y) = 0$. Therefore, Y is a rational surface. Note that linear equivalence and numerical equivalence between divisors are the same on a smooth rational surface. Since Y contains only rational double points and B' is contained in the smooth locus of Y, we have $-K_Y \equiv B'$ indeed. Then by (3.3),

$$\begin{aligned} K_S &\equiv \pi^*(-B') + 3F + 2B + 2\alpha(B) \\ &\equiv (-3B - 3\alpha(B)) + 3F + 2B + 2\alpha(B) \\ &\equiv F + A + \alpha(A). \end{aligned}$$

We obtain a contradiction to $p_g(S) = 0$ and complete the proof of Theorem 3.1.

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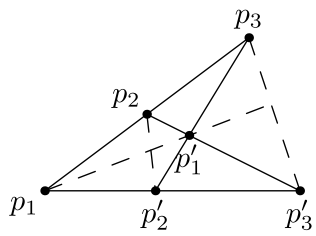

Figure 1.
Configurations of the points $p_1, \ldots, p'_3$.

§4. Inoue surfaces

As mentioned in the introduction, Inoue surfaces have been the first examples of surfaces of general type with $p_g = 0$ and $K^2 = 7$ (cf. [14]). Here we describe them as finite Galois $\mathbb{Z}_2^2$-covers of the 4-nodal cubic surface, following [19, Example 4.1]. At the end of this section, we prove Theorem 1.4.

EXAMPLE 4.1. Let $\sigma : W \to \mathbb{P}^2$ be the blowup of the six vertexes $p_1, p_2, p_3, p'_1, p'_2, p'_3$ of a complete quadrilateral on $\mathbb{P}^2$ (see Figure 1). Denote by E_i (respectively E'_i) the exceptional curve of W over p_i (respectively p'_i) and denote by L the pullback of a general line by σ. Then $\mathrm{Pic}(W) = \mathbb{Z}L \oplus \bigoplus_{i=1}^3(\mathbb{Z}E_i \oplus \mathbb{Z}E'_i)$.

The surface W has four disjoint (-2)-curves. They are the proper transforms of the four sides of the quadrilateral and their divisor classes are

$$Z_i \equiv L - E_i - E'_{i+1} - E'_{i+2}, \qquad Z \equiv L - E_1 - E_2 - E_3.$$

Let $\eta : W \to \Sigma$ be the morphism contracting there curves. Then Σ is the 4-nodal cubic surface.

Let Γ_1, Γ_2 and Γ_3 be the proper transforms of the three diagonals of the quadrilateral, that is, $\Gamma_i \equiv L - E_i - E'_i$ for $i = 1, 2, 3$. Note that they are exactly the (-1)-curves which are disjoint from any (-2)-curve. For each $i = 1, 2, 3$, W has a pencil of rational curves $|F_i| := |2L - E_{i+1} - E_{i+2} - E'_{i+1} - E'_{i+2}|$. Observe that $-K_W \equiv \Gamma_1 + \Gamma_2 + \Gamma_3 \equiv \Gamma_i + F_i$ for $i = 1, 2, 3$.

We define three effective divisors on W

$$\Delta_1 := \Gamma_1 + F_2 + Z_1 + Z_3, \qquad \Delta_2 := \Gamma_2 + F_3,$$
$$\Delta_3 := \Gamma_3 + F_1 + F'_1 + Z_2 + Z. \tag{4.1}$$

We require that F_i $(i = 1, 2, 3)$ and F_1' are smooth 0-curves such that the divisor $\Delta := \Delta_1 + \Delta_2 + \Delta_3$ has only nodes. It is directly shown that there are divisors $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ satisfying (2.1) in Proposition 2.4. Then there is a smooth finite G-cover $\overline{\pi} : V \to W$ branched on the divisors Δ_1, Δ_2 and Δ_3. The (set theoretic) inverse image of a (-2)-curve under $\overline{\pi}$ is a disjoint union of two (-1)-curves. Let $\varepsilon : V \to S$ be the blowdown of these eight (-1)-curves. Then there is a finite G-cover $\pi : S \to \Sigma$ such that the following diagram (4.2) commutes.

$$\begin{tikzcd} V \arrow[r, "\varepsilon"] \arrow[d, "\overline{\pi}"'] & S \arrow[d, "\pi"] \\ W \arrow[r, "\eta"] & \Sigma \end{tikzcd} \tag{4.2}$$

The surface S is a smooth minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 7$. It is called an Inoue surface. When the curves F_1, F_1', F_2 and F_3 vary, we obtain a 4-dimensional family of Inoue surfaces.

LEMMA 4.2. *Let W be as in Example 4.1.*

(a) *Let α be an automorphism on W. If the induced map $\alpha^* : H^2(W, \mathbb{C}) \to H^2(W, \mathbb{C})$ is the identity, then $\alpha = \mathrm{Id}_W$.*
(b) *Let $\alpha_{\mathbb{P}^2}$ be the involution on $\mathbb{P}^2$ such that $\alpha_{\mathbb{P}^2}(p_k) = p_k'$ for $k = 1, 3$. It induces an involution α_0 on W. Then $\mathrm{Fix}(\alpha_0)$ consists of the (-1)-curve Γ_2 and three isolated fixed points $\Gamma_1 \cap \Gamma_3$, $E_2 \cap F_3^*$ and $E_2' \cap F_3^*$, where F_3^* is the unique smooth α_0-invariant curve in the pencil $|F_3| = |2L - E_1 - E_1' - E_2 - E_2'|$.*

Proof. For (a), the assumption implies that the (-1)-curves E_i and E_i' $(i = 1, 2, 3)$ are α-invariant. So α comes from an automorphism on $\mathbb{P}^2$ which has $p_1, \ldots, p_3'$ as fixed points and thus it is the identity morphism.

For (b), note that $\mathrm{Fix}(\alpha_{\mathbb{P}^2}) = \overline{p_2 p_2'} \cup \{p_{13} := \overline{p_1 p_1'} \cap \overline{p_3 p_3'}\}$ and $\sigma(\mathrm{Fix}(\alpha_0)) = \mathrm{Fix}(\alpha_{\mathbb{P}^2})$. Because $\sigma^{-1}(\overline{p_2 p_2'}) = E_2 \cup E_2' \cup \Gamma_2$ and the divisorial part of α_0 is smooth, $\mathrm{Fix}(\alpha_0)$ has Γ_2 as the divisorial part. Then α_0 has three isolated fixed points by Proposition 2.1. The point $\sigma^{-1}(p_{13}) = \Gamma_1 \cap \Gamma_3$ is an isolated fixed point of α_0. Note that α_0 induces a nontrivial action on the pencil $|F_3|$. So $|F_3|$ contains exactly two α_0-invariant curves $\Gamma_1 + \Gamma_2$ and F_3^*. Since E_2 and E_2' are also α_0-invariant, the intersection points $E_2 \cap F_3^*$ and $E_2' \cap F_3^*$ are isolated fixed points of α_0. □

Proof of Theorem 1.4. Let $\tau \in \mathrm{Aut}(S)$. By Theorem 1.2, $\mathrm{Fix}(g_i)$ is τ-invariant for $i = 1, 2, 3$ and τ induces an automorphism α_Σ on the quotient surface $\Sigma = S/G$. So the branch locus $\pi(\mathrm{Fix}(g_i))$ (for $i = 1, 2, 3$) of $\pi : S \to \Sigma$ is α_Σ-invariant.

Assume that $\tau \notin G$, that is, $\alpha_\Sigma \neq \mathrm{Id}_\Sigma$. The automorphism α_Σ lifts to the minimal resolution W of Σ. Denote by α the induced automorphism on W. Then $\Delta_1, \Delta_2, \Delta_3$ (see the diagram (4.2)) are α-invariant because Δ_i is the inverse image of $\pi(\mathrm{Fix}(g_i))$ under the morphism $\eta : W \to \Sigma$. These divisors are given by (4.1). It follows that the (-1)-curves $\Gamma_1, \Gamma_2, \Gamma_3$, the 0-curves F_2, F_3 and the curves $F_1 + F_1'$, $Z_1 + Z_3$, $Z_2 + Z$ are α-invariant. Note that the Chern classes of $\Gamma_1, \Gamma_2, \Gamma_3$ and the Chern classes of Z_1, Z_2, Z_3, Z generate $H^2(S, \mathbb{C})$. The argument above implies that $(\alpha^2)^* = (\alpha^*)^2$ is the identity morphism. Then α is an involution by Lemma 4.2.

Since $F_i \equiv \Gamma_{i+1} + \Gamma_{i+2}$, the fibration $f_i : W \to \mathbb{P}^1$ induced by $|F_i|$ is α-equivalent for $i = 1, 2, 3$. Note that f_2 has three singular fibers $\Gamma_1 + \Gamma_3$, $Z_1 + 2E_2' + Z_3$ and $Z_2 + 2E_2 + Z$. According to the discussion above, these three fibers are α-invariant. Because any nontrivial automorphism on $\mathbb{P}^1$ has at most two fixed points, α respects the fibration f_2, that is, $f_2 = f_2\alpha$. In particular, E_2 and E_2' are α-invariant.

Note that f_1 has three singular fibers $\Gamma_2 + \Gamma_3$, $Z_1 + 2E_1 + Z$ and $Z_2 + 2E_1' + Z_3$. If $f_1 = f_1\alpha$, then all the (-2)-curves Z_1, Z_2, Z_3 and Z are α-invariant since α also respects f_2. Then α^* is the identity morphism and so is α by Lemma 4.2, a contradiction to our assumption. So α induces a nontrivial action on $|F_1| \cong \mathbb{P}^1$. Since the singular $\Gamma_2 + \Gamma_3$ is α-invariant, α must permute the other two singular fibers of f_1. Hence $\alpha(E_1) = E_1'$ and $\alpha(E_1') = E_1$. Similarly, by considering the action of α on $|F_3|$, we see that $\alpha(E_3) = E_3'$ and $\alpha(E_3') = E_3$.

We conclude that α is the involution α_0 in Lemma 4.2. We actually prove that if $\mathrm{Aut}(S) \neq G$, then $\mathrm{Aut}(S)/G \cong \langle \alpha_0 \rangle$, and in the Equation (4.1), the curve F_3 in Δ_2 is indeed the curve F_3^* in Lemma 4.2(b) and $F_1' = \alpha_0(F_1)$ in Δ_3. Combining with Theorem 1.2 and Corollary 1.3, we complete the proof of Theorem 1.4. ∎

REMARK 4.3. When the curves F_1 and F_2 vary, the Inoue surfaces corresponding to the branch divisors (4.1) with $F_3 = F_3^*$ and $F_1' = \alpha_0(F_1)$ form a 2-dimensional irreducible closed subset of the total 4-dimensional family of Inoue surfaces. Also Lemma 4.2 shows that $W/\langle \alpha_0 \rangle$ has three nodes. Moreover, it contains three (-2)-curves in the smooth locus and these curves are the images of $Z_1 + Z_3$, $Z_2 + Z$ and Γ_2 under the quotient

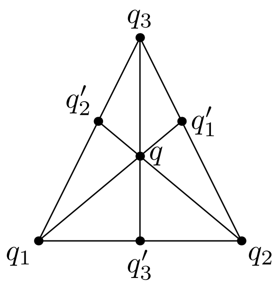

Figure 2.
Configurations of the points $q, q_1, \ldots, q_3'$.

map from W to $W/\langle\alpha_0\rangle$. This observation motivates us to construct some special Inoue surfaces in the next section.

§5. Special Inoue surfaces

We construct a 2-dimensional family of Inoue surfaces with automorphism groups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$. We use the notation in Section 2.2.

Let $q, q_1, q_2, q_3, q_1', q_2'$ and q_3' be seven points on $\mathbb{P}^2$ with the configuration as in Figure 2.

Let $\nu : Y \to \mathbb{P}^2$ be the blowup of these points. Denote by Q_i (respectively Q_i', Q) the exceptional curve of Y over q_i (respectively q_i', q) and by L the pullback of a general line by ν. Then $\mathrm{Pic}(W) = \mathbb{Z}L \oplus \mathbb{Z}Q \oplus (\bigoplus_{i=1}^3 \mathbb{Z}Q_i \oplus \mathbb{Z}Q_i')$. The surface Y has six disjoint (-2)-curves and their divisor classes are:

$$M_i = L - Q - Q_i - Q_i', \qquad N_i = L - Q_i' - Q_{i+1} - Q_{i+2} \quad \text{for } i = 1, 2, 3.$$

Let Λ_i be the proper transform of the line $\overline{q_{i+1}' q_{i+2}'}$, that is, $\Lambda_i \equiv L - Q_{i+1}' - Q_{i+2}'$ for $i = 1, 2, 3$.

We describe four base-point-free pencils of rational curves on Y. They are $|\Phi| := |2L - Q - Q_1 - Q_2 - Q_3|$ and $|\Phi_i| := |2L - Q - Q_i - Q_{i+1}' - Q_{i+2}'|$ $(i = 1, 2, 3)$. The singular members of $|\Phi|$ are $M_1 + 2Q_1' + N_1$, $M_2 + 2Q_2' + N_2$, $M_3 + 2Q_3' + N_3$ and those of $|\Phi_i|$ (fixed i) are $\Lambda_i + M_i + Q_i'$, $M_{i+1} + 2Q_{i+1} + N_{i+2}$, $M_{i+2} + 2Q_{i+2} + N_{i+1}$. Also note that $\Phi_i + N_i \equiv -K_Y$ for $i = 1, 2, 3$.

Let $\zeta : Y \to \Upsilon$ be the morphism contracting the five (-2)-curves M_1, M_2, N_2, M_3 and N_3. Then Υ has five nodes and contains a unique (-2)-curve $\zeta(N_1)$ in the smooth locus.

Now we define the following effective divisors on Y:

$$D_1 := \Lambda_1 + \Phi_1 + M_3, \qquad D_2 := \Lambda_2, \qquad D_3 := Q_1' + \Phi + N_3,$$
$$D_{g,\mathrm{i}} := N_1 + N_2, \qquad D_{g,-\mathrm{i}} := M_2, \qquad D_{g_1g,\mathrm{i}} := 0, \qquad D_{g_1g,-\mathrm{i}} := M_1. \tag{5.1}$$

We also define the following divisors:

$$\mathcal{L}_\chi := 4L - 2Q - 2Q_1 - Q_2 - Q_2' - Q_3 - 2Q_3',$$
$$\mathcal{L}_\rho := 4L - 2Q - 2Q_1 - Q_1' - 2Q_2 - Q_2' - Q_3 - Q_3'. \tag{5.2}$$

We require that $\Phi \in |\Phi|$ and $\Phi_1 \in |\Phi_1|$ are smooth curves such that the divisor $D = D_1 + \ldots + D_{g_1g,-\mathrm{i}}$ has only nodes. These divisors satisfy (2.2) in Proposition 2.4. So there is a finite Galois H-cover $\widehat{\pi} : X \to Y$ and X is normal.

We use [22, Propositions 3.1 and 3.3] to analyze the singular locus of X.

Lemma 5.1. *Let $m := \Lambda_2 \cap M_2$ and $n := \Lambda_2 \cap N_2$.*

(a) *The inverse image $\widehat{\pi}^{-1}(m)$ (resp. $\widehat{\pi}^{-1}(n)$) consists of two points $\widehat{m_1}$ and $\widehat{m_2}$ (resp. $\widehat{n_1}$ and $\widehat{n_2}$), each of which has stabilizer $\langle g\rangle$.*
(b) *The points $\widehat{m_1}, \widehat{m_2}, \widehat{n_1}$ and $\widehat{n_2}$ are exactly the singularities of X and they are nodes.*
(c) *The curve $\widehat{\pi}^{-1}(M_2)$ is a disjoint union of two irreducible smooth curves $\widehat{M}_{21}$ and $\widehat{M}_{22}$, and $\widehat{M}_{2j}$ has self-intersection number $(-\frac{1}{2})$ and $\widehat{m}_j \in \widehat{M}_{2j}$ for $j = 1, 2$. The curve $\widehat{\pi}^{-1}(N_2)$ consists of two irreducible smooth curves $\widehat{N}_{21}$ and $\widehat{N}_{22}$, and $\widehat{N}_{2j}$ has self-intersection number $(-\frac{1}{2})$ and $\widehat{n}_j \in \widehat{N}_{2j}$ for $j = 1, 2$.*
(d) *The curve $\widehat{\pi}^{-1}(M_3)$ is a disjoint union four (-1)-curves and so is $\widehat{\pi}^{-1}(N_3)$.*
(e) *The curve $\widehat{\pi}^{-1}(M_1)$ is a (-1)-curve.*

Proof. [22, Proposition 3.1] shows that X is smooth outside $\widehat{\pi}^{-1}(m)$ and $\widehat{\pi}^{-1}(n)$. Note that M_2 intersects only one irreducible component of $D - M_2$; that is $M_2\Lambda_2 = 1$. Because $\Lambda_2 = D_2$, $M_2 \leqslant D_{g,-\mathrm{i}}$ and $[H : \langle g\rangle] = 2$, we conclude that $\widehat{\pi}^{-1}(m)$ consists of two points, each of which has stabilizer $\langle g\rangle$. These two points are nodes of X according to [22, Proposition 3.3]. For the same reason, we have $\widehat{\pi}^{-1}(M_2) = \widehat{M}_{21} \cup \widehat{M}_{22}$ with $\widehat{M}_{21} \cap \widehat{M}_{22} = \emptyset$ and $\widehat{\pi}|_{\widehat{M}_{2j}} : \widehat{M}_{2j} \to M_2$ is an isomorphism. We also have $\widehat{\pi}^*(M_2) = 4\widehat{M}_{21} + 4\widehat{M}_{22}$. Thus (a)–(c) follow from the discussion above. Similar arguments apply to (d) and (e). For (d), just note that $M_3 (\leqslant D_1)$ and $N_3 (\leqslant D_3)$ are connected irreducible components of D. And (e) follows from the observation that

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$M_1 (= D_{g_1 g, -\mathrm{i}})$ intersects exactly two irreducible components of $D - M_1$ and $M_1 D_1 = M_1 D_3 = 1$. ∎

Now we explain how to obtain the smooth minimal model of X. On the minimal resolution $\widetilde{X}$ of X, the strict transforms of $\widehat{M}_{21}$, $\widehat{M}_{22}$, $\widehat{N}_{21}$ and $\widehat{N}_{22}$ are (-1)-curves. Each of these (-1)-curves intersects transversely at one point with exactly one of the four (-2)-curves over the nodes of X. So we can contract the four curves $\widehat{M}_{21}$, $\widehat{M}_{22}$, $\widehat{N}_{21}$ and $\widehat{N}_{22}$ of X to smooth points on another surface.

Let $\theta : X \to S$ be the morphism contracting the disjoint union of the nine (-1)-curves $\widehat{\pi}^{-1}(M_3)$, $\widehat{\pi}^{-1}(N_3)$, $\widehat{\pi}^{-1}(M_1)$ and the four curves $\widehat{M}_{21}$, $\widehat{M}_{22}$, $\widehat{N}_{21}$ and $\widehat{N}_{22}$. Then there is a smooth H-cover $\pi : S \to \Upsilon$ such that the outer square of the following diagram (5.3) commutes.

(5.3)
$$\begin{tikzcd}
X \arrow[r, "\theta_2"'] \arrow[rr, bend left, "\theta"] \arrow[d, "\widehat{\pi}_2"] \arrow[dd, bend right, "\widehat{\pi}"'] & \overline{V} \arrow[r] \arrow[d, "\overline{\pi}"] & S \arrow[dd, "\pi"] \\
X_1 \arrow[r, "\delta"'] \arrow[d, "\widehat{\pi}_1"] & \overline{W} & \\
Y \arrow[rr, "\zeta"] & & \Upsilon
\end{tikzcd}$$

We confirm that S is the smooth minimal model of X by the following proposition.

Proposition 5.2. *The surface S is an Inoue surface.*

Proof. From [22, Theorem 2.1], we obtain $\mathcal{L}_{\rho^2} = 2\mathcal{L}_\rho - D_2 - D_3 - D_{g,-\mathrm{i}} - D_{g_1 g,-\mathrm{i}}$. Then $2\mathcal{L}_{\rho^2} \equiv D_{g,\mathrm{i}} + D_{g,-\mathrm{i}} + D_{g_1 g,\mathrm{i}} + D_{g_1 g,-\mathrm{i}} = M_1 + N_1 + M_2 + N_2$ by Proposition 2.4(b) and (5.2). Let $\widehat{\pi}_1 : X_1 \to Y$ be the double cover branched along the four disjoint (-2)-curves M_1, N_1, M_2 and N_2. Note that ρ^2 is the unique character of H^* which is trivial on G. So the Galois group of $\widehat{\pi}_1$ is H/G and the cover $\widehat{\pi}$ factors through a G-cover $\widehat{\pi}_2 : X \to X_1$.

We have $2K_{X_1} = \widehat{\pi}_1^*(2K_Y + M_1 + N_1 + M_2 + N_2)$ and $K_{X_1}^2 = 0$. The inverse images of M_1, N_1, M_2 and N_2 under $\widehat{\pi}_1$ are (-1)-curves. Also $\widehat{\pi}_1^{-1}(M_3)$ is a disjoint union of two (-2)-curves and so is $\widehat{\pi}_1^{-1}(N_3)$. Let $\delta : X_1 \to \overline{W}$ be the morphism contracting three (-1)-curves $\widehat{\pi}_1^{-1} M_1, \widehat{\pi}_1^{-1} M_2$ and $\widehat{\pi}_1^{-1} N_2$. Then $\overline{W}$ is a weak Del Pezzo surface of degree three.

Let $\theta_2 : X \to \overline{V}$ be the morphism contracting the curves $\widehat{\pi}^{-1}(M_1)$, $\widehat{\pi}^{-1}(M_2)$ and $\widehat{\pi}^{-1}(N_2)$. We obtain a smooth Galois finite G-cover $\overline{\pi} : \overline{V} \to \overline{W}$

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and a commutative diagram (5.3). The branch locus of $\overline{\pi}$ is

$$\overline{\Delta_1} = \overline{\Lambda_1} + \overline{\Phi_1} + \overline{M_3}, \qquad \overline{\Delta_2} = \overline{N_1} + \overline{\Lambda_2}, \qquad \overline{\Delta_3} = \overline{Q_1'} + \overline{\Phi} + \overline{N_3}. \tag{5.4}$$

Here we denote by $\overline{\Lambda_1} = \delta\widehat{\pi}_1^{-1}(\Lambda_1)$, and so forth. We claim that

(i) $\overline{\Lambda_1}$, $\overline{N_1}$ and $\overline{Q_1'}$ are (-1)-curves;
(ii) $\overline{\Phi_1}$ and $\overline{\Lambda_2}$ are 0-curves, and $\overline{\Phi}$ is a disjoint union of two 0-curves in the same linear system;
(iii) $\overline{M_3}$ ($\overline{N_3}$) is a disjoint union of two (-2)-curves; these two (-2)-curves are disjoint from the (-1)-curves in (i);
(iv) $\overline{\Lambda_1} + \frac{1}{2}\overline{\Phi}$, $\overline{N_1} + \overline{\Phi_1}$ and $\overline{Q_1'} + \overline{\Lambda_2}$ and are linearly equivalent to $-K_{\overline{W}}$.

For example, because the general member of $|\Phi|$ is disjoint from $M_1 + N_1 + M_2 + N_2$, the curve $\widehat{\pi}_1^{-1}(\Phi)$ is a disjoint union of two 0-curves in the same linear system and so is $\overline{\Phi}$. In particular, $K_W\overline{\Phi} = -4$ and $\frac{1}{2}\overline{\Phi}$ is well defined in $\mathrm{Pic}(\overline{W})$. For the (-1)-curve Λ_1 on Y, since $\Lambda_1 M_1 = \Lambda_1 N_1 = 1$ and $\Lambda_1 M_2 = \Lambda_1 N_2 = 0$, the curve $\widehat{\pi}_1^{-1}(\Lambda_1)$ is a (-2)-curve, and it intersects with $\widehat{\pi}_1^{-1}(M_1)$ transversely at one point and it is disjoint from $\widehat{\pi}_1^{-1}(M_2)$ and $\widehat{\pi}_1^{-1}(N_2)$. So $\overline{\Lambda_1}$ is a (-1)-curve. Moreover, we have $\overline{\Lambda_1\Phi} = \widehat{\pi}_1^{*}(\Lambda_1)\widehat{\pi}_1^{*}(\Phi) = 2\Lambda_1\Phi = 4$. Finally, the algebraic index theorem yields $\overline{\Lambda_1} + \frac{1}{2}\overline{\Phi} \equiv -K_W$. Other statements can be proved in the same manner.

Comparing (5.4) to (4.1), we conclude that S is an Inoue surface. □

When Φ and Φ_1 vary, we obtain a 2-dimensional family of Inoue surfaces with automorphism groups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

REMARK 5.3. We may directly show that K_S is ample, $K_S^2 = 7$ and $p_g(S) = 0$ for the surface S in (5.3). According to the proof of [22, Proposition 4.2], we have

$$\begin{aligned} 4K_X &= \widehat{\pi}^*(4K_Y + 2D_1 + 2D_2 + 2D_3 + 3D_{g,\mathrm{i}} + 3D_{g,-\mathrm{i}} + 3D_{g_1g,\mathrm{i}} + 3D_{g_1g,-\mathrm{i}}) \\ &= \widehat{\pi}^*(-K_Y + \Phi + \Phi_1) + \widehat{\pi}^*(M_1 + 2M_2 + 2N_2 + 2M_3 + 2N_3). \end{aligned}$$

It follows that $4K_S = \pi^*(-K_\Upsilon + \phi + \phi_1)$ and $K_S^2 = 7$, where $|\phi|$ and $|\phi_1|$ are base-point-free pencils on Υ induced by $|\Phi|$ and $|\Phi_1|$. The linear system $|-K_Y + \Phi + \Phi_1|$ is base point free, and the corresponding morphism contracts exactly the nodal curves M_1, M_2, N_2, M_3 and N_3. Hence $|-K_\Upsilon + \phi + \phi_1|$ is ample and so is K_S. For each $\psi \in H^*$, we can calculate $\mathcal{L}_\psi$ by [22, Theorem 2.1] and then easily show that $H^0(Y, \mathcal{O}_Y(K_Y + \mathcal{L}_\psi)) = 0$. It follows that $p_g(S) = p_g(X) = 0$ by [22, Proposition 4.1].

REMARK 5.4. We remark that Theorem 1.4 contributes to the study of the moduli space of the Inoue surfaces. Let $\mathcal{M}_{1,7}^{\mathrm{can}}$ be the Gieseker moduli space of canonical models of surfaces of general type with $\chi(\mathcal{O}) = 1$ and $K^2 = 7$ (cf. [12]). Let S be any Inoue surface. Denote by $[S]$ the corresponding point in $\mathcal{M}_{1,7}^{\mathrm{can}}$ and by $\mathrm{B}(S)$ be the base of the Kuranishi family of deformations of S. Recall the facts that the tangent space of $\mathrm{B}(S)$ is $H^1(S, \Theta_S)$, where Θ_S is the tangent sheaf of S, and that the germ $(\mathcal{M}_{1,7}^{\mathrm{can}}, [S])$ is analytically isomorphic to $\mathrm{B}(S)/\mathrm{Aut}(S)$. It has been shown in [3] that the group G acts trivially on $H^1(S, \Theta_S)$ and $\mathrm{B}(S)$ is smooth of dimension 4.

Now assume that S is a special Inoue surface constructed here. We can use the same method as in the proof of [3, Theorem 5.1] to conclude that the invariant subspace of $H^1(S, \Theta_S)$ for the H-action has dimension 2. Note that $\mathrm{Aut}(S) = H$ and $H/G \cong \mathbb{Z}_2$. Combining the result of [3], we see that $(\mathcal{M}_{1,7}^{\mathrm{can}}, [S])$ is analytically isomorphic to $(\mathbb{C}^2 \times \mathrm{Spec}\, \mathbb{C}[x, y, z]/(xz - y^2), 0)$.

Acknowledgments. The author thanks Xiaotao Sun for all the support during the preparation of the article. The author is greatly indebted to Jinxing Cai for many discussions and helpful suggestions. The author also would like to express his appreciation to the anonymous referee for his/her valuable suggestions.

REFERENCES

[1] M. F. Atiyah and I. M. Singer, *The index of elliptic operators, III*, Ann. of Math. (2) **87** (1968), 546–604.

[2] A. Calabri, C. Ciliberto and M. Mendes Lopes, *Numerical Godeaux surfaces with an involution*, Trans. Amer. Math. Soc. **359**(4) (2007), 1605–1632.

[3] I. Bauer and F. Catanese, "*Inoue type manifolds and Inoue surfaces: a connected component of the moduli space of surfaces with $K^2 = 7, p_g = 0$*", in *Geometry and Arithmetic*, EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2012, 23–56.

[4] I. Bauer, *Bloch's conjecture for Inoue surfaces with $p_g = 0, K^2 = 7$*, Proc. Amer. Math. Soc. **142**(10) (2014), 3335–3342.

[5] F. Catanese, M. Franciosi, K. Hulek and M. Reid, *Embeddings of curves and surfaces*, Nagoya Math. J. **154** (1999), 185–220.

[6] F. Catanese, "*Singular bidouble covers and the construction of interesting algebraic surfaces*", in *Algebraic Geometry: Hirzebruch 70 (Warsaw, 1998)*, Contemp. Math. **241**, American Mathematical Society, Provindence, RI, 1999, 97–120.

[7] H. Cartan, "*Quotient d'un espace analytique par un groupe d'automorphismes*", in *Algebraic Geometry and Topology: A Symposium in Honor of S. Lefschetz*, Princeton University Press, Princeton, NJ, 1957, 90–102.

[8] D. Cartwright and T. Steger, *Enumeration of the 50 fake projective planes*, C. R. Math. Acad. Sci. Paris **348**(1–2) (2010), 11–13.

[9] Y. Chen, *A new family of surfaces of general type with $K^2 = 7$ and $p_g = 0$*, Math. Z. **275**(3–4) (2013), 1275–1286.
[10] Y. Chen, *Commuting involutions on surfaces of general type with $p_g = 0$ and $K^2 = 7$*, Manuscripta Math. (2015), doi:10. 1007/s00229-014-0725-3.
[11] I. Dolgachev, M. Mendes Lopes and R. Pardini, *Rational surfaces with many nodes*, Compos. Math. **132**(3) (2002), 349–363.
[12] D. Gieseker, *Global moduli for surfaces of general type*, Invent. Math. **43** (1977), 233–282.
[13] D. Hacon, J. McKernan and C. Xu, *On the birational automorphisms of varieties of general type*, Ann. of Math. (2) **177**(3) (2013), 1077–1111.
[14] M. Inoue, *Some new surfaces of general type*, Tokyo J. Math. **17**(2) (1994), 295–319.
[15] J. Keum, *Quotients of fake projective planes*, Geom. Topol. **12** (2008), 2497–2515.
[16] Y. Lee and Y. Shin, *Involutions on a surface of general type with $p_g = q = 0$, $K^2 = 7$*, Osaka J. Math. **51**(1) (2014), 121–139.
[17] S. Maggiolo, *Automorphisms of numerical Godeaux surfaces with torsion of order* 3, 4 *or* 5, preprint, 2010, arXiv:1002.3494.
[18] H. Matsumura, *On algebraic groups of birational transformations*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **34** (1963), 151–155.
[19] M. Mendes Lopes and R. Pardini, *The bicanonical map of surfaces with $p_g = 0$ and $K^2 \geqslant 7$*, Bull. Lond. Math. Soc. **33**(3) (2001), 265–274.
[20] M. Mendes Lopes and R. Pardini, *The bicanonical map of surfaces with $p_g = 0$ and $K^2 \geqslant 7$, II*, Bull. Lond. Math. Soc. **35**(3) (2003), 337–343.
[21] Y. Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. **268** (1984), 159–171.
[22] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991), 191–213.
[23] G. Prasad and S.-K. Yeung, *Fake projective planes*, Invent. Math. **168**(2) (2007), 321–370; Addendum, Invent. Math. **182**(1) (2010), 213–227.
[24] C. Rito, *Some bidouble planes with $p_g = q = 0$ and $4 \leqslant K^2 \leqslant 7$*, Int. J. Math. **26**(5) 1550035 (2015).
[25] G. Xiao, *Bound of automorphisms of surfaces of general type. I*, Ann. of Math. (2) **139**(1) (1994), 51–77.
[26] G. Xiao, *Bound of automorphisms of surfaces of general type. II*, J. Algebraic Geom. **4**(4) (1995), 701–793.

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