A Bott–Borel–Weil Theorem for Diagonal Ind-groups

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Abstract. A diagonal ind-group is a direct limit of classical affine algebraic groups of growing rank under a class of inclusions that contains the inclusion

$$SL(n) \to SL(2n), \quad M \mapsto \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$$

as a typical special case. If G is a diagonal ind-group and $B \subset G$ is a Borel ind-subgroup, we consider the ind-variety G/B and compute the cohomology $H^\ell(G/B, \mathcal{O}_{-\lambda})$ of any G-equivariant line bundle $\mathcal{O}_{-\lambda}$ on G/B. It has been known that, for a generic λ, all cohomology groups of $\mathcal{O}_{-\lambda}$ vanish, and that a non-generic equivariant line bundle $\mathcal{O}_{-\lambda}$ has at most one nonzero cohomology group. The new result of this paper is a precise description of when $H^j(G/B, \mathcal{O}_{-\lambda})$ is nonzero and the proof of the fact that, whenever nonzero, $H^j(G/B, \mathcal{O}_{-\lambda})$ is a G-module dual to a highest weight module. The main difficulty is in defining an appropriate analog W_B of the Weyl group, so that the action of W_B on weights of G is compatible with the analog of the Demazure "action" of the Weyl group on the cohomology of line bundles. The highest weight corresponding to $H^j(G/B, \mathcal{O}_{-\lambda})$ is then computed by a procedure similar to that in the classical Bott–Borel–Weil theorem.

Introduction

The classical Bott–Borel–Weil theorem is a cornerstone of geometric representation theory. In the late 1990's Joseph A. Wolf and his collaborators became interested in extending the theorem to direct limit Lie groups, and since then have made essential progress; see [NRW, W]. In the context of direct limit algebraic groups, *i.e.,* ind-groups, the problem has been addressed in our joint paper [DPW]. In that paper a general theorem has been proved (concerning infinite-rank equivariant bundles on locally proper homogeneous ind-varieties) under the condition that the ind-group considered is root reductive (see the definition in Section 1). The known results become much sketchier when this condition is dropped. The purpose of this paper is to consider in detail the most interesting class of ind-groups beyond the root reductive ones: diagonal ind-groups.

Recall that a locally affine ind-group G is the direct limit of embeddings of connected affine algebraic groups

$$G_1 \to G_2 \to \cdots .$$

The Bott–Borel–Weil paradigm for ind-groups is concerned with the computation of the cohomology of a G-equivariant line bundle $\mathcal{O}_{-\mu}$ on the ind-variety G/B, where $B = \varinjlim B_n$ is the direct limit of Borel subgroups $B_n \subset G_n$ with $B_{n-1} = B_n \cap G_{n-1}$. In the classical case G is a connected affine algebraic group, and the result (due to Borel–Weil [S] and Bott [B], see also [D1,D2]) is that the simple (finite-dimensional) G-module $V_B(\lambda)^*$ with B-highest weight λ occurs as the unique nonzero cohomology group of each of the sheaves $\mathcal{O}_{-w\cdot\lambda}$, where w runs over the Weyl group W and $\cdot$ stands for the "dot action" of w on λ. More precisely, $V_B(\lambda)^*$ occurs as the cohomology group of $\mathcal{O}_{-w\cdot\lambda}$ in degree $\ell(w)$, where $\ell(w)$ is the length of w with respect to the simple roots of B.

In contrast with this result, in the infinite-dimensional case it is not difficult to see that a generic equivariant line bundle $\mathcal{O}_{-\mu}$ is acyclic, *i.e.*, all its cohomology groups vanish. Wolf has introduced the condition of cohomological finiteness of a weight μ (see [W] and compare with [DPW]), which is equivalent to the condition that $\mathcal{O}_{-\mu}$ has a unique non-vanishing cohomology group. If μ is dominant, then this cohomology group is $H^0(G/B, \mathcal{O}_{-\mu})$, and in this case it is easy to show that $H^0(G/B, \mathcal{O}_{-\mu})$ is the (algebraic) dual of the simple B-highest weight G-module $V_B(\mu)$.

It is not known in general whether all higher cohomology groups $H^j(G/B, \mathcal{O}_{-\mu})$ are also dual to B-highest weight modules. This problem has been open since the late 1990's, and the main result of this paper is that for any locally simple diagonal ind-group G (see the definition in Section 1), all nonzero cohomology groups $H^j(G/B, \mathcal{O}_{-\lambda})$ are indeed dual to simple B-highest weight modules. The proof is a mixture of combinatorics and geometry. The most important new idea is to consider the intermediate algebraic groups $\widetilde{G}_n \cong G_n \times G_n \times \cdots \times G_n$,

$$G_n \to \widetilde{G}_n \to G_{n+1},$$

introduced in Section 1. They arise naturally from the diagonal embeddings $G_n \to G_{n+1}$. The corresponding homogeneous spaces $\widetilde{G}_n/\widetilde{B}_n$, where $\widetilde{B}_n = B_{n+1} \cap \widetilde{G}_n$, play a key role in the proof. More precisely, the realization of $\mathcal{O}_{-\lambda}$ as a line bundle on both G/B and on $\varinjlim \widetilde{G}_n/\widetilde{B}_n$ enables us to reduce the problem of studying the cohomologies $H^j(G/B, \mathcal{O}_{-\lambda})$ to two finite-dimensional problems: one concerns the embeddings $G_n \to \widetilde{G}_n$ and the other one concerns the embeddings $\widetilde{G}_n \to G_{n+1}$. For the second problem we use a recent result of Valdemar Tsanov, which allows us to obtain a strong condition on the weight λ so that $H^j(G/B, \mathcal{O}_{-\lambda}) \neq 0$; under this condition we then apply a result of Mike Roth and the first author to the embedding $G_n \to \widetilde{G}_n$. The final result, Theorem 4.2, is absolutely similar to the classical Bott–Borel–Weil Theorem with the only exception that the "Weyl group" W_B, relevant for G/B, depends on the choice of Borel ind-subgroup B.

1   Diagonal Ind-groups: Definitions and Notation

We work over an algebraically closed field $\mathbb{K}$ of characteristic 0. If V is a vector space, we set

$$V^{\oplus k} = \underbrace{V \oplus \cdots \oplus V}_{k \text{ times}}.$$

Throughout this paper, *classical group* will be an abbreviation for a connected (affine) algebraic group G whose Lie algebra is a simple classical Lie algebra. An embedding $G \to G'$ of classical groups is *diagonal* if the induced injection of Lie algebras $\mathfrak{g} \to \mathfrak{g}'$ has the following property: the natural representation of $\mathfrak{g}'$ considered as a $\mathfrak{g}$-module is isomorphic to a direct sum of copies of the natural representation of $\mathfrak{g}$, of its dual, and of the trivial representation. (If $\mathfrak{g} = so, sp$, the natural representation is self-dual, hence in this case the natural representation of $\mathfrak{g}'$ must simply be a direct sum of copies of the natural and trivial representations.) If $\mathfrak{g}$ and $\mathfrak{g}'$ are reductive Lie algebras, an injective Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g}'$ is a *root injection*, if for any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ there exists a Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{g}'$ containing the image of $\mathfrak{h}$ and such that any $\mathfrak{h}$-root space of $\mathfrak{g}$ is mapped to precisely one $\mathfrak{h}'$-root space of $\mathfrak{g}'$. An embedding $G \to G'$ of reductive, affine, algebraic groups is a *root embedding* if the corresponding injection $\mathfrak{g} \to \mathfrak{g}'$ is a root injection.

By definition a *diagonal ind-group G* is the direct limit of a sequence of diagonal embeddings of classical groups

$$(1.1) \qquad G_1 \to \cdots \to G_n \to G_{n+1} \to \cdots .$$

The group G is called *pure* if, for large enough n, the natural representation of $\mathfrak{g}_{n+1}$ contains no trivial $\mathfrak{g}_n$-constituents. The Lie algebras of diagonal ind-groups have been classified by Baranov and Zhilinskii [BZh].

Example 1.1 A diagonal embedding of classical groups $G \to G'$ of type A can be realized in matrix form as

$$
M \longmapsto
\begin{pmatrix}
M & & & & & & & \\
 & \ddots & & & & & & \\
 & & M & & & & & \\
 & & & (M^{\mathsf{T}})^{-1} & & & & \\
 & & & & \ddots & & & \\
 & & & & & (M^{\mathsf{T}})^{-1} & & \\
 & & & & & & 1 & \\
 & & & & & & & \ddots \\
 & & & & & & & & 1
\end{pmatrix}
$$

with k copies of M, l copies of $(M^{\mathsf{T}})^{-1}$, and z copies of the one-by-one matrix with entry one. Therefore, any diagonal ind-group of type A is obtained by iterating such embeddings with varying parameters k, l, and z. In particular, the ind-group $SL(\infty)$ can be defined as a diagonal ind-group of type A with $k = z = 1$, $l = 0$ at each step. To define the diagonal ind-group $SL(2^\infty)$ we set $G_1 := SL(2)$ and then put $k = 2$, $l = z = 0$ at each step. It is easy to check that, up to isomorphism, $SL(\infty)$ does not depend on the choice of n_1, where $G_1 = SL(n_1)$. The ind-group $SL(2^\infty)$ is pure, while $SL(\infty)$ is not.

In this paper we consider two kinds of G-modules, defined respectively as direct or inverse limits of finite-dimensional G_n-modules. Fix $G = \varinjlim G_n$, and let

$$(1.2) \qquad\qquad V_1 \to \cdots \to V_n \to V_{n+1} \to \cdots$$

(respectively,

$$(1.3) \qquad\qquad \cdots \to Y_{n+1} \to Y_n \to \cdots \to Y_1\ )$$

be a direct (respectively, inverse) system of finite-dimensional G_n-modules. A *G-module* will mean the direct limit of a system (1.2) endowed with G_n-module structures for all n, up to an isomorphism, and a *dual G-module* will mean the projective limit of a system (1.3) endowed with G_n-module structures. It is clear that if $V = \varinjlim V_n$ is a G-module, then $V^* = \varprojlim V_n^*$ is a dual G-module. Conversely, if $Y = \varprojlim Y_n$ is a dual G-module, then $\varinjlim Y_n^*$ is a G-module.

For the rest of the paper we fix an exhaustion $G = \varinjlim G_n$ of G by simply-connected classical groups of the same type $A, B, C,$ or D. In particular, every direct system (1.1) we consider has a well-defined type. In general G may have exhaustions of different type, however we will use the term *type of G* to refer to the type of the fixed exhaustion. The corresponding exhaustion of $\mathfrak{g}$ is then $\mathfrak{g} = \varinjlim \mathfrak{g}_n$. We denote the rank of $\mathfrak{g}_n$ by r_n.

For the purposes of this paper, we define a *Cartan subgroup H* of G as a direct limit of Cartan subgroups $H_n \subset G_n$. The corresponding Lie algebra $\mathfrak{h}$ is then the direct limit of Cartan subalgebras $\mathfrak{h}_n \subset \mathfrak{g}_n$ such that $\mathfrak{h}_n = \mathfrak{h}_{n+1} \cap \mathfrak{g}_n$. We fix once and for all a Cartan subgroup $H = \varinjlim H_n$ of G with corresponding Cartan subalgebra $\mathfrak{h} = \varinjlim \mathfrak{h}_n$ of $\mathfrak{g}$. The weights of $\mathfrak{g}_n$ are expressed in terms of the standard functions $\varepsilon_n^1, \dots, \varepsilon_n^{r_n+1} \subset \mathfrak{h}_n^*$ if G is of type A or $\varepsilon_n^1, \dots, \varepsilon_n^{r_n} \subset \mathfrak{h}_n^*$ otherwise. These functions are determined by the choice of the Cartan subalgebra $\mathfrak{h}_n \subset \mathfrak{g}_n$. The weights of the natural representation of $\mathfrak{g}_n$ are as follows: for G of type A they are $\varepsilon_n^1, \dots, \varepsilon_n^{r_n+1}$; for G of type B they are $\pm\varepsilon_n^1, \dots, \pm\varepsilon_n^{r_n}, 0$; and for G of type C or D they are $\pm\varepsilon_n^1, \dots, \pm\varepsilon_n^{r_n}$. Since $\mathfrak{h}_n \subset \mathfrak{h}_{n+1}$, the $\mathfrak{h}_{n+1}$-weight spaces of the natural representation of $\mathfrak{g}_{n+1}$ restrict to $\mathfrak{h}_n$-weight spaces. In particular, ε_{n+1}^i restricts to $\pm\varepsilon_n^j$ for some j, or to 0.

Denote the injection $\mathfrak{g}_n \to \mathfrak{g}_{n+1}$ by δ_n. We will now define a subalgebra $\widetilde{\mathfrak{g}}_n \cong \mathfrak{g}_n^{\oplus s_n}$ of $\mathfrak{g}_{n+1}$, where s_n is the total multiplicity of all nontrivial simple constituents of the natural representation of $\mathfrak{g}_{n+1}$ considered as a $\mathfrak{g}_n$-module. Note first that the $\mathfrak{h}_{n+1}$-weight decomposition of the natural representation of $\mathfrak{g}_{n+1}$ determines a unique decomposition of each nontrivial isotypic $\mathfrak{g}_n$-component as a direct sum of simple constituents. To define the subalgebra $\widetilde{\mathfrak{g}}_n$ it suffices to define its simple ideals:

(i)   if G is of type A, each simple ideal of $\widetilde{\mathfrak{g}}_n$ equals the traceless endomorphisms of a simple nontrivial constituent of the natural representation of $\mathfrak{g}_{n+1}$;

(ii) if G is of type $B, C,$ or D, each simple ideal of $\widetilde{\mathfrak{g}}_n$ is the Lie algebra of orthogonal or respectively symplectic endomorphisms of a simple nontrivial constituent of the natural representation of $\mathfrak{g}_{n+1}$.

In all cases, there is an obvious injective homomorphism $\varphi_n\colon \mathfrak{g}_n \to \widetilde{\mathfrak{g}}_n$ such that the

diagram

(1.4)

$$\mathfrak{g}_n \xrightarrow{\delta_n} \mathfrak{g}_{n+1}$$

with φ_n and κ_n to $\widetilde{\mathfrak{g}}_n$

is commutative, κ_n being the inclusion. Moreover, if G is of type A, C, or D, the map κ_n is a root injection.

If G is of type B and $s_n > 1$, κ_n is no longer a root injection; however, we can still factor κ_n as

$$\widetilde{\mathfrak{g}}_n \xrightarrow{\psi_n} \bar{\mathfrak{g}}_n \xrightarrow{\theta_n} \mathfrak{g}_{n+1}$$

so that θ_n is a root injection and ψ_n is "close" to a root injection. To construct this factorization, recall that $\mathfrak{g}_n \cong B_{r_n}$, $\widetilde{\mathfrak{g}}_n \cong B_{r_n}^{\oplus s_n}$, $\mathfrak{g}_{n+1} \cong B_{r_{n+1}}$, and assume that the natural representation V_{n+1} of $\mathfrak{g}_{n+1}$ when considered as a $\mathfrak{g}_n$-module contains s_n copies of the natural representation of $\mathfrak{g}_n$ and z_n copies of the trivial representation. Note that $2r_{n+1} + 1 = s_n(2r_n + 1) + z_n$, hence s_n and z_n are distinct modulo 2. The smallest interesting case is when $s_n = 2$ and $z_n = 1$. In this case the $\mathfrak{g}_n$-module decomposition of the natural representation of $\mathfrak{g}_{n+1}$ is $V_{n+1} = V^1 \oplus V^2 \oplus \mathbb{K}$, where V^1 and V^2 are the two copies of the natural representation of $\mathfrak{g}_n$. We set

$$\bar{\mathfrak{g}}_n := so(V^1 \oplus V^2) \cong D_{2r_n+1}$$

and consider the natural injections $\widetilde{\mathfrak{g}}_n \xrightarrow{\psi_n} \bar{\mathfrak{g}}_n \xrightarrow{\theta_n} \mathfrak{g}_{n+1}$. The assumption that $\mathfrak{h}_n$ is contained in $\mathfrak{h}_{n+1}$ ensures that $\mathfrak{h}_{n+1}$ is contained in $\bar{\mathfrak{g}}_n$ and, consecutively, θ_n is a root injection. Furthermore, for each long root of $\widetilde{\mathfrak{g}}_n$, the corresponding root space is mapped via ψ_n into a root space of $\bar{\mathfrak{g}}_n$. In the case when s_n and z_n are arbitrary, *i.e.,* when $V' = V^1 \oplus \cdots \oplus V^{s_n} \oplus \mathbb{K}^{z_n}$, we combine $V^1, \ldots, V^{s_n}$ into pairs when s_n is even, and into pairs and a single element when s_n is odd, and set

$$\bar{\mathfrak{g}}_n := so(V^1 \oplus V^2) \oplus \cdots \oplus so(V^{s_n-1} \oplus V^{s_n})$$

in the former case and

$$\bar{\mathfrak{g}}_n := so(V^1 \oplus V^2) \oplus \cdots \oplus so(V^{s_n-2} \oplus V^{s_n-1}) \oplus so(V^{s_n})$$

in the latter. The injections $\widetilde{\mathfrak{g}}_n \xrightarrow{\psi_n} \bar{\mathfrak{g}}_n \xrightarrow{\theta_n} \mathfrak{g}_{n+1}$ are defined in the obvious way; θ_n is a root injection, and ψ_n maps roots spaces corresponding to long roots of $\widetilde{\mathfrak{g}}_n$ to root spaces of $\bar{\mathfrak{g}}_n$. As a result of this construction, we obtain a refinement of diagram (1.4) as follows:

(1.5)

$$\mathfrak{g}_n \xrightarrow{\delta_n} \mathfrak{g}_{n+1} \quad .$$

with φ_n, κ_n, θ_n, ψ_n and $\widetilde{\mathfrak{g}}_n$, $\bar{\mathfrak{g}}_n$

Remark 1.2 Note that for $s_n \geq 3$ the subalgebra $\bar{\mathfrak{g}}_n$ depends on the way we combine $V^1, \ldots, V^{s_n}$ into pairs. In the proof of Corollary 4.4 we make use of different choices for $\bar{\mathfrak{g}}_n$ in the case when $s_n \geq 3$ is odd.

2 Borel Subalgebras, Dominant Weights, and Highest Weight Modules

For the purposes of this paper we adopt the following definition of a Borel subgroup: *B is a Borel subgroup of G* if $B = \varinjlim B_n$, where B_n is a Borel subgroup of G_n for every n. All Borel subgroups we consider contain the fixed Cartan subgroup H. The corresponding Lie algebra $\mathfrak{b}$ then contains $\mathfrak{h}$ and is the direct limit of Borel subalgebras $\mathfrak{b}_n \subset \mathfrak{g}_n$ containing the fixed Cartan subalgebras $\mathfrak{h}_n$. Note that we have $\mathfrak{b}_n = \mathfrak{b}_{n+1} \cap \mathfrak{g}_n$.

The Borel subalgebras of $\mathfrak{g}_n$ that contain $\mathfrak{h}_n$ correspond to linear orders on the weights of the natural representation of $\mathfrak{g}_n$. More precisely (see also [DP]), the Borel subalgebras of $\mathfrak{g}_n$ correspond to

- linear orders on the set $\{\varepsilon_n^1, \ldots, \varepsilon_n^{r_n+1}\}$ for type A;
- linear orders compatible with multiplication by -1 on the set $\{\pm\varepsilon_n^1, \ldots, \pm\varepsilon_n^{r_n}, 0\}$ for type B;
- linear orders compatible with multiplication by -1 on the set $\{\pm\varepsilon_n^1, \ldots, \pm\varepsilon_n^{r_n}\}$ for type C;
- linear orders compatible with multiplication by -1 on the set $\{\pm\varepsilon_n^1, \ldots, \pm\varepsilon_n^{r_n}\}$ for type D.

Here "compatible with multiplication by -1" means that $\varepsilon_n^i < \pm\varepsilon_n^j$ is equivalent to $\mp\varepsilon_n^j < -\varepsilon_n^i$. The above correspondence is a bijection in types A, B, and C; in type D each Borel subalgebra corresponds to exactly two orders as above, since the smallest element $\pm\varepsilon_n^i$ such that $\pm\varepsilon_n^i > \mp\varepsilon_n^i$ can be interchanged with its opposite without changing the Borel subalgebra.

The condition $\mathfrak{b}_n = \mathfrak{b}_{n+1} \cap \mathfrak{g}_n$ is equivalent to the fact that the order on the weights of the natural representation of $\mathfrak{g}_{n+1}$ restricts to the order (or one of the two orders in type D) on the weights of the natural representation of $\mathfrak{g}_n$. In this way we can say that a Borel subalgebra $\mathfrak{b} = \varinjlim \mathfrak{b}_n$ is determined by a projective system of linear orders on the weights of the natural representations of $\mathfrak{g}_n$. Note that in type A the weights of $\mathfrak{g}_{n+1}$ corresponding to constituents isomorphic to the dual of the natural representation of $\mathfrak{g}_n$ restrict to $-\varepsilon_n^i$ with $1 \leq i \leq r_n + 1$.

Example 2.1 (i) Let $G = SL(2^\infty)$. Then $\mathfrak{g}_n = sl(2^n)$ with weights $\{\varepsilon_n^1, \ldots, \varepsilon_n^{2^n}\}$ of the natural representation. The weights ε_{n+1}^i and $\varepsilon_{n+1}^{2^n+i}$ restrict to ε_n^i for $1 \leq i \leq 2^n$. The projective system of orders

$$\varepsilon_n^1 > \varepsilon_n^2 > \cdots > \varepsilon_n^{2^n}$$

defines the Borel subgroup of G consisting of upper triangular matrices in the realization of G from Example 1.1. We will call this Borel subgroup *the upper triangular Borel subgroup* of $SL(2^\infty)$.

(ii) A more interesting example of a Borel subgroup of $SL(2^n)$ is provided by the projective systems of orders

$$\varepsilon_n^1 > \varepsilon_n^{2^{n-1}+1} > \varepsilon_n^2 > \varepsilon_n^{2^{n-1}+2} > \cdots > \varepsilon_n^{2^{n-1}} > \varepsilon_n^{2^n}.$$

We will call this Borel subgroup *the interlacing Borel subgroup* of $SL(2^\infty)$.

(iii) Let $G_n = Sp(2(2^n - 1))$ and let the embedding $G_n \to G_{n+1}$ be determined by the condition that the natural representation of $\mathfrak{g}_{n+1}$ contains two copies of the natural representation of $\mathfrak{g}_n$ and two copies of the trivial representation. The resulting ind-group is not pure; we denote it by $Sp(2^\infty - 1)$. The weights of the natural representation of $\mathfrak{g}_n$ are $\pm \varepsilon_n^1, \ldots, \pm \varepsilon_n^{2^n - 1}$. We assume that ε_{n+1}^{1+i} and $\varepsilon_{n+1}^{1+2^n+i}$ restrict to ε_n^{1+i} for $1 \le i \le 2^n - 2$, while ε_{n+1}^1 restricts to 0. The projective system of orders

$$\varepsilon_n^1 > \varepsilon_n^2 > \varepsilon_n^{2+2^{n-1}} > \cdots > \varepsilon_n^{2^{n-1}-1} > \varepsilon_n^{2^n-1}$$

$$> -\varepsilon_n^{2^n-1} > -\varepsilon_n^{2^{n-1}-1} > \cdots > -\varepsilon_n^{2+2^{n-1}} > -\varepsilon_n^2 > -\varepsilon_n^1$$

defines an *interlacing Borel subgroup B* of $Sp(2^\infty - 1)$.

A *weight λ of G* is by definition an inverse system of weights of G_n, *i.e.,* a sequence $\{\lambda_n\}$ of integral weights of $\mathfrak{g}_n$ such that λ_{n+1} restricts to λ_n for every n. We use the notation $\lambda = \varprojlim \lambda_n$ to indicate that the sequence $\{\lambda_n\}$ defines the weight λ. $\mathcal{P}$ stands for the set of weights of G. As in the finite-dimensional case, for every Borel subgroup $B \subset G$, $\mathcal{P}$ is in a natural bijection with the one-dimensional B-modules. A weight $\lambda \in \mathcal{P}$ is *B-dominant* (or simply *dominant* if B is clear from the context) if λ_n is a B_n-dominant weight for every n; the set of B-dominant weights will be denoted by $\mathcal{P}_B^+$ (respectively, by $\mathcal{P}^+$). The fundamental $\mathfrak{b}_n$-weights of $\mathfrak{g}_n$ (in the standard order on the nodes of the Dynkin diagram of $\mathfrak{g}_n$) will be denoted by $\omega_n^1, \ldots, \omega_n^{r_n}$.

Example 2.2 We discuss $\mathcal{P}_B^+$ for each of the Borel subgroups from Example 2.1.
(i) Consider $\lambda = \varprojlim \lambda_n \in \mathcal{P}$. Let

$$(2.1) \qquad \lambda_n = \lambda_n^1 \varepsilon_n^1 + \cdots + \lambda_n^{2^n} \varepsilon_n^{2^n} = a_n^1 \omega_n^1 + \cdots + a_n^{2^n-1} \omega_n^{2^n-1}.$$

Since $a_n^i = \lambda_n^i - \lambda_n^{i+1}$ for $1 \le i \le 2^n - 1$, the fact that λ_{n+1} restricts to λ_n is equivalent to the equations

$$(2.2) \qquad a_n^1 = a_{n+1}^1 + a_{n+1}^{2^n+1}, a_n^2 = a_{n+1}^2 + a_{n+1}^{2^n+2}, \ldots, a_n^{2^n-1} = a_{n+1}^{2^n-1} + a_{n+1}^{2^{n+1}-1},$$

and $\lambda \in \mathcal{P}^+$ is equivalent to the condition that $a_n^i \in \mathbb{Z}_{\ge 0}$ for every n and every $1 \le i \le 2^n - 1$. As (2.2) shows, every B_n-dominant weight λ_n of G_n is the restriction of infinitely many B_{n+1}-dominant weights of G_{n+1}. More precisely, there are finitely many choices for the parameters $a_{n+1}^1, \ldots, a_{n+1}^{2^n-1}, a_{n+1}^{2^n+1}, \ldots, a_{n+1}^{2^{n+1}-1}$, and the parameter $a_{n+1}^{2^n}$ can be chosen as any element of $\mathbb{Z}_{\ge 0}$. In particular, $\mathcal{P}^+$ is not finitely generated and contains the lattice points of an open n-dimensional cone for every n.

(ii) As in (i) above, each $\lambda \in \mathcal{P}$ can be written as in (2.1). The restriction of λ_{n+1} to λ_n is equivalent to

$$
\begin{aligned}
a_n^1 &= a_{n+1}^1 &+& \quad 2a_{n+1}^2 &+& \quad a_{n+1}^3, \\
a_n^2 &= a_{n+1}^3 &+& \quad 2a_{n+1}^4 &+& \quad a_{n+1}^5, \\
&\vdots & &\quad\vdots & &\quad\vdots \\
a_n^{2^n-1} &= a_{n+1}^{2^{n+1}-3} &+& \quad 2a_{n+1}^{2^{n+1}-2} &+& \quad a_{n+1}^{2^{n+1}-1}.
\end{aligned}
\tag{2.3}
$$

Set $b_n := a_n^1 + \cdots + a_n^{2^n-1}$. Adding the equations in (2.3) we obtain

$$
b_n = b_{n+1} + \left(a_{n+1}^2 + \cdots + a_{n+1}^{2^{n+1}-2} \right).
\tag{2.4}
$$

Thus $b_1 \geq b_2 \geq b_3 \geq \cdots$ and, consequently, $b_{n_0} = b_{n_0+1} = b_{n_0+2} = \cdots$ for some n_0. Again (2.4) shows that $a_n^2 = \cdots = a_n^{2^n-2} = 0$ and $a_n^1 = a_{n+1}^1$, $a_n^{2^n-1} = a_{n+1}^{2^{n+1}-1}$ for every $n > n_0$. Putting these facts together, we see that if $\lambda \in \mathcal{P}^+$, then $\lambda = \varprojlim(a'\omega_n^1 + a''\omega_n^{2^n-1})$. In particular, $\mathcal{P}^+$ consists of the lattice points in a cone of dimension two.

(iii) In this case we are going to show that $\mathcal{P}^+ = 0$. Again, consider $\lambda = \varprojlim \lambda_n \in \mathcal{P}$. Let

$$
\lambda_n = \lambda_n^1 \varepsilon_n^1 + \cdots + \lambda_n^{2^n-1} \varepsilon_n^{2^n-1} = a_n^1 \omega_n^1 + \cdots + a_n^{2^n-1} \omega_n^{2^n-1}.
$$

Since $a_n^i = \lambda_n^i - \lambda_n^{i+1}$ for $1 \leq i \leq 2^n - 2$ and $a_n^{2^n-1} = \lambda_n^{2^n-1}$, the fact that λ_{n+1} restricts to λ_n is equivalent to the equations

$$
\begin{aligned}
a_n^1 &= a_{n+1}^2 &+& \quad 2a_{n+1}^3 &+& \quad a_{n+1}^4, \\
a_n^2 &= a_{n+1}^4 &+& \quad 2a_{n+1}^5 &+& \quad a_{n+1}^6, \\
&\vdots & &\quad\vdots & &\quad\vdots \\
a_n^{2^n-2} &= a_{n+1}^{2^{n+1}-4} &+& \quad 2a_{n+1}^{2^{n+1}-3} &+& \quad a_{n+1}^{2^{n+1}-2}, \\
a_n^{2^n-1} &= a_{n+1}^{2^{n+1}-2} &+& \quad 2a_{n+1}^{2^{n+1}-1}.
\end{aligned}
\tag{2.5}
$$

Assume that $\mathcal{P}^+ \neq 0$ and let $\lambda \in \mathcal{P}^+$ be a nonzero weight. Choose n_0 so that $\lambda_{n_0-1} \neq 0$ and set

$$
b_k := a_{n_0+k}^{2^k} + a_{n_0+k}^{2^k+1} + \cdots + a_{n_0+k}^{2^{n+k}-1} \text{ for } k \geq 0.
$$

Adding the appropriate equations from (2.5) we obtain

$$
b_k = b_{k+1} + \left(a_{n_0+k+1}^{2^{k+1}+1} + \cdots + a_{n_0+k+1}^{2^{n_0+k+1}-1} \right),
\tag{2.6}
$$

which implies $b_0 \geq b_1 \geq \cdots$. Hence there exists k_0 such that $b_{k_0} = b_{k_0+1} = \cdots$. We may assume that k_0 is the smallest such integer.

If $k_0 = 0$, then (2.6) shows that

$$
a_{n_0+1}^3 = \cdots = a_{n_0+1}^{2^{n_0+1}-1} = 0,
$$

which substituted in (2.5) implies

$$a_{n_0}^2 = \cdots = a_{n_0}^{2^{n_0}-1} = 0.$$

After another look at (2.5), we obtain $\lambda_{n_0-1} = 0$, which contradicts the assumption that $\lambda_{n_0-1} \neq 0$.

If $k_0 > 1$, then (2.6) shows that

$$a_{n_0+k_0+1}^{2^{k_0+1}+1} = \cdots = a_{n_0+k_0+1}^{2^{n_0+k_0+1}-1} = 0,$$

which substituted back into (2.5) gives

$$a_{n_0+k_0}^{2^{k_0}+1} = \cdots = a_{n_0+k_0}^{2^{n_0+k_0}-1} = 0.$$

The last equation together with (2.6) implies $b_{k_0-1} = b_{k_0}$, which contradicts the choice of k_0. This proves that $\mathcal{P}^+ = 0$.

Despite Example 2.2(iii), we can prove that $\mathcal{P}^+ \neq 0$ under some natural assumptions. On the other hand, there are no strictly dominant weights unless G is root reductive.

Proposition 2.3

(i)   *If G is pure, then $\mathcal{P}_B^+ \neq 0$ for any Borel subgroup $B \subset G$.*

(ii)   *For any G there exists a Borel subgroup B such that $\mathcal{P}_B^+ \neq 0$.*

(iii)   *Assume that $\mathcal{P}_B^+$ contains a strictly dominant weight λ, i.e., such that $(\lambda_n, \alpha) > 0$ for every root α of $\mathfrak{b}_n$. Then G is root reductive.*

Proof (i) If G is pure of type B, C, or D, then ω_{n+1}^1 restricts to ω_n^1 and thus $\mathcal{P}_B^+$ contains $\varprojlim(a\omega_n^1)$ for every $a \in \mathbb{Z}_{\geq 0}$. If G is pure of type A, then ω_{n+1}^1 restricts to ω_n^1 or $\omega_n^{r_n}$, and $\omega_{n+1}^{r_n}$ restricts to ω_n^1 or $\omega_n^{r_n}$, which implies that every B_n-dominant weight of the form $a'\omega_n^1 + a''\omega_n^{r_n}$ extends to a B_{n+1}-dominant weight. This shows that $\mathcal{P}_B^+ \neq 0$.
(ii) Using induction we can construct a projective system of orders on the weights of the natural representation of G_n in such a way that the maximal element among the weights of the natural representation of G_{n+1} restricts to a weight of the natural representation of G_n and not to zero. This projective system defines a Borel subgroup B, and, for every $a \in \mathbb{Z}_{>0}$, $\varprojlim(a\omega_n^1)$ is a nonzero element of $\mathcal{P}_B^+$.
(iii) Let α be a long root of $\mathfrak{b}_m$, and let $\alpha^1, \ldots, \alpha^{s_{m,n}}$ be the roots of $\mathfrak{b}_n$, $n > m$, which restrict to α. Assuming that λ_n is a strictly dominant $\mathfrak{g}_n$-weight we conclude that $(\lambda_n, \alpha^i) \geq 1/2$ for $1 \leq i \leq s_{m,n}$. This gives

$$(\lambda_m, \alpha) = (\lambda_n, \alpha^1) + \cdots + (\lambda_n, \alpha^{s_{m,n}}) \geq \tfrac{1}{2}s_{m,n},$$

which is only possible if there is $n_0 > m$ so that $s_{m,n} = s_{m,n_0}$ for $n \geq n_0$. The latter condition implies that G is root reductive. ∎

Every $\lambda \in \mathcal{P}_B^+$ defines an irreducible G-module $V_B(\lambda)$ in the following way. The weight λ determines the direct system of finite dimensional modules $V_{B_n}(\lambda_n) \xrightarrow{e_n} V_{B_{n+1}}(\lambda_{n+1})$, where e_n maps the B_n-highest weight space of $V_{B_n}(\lambda_n)$ into the B_{n+1}-highest weight space of $V_{B_{n+1}}(\lambda_{n+1})$. Then $V_B(\lambda)$ is defined as $\varinjlim V_{B_n}(\lambda_n)$.

Example 2.4 Let $G = SL(2^\infty)$ and let B be any Borel subalgebra of G. Set $\lambda_n :=$ $\omega_n^1 + \omega_n^{2^n-1}$. The sequence $\{\lambda_n\}$ is a B-dominant weight of G and hence the G-module $V_B(\lambda)$ is well defined. Furthermore, in this case it is easy to check that $V_B(\lambda)$ is a weight module, *i.e.,* $V_B(\lambda) = \oplus_\mu V_B(\lambda)^\mu$, where

$$V_B(\lambda)^\mu = \{v \in V_B(\lambda) \mid h \cdot v = \mu(h)v \text{ for every } h \in \mathfrak{h}\}.$$

This observation implies that, despite the fact that each of the modules $V_{B_n}(\lambda_n)$ is isomorphic to the adjoint representation of G_n, $V_B(\lambda)$ is not isomorphic to the adjoint representation of G as the latter is not a weight module.

3 The Weyl Group W_B

In this section we use the filtration (1.4) to construct a group W_B that plays the role that the Weyl group plays in the classical Bott–Borel–Weil theorem.

First we consider the case when G is not of type B. Let W_n denote the Weyl group of $\mathfrak{g}_n$ and let $pr_n^i \colon \widetilde{\mathfrak{g}}_n = \mathfrak{g}_n^{\oplus s_n} \to \mathfrak{g}_n$ be the projection onto the i-th direct summand for $1 \leq i \leq s_n$. Then for each pair n, i, the composition

$$(3.1) \qquad\qquad \mathfrak{g}_n \longrightarrow \widetilde{\mathfrak{g}}_n \overset{pr_n^i}{\longrightarrow} \mathfrak{g}_n^i \longrightarrow \mathfrak{g}_{n+1}$$

is a root injection and hence yields an injective homomorphism of Weyl groups $\tau_n^i \colon W_n \to W_{n+1}$. For every sequence $\{t_n\}_{n=1}^\infty$ with $1 \leq t_n \leq s_n$, the injections $\tau_n^{t_n}$ form a direct system. Note that, if $\{t_n'\}$ and $\{t_n''\}$ are two sequences that differ in finitely many positions only, then $\varinjlim_{\tau_n^{t_n'}} W_n = \varinjlim_{\tau_n^{t_n''}} W_n$. We define an equivalence relation between sequences by setting $\{t_n'\} \sim \{t_n''\}$ if $t_n' = t_n''$ for large enough n, and denote the set of equivalence classes by $\mathcal{T}$:

$$\mathcal{T} = \{\{t_n\} \mid 1 \leq t_n \leq s_n\}/\sim .$$

The set $\mathcal{T}$ consists of a single element if G is root reductive, and is uncountable otherwise. For any element $t \in \mathcal{T}$ we put $W^t := \varinjlim_{\tau_n^{t_n}} W_n$, where $\{t_n\}$ is a representative of t. It is easy to see that W^t depends only on the type of G. Namely, W^t is isomorphic to S_∞, the group of finite permutations of $\mathbb{N}$, if G is of type A; to the group of signed finite permutations of $\mathbb{N}$ if G is of type C; and to the group of signed finite permutations of $\mathbb{N}$ with even number of minus signs if G is of type D. Finally we put $W := \dot{\times}_{t \in \mathcal{T}} W^t$, where $\dot{\times}$ stands for restricted direct product.

If G is root reductive of type B, the definitions above still make sense. Moreover, $\mathcal{T}$ consists of a single element t, and $W = W^t$ is isomorphic to the group of signed finite permutations of $\mathbb{N}$.

If G is of type B but is not root reductive, κ_n is not a root injection for infinitely many n, and we need to modify the definitions above. Let $\mathring{W}_n$ denote the subgroup of W_n generated by reflections along the long simple roots of $\mathfrak{b}_n$. It is clear that (3.1) maps root spaces corresponding to long roots of $\mathfrak{g}_n$ into root spaces corresponding

to long roots of $\mathfrak{g}_{n+1}$. Hence, again we have an injective homomorphism of groups $\tau_n^i \colon \mathring{W}_n \to \mathring{W}_{n+1}$. We can now proceed as above to define

$$W^t := \varinjlim{}_{\tau_n^{t_n}} \mathring{W}_n \quad \text{and} \quad W := \dot{\times}_{t \in \mathcal{T}} W^t.$$

Note that W^t is isomorphic to S_∞, the group of finite permutations of $\mathbb{N}$.

Next we define a *length function* $\ell_B \colon W \to \mathbb{N} \cup \{\infty\}$. Let ℓ_n denote the length function on W_n determined by $\mathfrak{b}_n$.

Lemma 3.1 *For every $n \in \mathbb{N}$, every $1 \le i \le s_n$, and every $w \in W_n$ we have*

$$\ell_n(w) \le \ell_{n+1}(\tau_n^i(w)).$$

Furthermore, if $\ell_n(w) = \ell_{n+1}(\tau_n^i(w))$, then for every reduced factorization $w = \sigma_1 \cdots \sigma_j$ into a product of simple reflections, $\tau_n^i(w) = \tau_n^i(\sigma_1) \cdots \tau_n^i(\sigma_j)$ is a reduced factorization of $\tau_n^i(w)$.

Proof Let $\Delta_n = \Delta_n^+ \sqcup \Delta_n^-$, where Δ_n^+ are the roots of $\mathfrak{b}_n$, and let $\gamma \colon \Delta_n \to \Delta_{n+1}$ be the map corresponding to the injection (3.1). Then $\gamma(\Delta_n^\pm) \subset \Delta_{n+1}^\pm$. Set

$$\Phi_w := (w^{-1}\Delta_n^-) \cap \Delta_n^+ \quad \text{and} \quad \Phi_{\tau_n^i(w)} := (\tau_n^i(w)^{-1}\Delta_{n+1}^-) \cap \Delta_{n+1}^+.$$

The inclusion $\gamma(\Phi_w) \subset \Phi_{\tau_n^i(w)}$ implies

$$\ell_n(w) = |\Phi_w| \le |\Phi_{\tau_n^i(w)}| = \ell_{n+1}(\tau_n^i(w)).$$

Moreover, the equality $\ell_n(w) = \ell_{n+1}(\tau_n^i(w))$ implies that $\gamma(\Phi_w) = \Phi_{\tau_n^i(w)}$, and thus γ sends every simple root in Φ_w into a simple root of $\mathfrak{b}_{n+1}$. Consider a reduced factorization $w = \sigma_1 \dots \sigma_j$ and let σ_j be the reflection along the simple root α of $\mathfrak{b}_n$. Then $\tau_n^i(\sigma_j)$ is the reflection along the simple root $\gamma(\alpha)$ of $\mathfrak{b}_{n+1}$. Set $w' := \sigma_1 \cdots \sigma_{j-1}$. Then we have $\ell_n(w') = \ell_{n+1}(\tau_n^i(w'))$, and we complete the proof by induction. ∎

Lemma 3.1 (and the observation that $\tau_n^i(\mathring{W}_n) \subset \mathring{W}_{n+1}$ if G is of type B) implies that every element $w^t \in W^t$ has a well-defined, possibly infinite, *length* $\ell_B(w^t)$. We extend the definition of length to elements of W by setting $\ell_B(w) := \sum_{t \in \mathcal{T}} \ell_B(w^t)$ for $w = (w^t)_{t \in \mathcal{T}}$. We now define W_B as the subgroup of W consisting of all elements $w \in W$ of finite length $\ell_B(w)$. For $w = (w^t) \in W_B$ we say that the *support* of w is $\{t^1, \dots, t^l\} \subset \mathcal{T}$ if $w^t = 1_{W^t}$ precisely when $t \notin \{t^1, \dots, t^l\}$. Assume that n_0 is such that $w^{t^i} \in W_{n_0}$ for $1 \le i \le l$. Then $w^{t^i} \in W_n$ for $n \ge n_0$ and $1 \le i \le l$. It is not necessarily true that $w^{t^1}, \dots, w^{t^l}$ commute in W_n. If, however, the sequences $(t_{n_0}^i, \dots, t_{n-1}^i)$ for $i = 1, \dots, l$ are distinct, then $w^{t^1}, \dots, w^{t^l}$ commute in W_n, and define an element $w(n) := w^{t^1} \dots w^{t^l} \in W_n$. Since $t^1, \dots, t^l$ are distinct, there exists n_1 such that the sequences $(t_{n_0}^i, \dots, t_{n_1-1}^i)$ for $i = 1, \dots, l$ are distinct and hence $w^{t^1}, \dots, w^{t^l}$ commute in W_n for every $n \ge n_1$. For the rest of the paper, whenever for an element $w \in W_B$ we consider the elements $w(n)$, we will assume that $n \ge n_1$. If G is of type B, then $w(n) \in \mathring{W}_n$.

Example 3.2 For $G = SL(2^\infty)$ we have $W \cong \dot{\times}_{\mathcal{T}} S_\infty$. If B is the upper triangular Borel subgroup of G, then $W_B = W$. If, on the other hand, B is the interlacing Borel subgroup, W_B is trivial.

Proposition 3.3 *If W_B contains an element of length l, then W_B contains elements of all lengths from 0 through l. W_B may be finite or infinite and may or may not contain an element of maximal length. In addition, for fixed G and variable B, any non-negative integer can appear as a maximal possible length of an element in W_B.*

Proof The first statement follows from the generalization of Lemma 3.1 discussed above. Let $w \in W$ with $\ell_B(w) = l$. For any reduced expression for $w(n)$, any subword of $w(n)$ is well-defined and represents an element of W_B. Subwords of $w(n)$ will provide elements of W of any length between 0 and l.

The remaining statements are rather straightforward, and we omit their proofs here.   ∎

It is not difficult to see that in general the group W_B does not act on $\mathcal{P}$, *i.e.*, there exist $w \in W_B$ and $\lambda = \varprojlim \lambda_n$ for which $\{w(n)(\lambda_n)\}$ is not an inverse system of weights. Here is a simple example.

Example 3.4 Let B be the upper triangular Borel subgroup of $SL(2^\infty)$. Consider $w \in W_B$ given by

$$w^t = \begin{cases} (12) & \text{if } t = (1, 1, \dots), \\ 1_{S_\infty} & \text{otherwise}, \end{cases}$$

where the transposition (12) is understood as an element of the symmetric group S_∞. Let $\lambda = \varprojlim \lambda_n$ be a weight such that $a_n^1 = n$ and $a_n^{2^{n-1}+1} = -1$ in the notation of Example 2.2(i). Then $w(n+1)(\lambda_{n+1}) = \lambda_{n+1} - (n+1)(\varepsilon_n^1 - \varepsilon_n^2)$ restricts to $\lambda_n - (n+1)(\varepsilon_n^1 - \varepsilon_n^2)$, while $w(n)(\lambda_n) = \lambda_n - n(\varepsilon_n^1 - \varepsilon_n^2)$, which shows that $w(n)(\lambda_n)$ is not an inverse system of weights.

Despite this example, we are going to show that if $\lambda \in \mathcal{P}^+$, then $w(\lambda)$ is a well-defined element of $\mathcal{P}$ for any $w \in W_B$. We will also define an analog of the "dot" action in the finite-dimensional case. Recall that, for a finite-dimensional reductive Lie algebra $\mathfrak{g}'$ with fixed Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{g}'$ and Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}'$, $\mathfrak{b}' \supset \mathfrak{h}'$, the dot action of a Weyl group element w' on a weight $\mu' \in (\mathfrak{h}')^*$ is defined as $w'(\mu' + \rho_{\mathfrak{b}'}) - \rho_{\mathfrak{b}'}$, where $\rho_{\mathfrak{b}'}$ is the half-sum of roots of $\mathfrak{b}'$. One writes $w' \cdot \mu' := w'(\mu' + \rho_{\mathfrak{b}'}) - \rho_{\mathfrak{b}'}$. In the case of a diagonal ind-group G, for any $\lambda = \varprojlim \lambda_n$ and $w \in W_B$ it is natural to consider the weights $\{w(n)(\lambda_n + \rho_n) - \rho_n\}$, where ρ_n denotes the half-sum of the roots of $\mathfrak{b}_n$.

To prove results about the action of W_B on weights we need additional notation. If α' is a root of $\mathfrak{b}_m$ and α'' is a root of $\mathfrak{b}_n$ with $n \geq m$, we say that α'' is *a successor* of α' if $\alpha'' \in \mathfrak{h}_n^*$ restricts to $\alpha' \in \mathfrak{h}_m^*$. If, in addition, $n = m + 1$, we say that α'' is *an immediate successor* of α'. Every root of $\mathfrak{b}_m$ has exactly s_m immediate successors. The set of successors $\mathcal{S}^\alpha$ of a root α of $\mathfrak{b}_m$ has a natural structure of a directed tree — every element is connected with its immediate successors. If α is a root of $\mathfrak{b}_m$, then $\mathcal{S}^\alpha = \sqcup_{n \geq m} \mathcal{S}_n^\alpha$, where $\mathcal{S}_n^\alpha$ is the set of successors of α of level n, *i.e.*, those successors

of α which are roots of $\mathfrak{b}_n$. Furthermore, given $\lambda \in \mathcal{P}$, we assign integer labels to all nodes of this tree in a natural way: the node $\alpha' \in \mathcal{S}_n^\alpha$ is labeled by $2(\lambda_n, \alpha')/(\alpha', \alpha')$. It is clear that the sum of the labels of the elements of $\mathcal{S}_n^\alpha$ is the same for all n and equals $(2\lambda_m, \alpha)/(\alpha, \alpha)$.

Proposition 3.5
(i) *If $w \in W_B$, then $w \cdot 0$ is a well-defined element of $\mathcal{P}$, i.e., $\{w(n) \cdot 0 = w(n)(\rho_n) - \rho_n\}$ is an inverse system of weights of G.*
(ii) *If $w \in W_B$ and $\lambda \in \mathcal{P}_B^+$, then $w(\lambda)$ is a well-defined element of $\mathcal{P}$.*
(iii) *If $w \in W_B$ and $\lambda \in \mathcal{P}_B^+$, then $w \cdot \lambda$ is a well-defined element of $\mathcal{P}$.*

Proof (i) Since $w \in W_B$, we have $w^{-1} \in W_B$ and $w(n)^{-1} = w^{-1}(n)$. The proof of Lemma 3.1 applied to w^{-1} implies that the set $\Phi_{w(n+1)^{-1}}$ projects onto the set $\Phi_{w(n)^{-1}}$, and the formulas (cf. [DR])

$$w(n) \cdot 0 = w(n)(\rho_n) - \rho_n = - \sum_{\alpha \in \Phi_{w(n)^{-1}}} \alpha$$

and

$$w(n+1) \cdot 0 = w(n+1)(\rho_{n+1}) - \rho_{n+1} = - \sum_{\alpha \in \Phi_{w(n+1)^{-1}}} \alpha$$

show that $w(n+1) \cdot 0$ restricts to $w(n) \cdot 0$.
(ii) Let $w = (w^t) \in W_B$ have support $t^1, \ldots, t^l$ and let m be such that w^{t^i} for $1 \le i \le l$ all belong to W_n and commute in W_n for $n \ge m$. Let $w(m) = \sigma_{\alpha^1} \cdots \sigma_{\alpha^q}$ be a reduced expression of $w(n)$. Then, for $n \ge m$, $w(n) = \sigma_{\alpha_n^1} \cdots \sigma_{\alpha_n^q}$ is a reduced expression of $w(n)$, and α_n^i is a successor of α^i for $1 \le i \le q$. Furthermore, the sequence $\alpha^i = \alpha_m^i, \alpha_{m+1}^i, \alpha_{m+2}^i, \ldots$ is a path in $\mathcal{S}^{\alpha^i}$. Since λ is dominant, *i.e.*, all labels in $\mathcal{S}^{\alpha^i}$ corresponding to λ are non-negative integers, there exists $n_0 \ge m$ such that the labels on each of the paths $\alpha^i = \alpha_m^i, \alpha_{m+1}^i, \alpha_{m+2}^i, \ldots$ of level $n \ge n_0$ stabilize. For $n \ge n_0$ we have
(3.2)

$$w(n)(\lambda_n) = \lambda_n - \left(\sum_{1 \le i \le q} \frac{2(\lambda_n, \alpha_n^i)}{(\alpha_n^i, \alpha_n^i)} \alpha_n^i - \sum_{1 \le i < j \le q} \frac{2(\lambda_n, \alpha_n^j)}{(\alpha_n^j, \alpha_n^j)} \frac{2(\alpha_n^j, \alpha_n^i)}{(\alpha_n^i, \alpha_n^i)} \alpha_n^i + \cdots \right).$$

Now consider the restriction of $w(n+1)(\lambda_{n+1})$ to $\mathfrak{h}_n^*$. By the definition of λ, λ_{n+1} restricts to λ_n and, by the stabilization of the labels along the paths $\alpha^i = \alpha_m^i, \alpha_{m+1}^i, \alpha_{m+2}^i, \ldots$,

$$\sum_{1 \le i \le q} \frac{2(\lambda_{n+1}, \alpha_{n+1}^i)}{(\alpha_{n+1}^i, \alpha_{n+1}^i)} \alpha_{n+1}^i - \sum_{1 \le i < j \le q} \frac{2(\lambda_{n+1}, \alpha_{n+1}^j)}{(\alpha_{n+1}^j, \alpha_{n+1}^j)} \frac{2(\alpha_{n+1}^j, \alpha_{n+1}^i)}{(\alpha_{n+1}^i, \alpha_{n+1}^i)} \alpha_{n+1}^i + \cdots$$

restricts to

$$\sum_{1 \le i \le q} \frac{2(\lambda_n, \alpha_n^i)}{(\alpha_n^i, \alpha_n^i)} \alpha_n^i - \sum_{1 \le i < j \le q} \frac{2(\lambda_n, \alpha_n^j)}{(\alpha_n^j, \alpha_n^j)} \frac{2(\alpha_n^j, \alpha_n^i)}{(\alpha_n^i, \alpha_n^i)} \alpha_n^i + \cdots.$$

These observations together with (3.2) and its analog with $n + 1$ in place of n imply that $w(n + 1)(\lambda_{n+1})$ restricts to $w(n)(\lambda_n)$. This completes the proof of (ii).

(iii) The statement follows from (i), (ii), and the obvious formula

$$w(n) \cdot \lambda_n = w(n)(\lambda_n) + w(n) \cdot 0. \qquad \blacksquare$$

Proposition 3.6

(i) Let $\sigma \in W_B$ be an element of length one and let $\lambda \in \mathcal{P}$ be such that $\sigma(n) \cdot \lambda_n$ is dominant for large enough n. Then there exists n' such that, for $n \geq n'$, $\sigma(n) = \sigma_{\alpha_n}$, (λ_n, α_n) does not depend on n, and $(\lambda_n, \alpha) = 0$ for every successor $\alpha \in \mathcal{S}_n^{\alpha_{n'}}$ different from α_n.

(ii) Let $w \in W_B$ and $\lambda \in \mathcal{P}$ be such that $\mu_n := w(n) \cdot \lambda_n$ is $\mathfrak{b}_n$-dominant for large enough n. Then $\mu := \varprojlim \mu_n$ is a well-defined element of $\mathcal{P}_B^+$.

Proof (i) Since $\ell_B(\sigma) = 1$, $\sigma(n) = \sigma_{\alpha_n}$ where α_{n+1} is an immediate successor of α_n for $n \geq n_0$ and both α_n and α_{n+1} are simple roots of the respective Borel subalgebras $\mathfrak{b}_n$ and $\mathfrak{b}_{n+1}$. The label of λ at α_n in $\mathcal{S}^{\alpha_{n_0}}$ is negative, while all other labels of λ in $\mathcal{S}^{\alpha_{n_0}}$ are non-negative. This implies that the labels of λ along the path $\alpha_{n_0}, \alpha_{n_0+1}, \ldots$ are non-increasing. Let $\beta_{n_0+1} = \alpha_{n_0+1} + \alpha'_{n_0+1}$ be a root of $\mathfrak{b}_{n_0+1}$ higher than α_{n_0+1} and let $\beta_n = \alpha_n + \alpha'_n$ be a successor of β_{n_0+1}. Note that β_n is uniquely determined by α_n. We have

$$(\lambda_n, \beta_n) = (\lambda_n, \alpha_n) + (\lambda_n, \alpha'_n),$$

which implies $(\lambda_n, \alpha_n) \geq -(\lambda_n, \beta_n)$. Since $\{(\lambda_n, \beta_n)\}$ is a non-increasing sequence of non-negative integers or half-integers, we conclude that the sequence $\{(\lambda_n, \alpha_n)\}$ is bounded, and hence it stabilizes. Noting that $(\lambda_{n+1}, \alpha_{n+1}) = (\lambda_n, \alpha_n)$ implies that $(\lambda_{n+1}, \alpha) = 0$ for every immediate successor of α_n other than α_{n+1} concludes the proof of (i).

(ii) Write $w(n) = \sigma^1_{\alpha_n} \ldots \sigma^q_{\alpha_n}$ as in the proof of Proposition 3.5(ii). As in (i) we prove that the labels along the paths $\{\alpha^i_n\}$ for $1 \leq i \leq q$ stabilize, and then repeat the argument in the proof of Proposition 3.5(ii). $\blacksquare$

4 G/B **and the Bott–Borel-Weil Theorem**

Recall that an *ind-variety* $X = \varinjlim X_n$ is determined by a sequence of morphisms of algebraic varieties

$$X_1 \xrightarrow{\varphi_1} \cdots \to X_n \xrightarrow{\varphi_n} X_{n+1} \to \cdots;$$

see for instance [Sh, DPW]. We denote by $\mathcal{O}_{X_n}$ the structure sheaf of X_n and we define the *structure sheaf $\mathcal{O}$ of X* as the inverse limit $\varprojlim \mathcal{O}_{X_n}$. More generally, a *sheaf $\mathcal{F}$ on X* is by definition the limit of an inverse system of sheaves $\mathcal{F}_n$ on X_n, and $\mathcal{F}$ is a *sheaf of $\mathcal{O}$-modules* whenever $\{\mathcal{F}_n\}$ is an inverse system of sheaves of $\mathcal{O}_{X_n}$-modules. A sheaf $\mathcal{F}$ of $\mathcal{O}$-modules is *locally free of rank r* whenever each $\mathcal{F}_n$ is locally free of rank r. In what follows we will also call a locally free sheaf of $\mathcal{O}$-modules a *vector bundle* on X.

Assume now that all X_n are proper. Then it is well known that the cohomology $H^{\cdot}(X, E)$ of any vector bundle $E = \varprojlim E_n$ of finite-rank on X is canonically isomorphic to the inverse limit $\varprojlim H^{\cdot}(X_n, E_n)$; see [W, DPW].

In this paper we consider the ind-varieties G/B and G/P, where $G = \varinjlim G_n$ is a diagonal ind-group and $B = \varinjlim B_n$ or $P = \varinjlim P_n$ are respectively direct limits of Borel subgroups $B_n \subset G_n$ or parabolic subgroups $P_n \subset G_n$. More precisely, if $B = \varinjlim B_n$ with $B_n = G_n \cap B_{n+1}$, or $P = \varinjlim P_n$ with $P_n = G_n \cap P_{n+1}$, the embeddings $G_n \to G_{n+1}$ induce closed immersions $G_n/B_n \to G_{n+1}/B_{n+1}$ and $G_n/P_n \to G_{n+1}/P_{n+1}$ of proper smooth varieties. In what follows, we denote the corresponding ind-varieties by G/B and G/P.

If $\lambda \in \mathcal{P}$, the line bundles $(\mathcal{O}_{G_n/B_n})_{-\lambda_n}$ form an inverse system, and hence determine a line bundle (or a locally free sheaf of $\mathcal{O}$-modules of rank one) $\mathcal{O}_{-\lambda}$ on G/B. Recall that, by definition, $(\mathcal{O}_{G_n/B_n})_{-\lambda_n}$ is the G_n-equivariant line bundle on G_n/B_n whose geometric fibre at the closed point $B_n \in G_n/B_n$ is the B_n-module $\mathbb{K}^{-\lambda_n}$.

More generally, if E_n is a G_n-equivariant vector bundle (or, for short, G_n-*bundle*) on G_n/P_n, then the vector bundle $E = \varprojlim E_n$ on G_n/P_n is by definition G-equivariant, and each cohomology group $H^j(G/P, E)$ is a dual G-module, being an inverse limit of G_n-modules $H^j(G_n/P_n, E_n)$.

The Bott–Borel–Weil theorem computes the cohomology $H^{\cdot}(G_n/B_n, (\mathcal{O}_{G_n/B_n})_{-\lambda_n})$ for each weight λ_n; see [B, S, D1, D2]. It is the following result.

Theorem 4.1 (Bott–Borel–Weil, [S, B]) *There exists a (necessarily unique) $w_n \in W_n$ such that $w_n \cdot \lambda_n$ is a B_n-dominant weight of G_n if and only if $H^{\cdot}(G_n/B_n, (\mathcal{O}_{G_n/B_n})_{-\lambda_n}) \neq 0$. In this case*

$$H^j(G_n/B_n, (\mathcal{O}_{G_n/B_n})_{-\lambda_n}) \cong \begin{cases} V_{B_n}(w_n \cdot \lambda_n)^* & \text{for } j = \ell_n(w_n), \\ 0 & \text{for } j \neq \ell_n(w_n). \end{cases}$$

If w_n as above does not exists, then $H^{\cdot}(G_n/B_n, (\mathcal{O}_{G_n/B_n})_{-\lambda_n}) = 0$.

An immediate corollary of Theorem 4.1 is that, for a fixed $\lambda \in \mathcal{P}$, the cohomology $H^{\cdot}(G/B, \mathcal{O}_{-\lambda})$ can be nonzero in at most one degree. This follows from Theorem 4.1 and from the fact that $H^j(G/B, \mathcal{O}_{-\lambda}) = \varprojlim H^j(G_n/B_n, (\mathcal{O}_{G_n/B_n})_{-\lambda_n})$ for any j. The following theorem provides a much stronger statement. It is an analog of the Bott–Borel–Weil theorem and is the central result in this paper.

Theorem 4.2 *Let G be a diagonal ind-group, let B be a Borel subgroup of G, and let $\lambda \in \mathcal{P}$. Then $H^j(G/B, \mathcal{O}_{-\lambda}) \neq 0$ for at most one value of j. More precisely, $H^j(G/B, \mathcal{O}_{-\lambda}) \neq 0$ if and only if there exists $w \in W_B$ such that $w \cdot \lambda \in \mathcal{P}_B^+$. In the latter case we have an isomorphism of dual G-modules*

$$H^j(G/B, \mathcal{O}_{-\lambda}) \cong V_B(w \cdot \lambda)^*.$$

Before we prove Theorem 4.2 we state two results necessary for the proof. Let $G' \subset G''$ be reductive algebraic groups with Lie algebras $\mathfrak{g}' \subset \mathfrak{g}''$ respectively. Assume that $B'' \subset G''$ is a Borel subgroup of G'' and that $B' := G' \cap B''$ is a Borel subgroup of G'. Then we have a closed immersion of homogeneous spaces $G'/B' \to G''/B''$. Denote the Weyl groups of G' and G'' by W' and W'' respectively. If G' (respectively, G'') is of type B or product of groups of type B, denote by $\mathring{W}'$ (respectively, $\mathring{W}''$) the subgroup of W' (respectively, W'') generated by reflections along the long simple

roots of the corresponding Borel subalgebra. Let λ'' be a weight of B'' that restricts to the weight λ' of G'. Assume that there exist $w' \in W'$ and $w'' \in W''$ both of length j and such that $w'' \cdot \lambda''$ and $w' \cdot \lambda'$ are dominant weights. The natural map

$$(4.1) \qquad\qquad H^j(G''/B'', \mathcal{O}_{-\lambda''}) \to H^j(G'/B', \mathcal{O}_{-\lambda'})$$

is a homomorphism of nontrivial G'-modules.

Proposition 4.3 (Tsanov, [T])   *In the notation above the following statements hold.*

(i)   *Assume that $\mathfrak{g}'$ is a root subalgebra of $\mathfrak{g}''$ and consider W' as a subgroup of W''. Then* (4.1) *is nonzero if and only if $w'' = w' \in W'$.*

(ii) *Assume that $\mathfrak{g}' \cong B_r \oplus B_r$ and $\mathfrak{g}'' \cong D_{2r+1}$ as in Section 1. If $w'' = w' \in \mathring{W}'$, then* (4.1) *is nonzero.*

Corollary 4.4   *Let $G' = G_1 \times \cdots \times G_s \cong (B_{r'})^s$ with $s > 1$, let $G'' \cong B_{r''}$ and assume that the embedding $\kappa\colon G' \to G''$ is analogous to κ_n from* (1.5). *In other words, the natural representation V'' of G'' decomposes as $V'_1 \oplus \cdots \oplus V'_s \oplus \mathbb{K}^z$, where each V'_i is isomorphic to the natural representation of $B_{r'}$. Consider the diagram*

$$(4.2)$$

where $\bar{\mathfrak{g}}'$ is defined analogously to $\bar{\mathfrak{g}}_n$ from (1.5).

(i)   *If* (4.1) *is nonzero, then $w'' \in \mathring{W}''$.*

(ii) *If $w' \in \mathring{W}'$, then* (4.1) *is nonzero if and only if $w'' = w'$.*

Proof   The second statement follows from Proposition 4.3. Here is the proof of (i). Denote by $\bar{G}'$ the simply-connected algebraic group with Lie algebra $\bar{\mathfrak{g}}'$. Assume that (4.1) is nonzero. The fact that (4.1) is nonzero implies that the map

$$H^j(G''/B'', \mathcal{O}_{-\lambda''}) \to H^j(\bar{G}'/\bar{B}', \mathcal{O}_{-\bar{\lambda}'})$$

is nonzero, where $\bar{\lambda}'$ is the restriction of λ'' to $\bar{G}'$. Since θ is a root injection, Proposition 4.3(i) implies that $w'' \in \mathring{W}''$ if s is even. If s is odd, Proposition 4.3(i) implies that w'' is contained in the subgroup ${}^1\mathring{W}''$ of W'' generated by reflections along long simple roots corresponding to the components of $\bar{G}'$ of type D and by reflections along the simple roots corresponding to the component of $\bar{G}'$ of type B. We can use a different way of combining the G'-constituents $V'_1, \ldots, V'_s$ of the natural representation of G'' to obtain a diagram analogous to (4.2) but with different $\bar{\mathfrak{g}}'$ and different maps κ and θ. Repeating the argument above we conclude that $w'' \in {}^2\mathring{W}''$ for a subgroup ${}^2\mathring{W}''$ analogous to ${}^1\mathring{W}''$. Note that ${}^1\mathring{W}'' \cap {}^2\mathring{W}'' = \mathring{W}''$ as long as we choose a different component of type B of $\bar{\mathfrak{g}}'$. This completes the proof.   ■

Proposition 4.5 ([DR, Corollary 5.4.1]) *Let $\mathfrak{g}''$ equal the direct sum of s isomorphic copies $\mathfrak{g}_1'', \ldots, \mathfrak{g}_s''$ of $\mathfrak{g}'$ so that $\mathfrak{g}'$ projects isomorphically onto each subalgebra $\mathfrak{g}_i''$. Assume that*

$$H^j(G''/B'', \mathcal{O}_{-\lambda''}) =$$

$$H^j(G_1''/B_1'', \mathcal{O}_{-\lambda_1''}) \otimes H^0(G_2''/B_2'', \mathcal{O}_{-\lambda_2''}) \otimes \cdots \otimes H^0(G_s''/B_s'', \mathcal{O}_{-\lambda_s''}),$$

where $G'' = G_1'' \times \cdots \times G_s'' \cong (G')^s$ and $\lambda_1'', \ldots, \lambda_s''$ are the restrictions of λ'' to $\mathfrak{g}_1'', \ldots, \mathfrak{g}_s''$. Then (4.1) is a nonzero homomorphism.

Proof of Theorem 4.2 For $n > m$, diagram (1.4) induces the commutative diagram
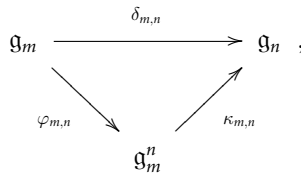
where $\delta_{m,n} = \delta_{n-1} \circ \cdots \circ \delta_m$ and $\varphi_{m,n}$ and $\kappa_{m,n}$ are defined in the obvious way. By definition $\widetilde{\mathfrak{g}}_m := \mathfrak{g}_m^{m+1}$ and $\mathfrak{g}_m^n \cong \mathfrak{g}_m^{\oplus s_m^n}$, where $s_m^n := s_m \cdots s_{n-1}$. Given $\lambda \in \mathcal{P}_B$, we denote the restriction of λ to $\mathfrak{g}_m^n$ by λ_m^n. Furthermore, for $n > m > k$ there exist Lie algebra homomorphisms $\delta_{k,m,n}$, $\varphi_k^{m,n}$, and $\kappa_{k,m}^n$ such that the diagram

(4.3)
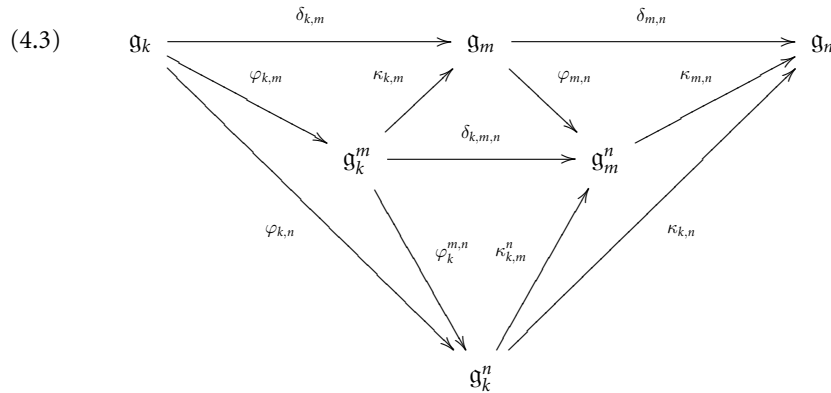
commutes. To simplify notation we set

$$\varphi_m^n := \begin{cases} \varphi_m & \text{if } n = m, \\ \varphi_m^{n,n+1} & \text{if } n > m, \end{cases} \qquad \kappa_m^n := \begin{cases} \kappa_m & \text{if } n = m+1, \\ \kappa_{m,m+1}^n & \text{if } n > m+1, \end{cases}$$

and $\mathfrak{g}_n^n := \mathfrak{g}_n$.

Diagram (4.3) gives rise to a commutative diagram

$$
(4.4) \quad
\begin{array}{ccccccc}
\cdots\ \mathfrak{g}_m & \to & \mathfrak{g}_{m+1} & \to & \mathfrak{g}_{m+2} & \to & \mathfrak{g}_{m+3}\ \cdots \\
\searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \\
\cdots\quad \mathfrak{g}_m^{m+1} & & \mathfrak{g}_{m+1}^{m+2} & & \mathfrak{g}_{m+2}^{m+3} & & \cdots \\
& \searrow & \nearrow & \searrow & \nearrow & & \\
& \cdots\quad \mathfrak{g}_m^{m+2} & & \mathfrak{g}_{m+1}^{m+3} & & \cdots & \\
& & \searrow & \nearrow & & & \\
& & \cdots\quad \mathfrak{g}_m^{m+3} & & \cdots & & \\
\end{array}
$$

$$\cdots$$

The idea of the proof is to study the maps between the cohomology groups $H^j(G_m^n/B_m^n, \mathcal{O}_{-\lambda_m^n})$ induced from (4.4), where G_m^n is the simply-connected classical group with Lie algebra $\mathfrak{g}_m^n$ and $B_m^n = B \cap G_m^n$. More precisely, every injection in the direct system $\mathfrak{g}_k \to \mathfrak{g}_k^{k+1} \to \mathfrak{g}_k^{k+2} \to \cdots$ splits into the direct product of embeddings to which Proposition 4.5 applies. Similarly, Proposition 4.3 and Corollary 4.4 apply to every injection in the sequence

$$
\mathfrak{g}_m^n \to \mathfrak{g}_{m+1}^n \to \cdots \to \mathfrak{g}_{n-1}^n \to \mathfrak{g}_n.
$$

Assume first that $H^j(G/B, \mathcal{O}_{-\lambda}) \neq 0$. Then there exists k such that all maps between the cohomology groups $H^j(G_m^n/B_m^n, \mathcal{O}_{-\lambda_m^n})$ induced from (4.4) for $n \geq m \geq k$ are nonzero. Note that

$$
H^j(G_k^m/B_k^m, \mathcal{O}_{-\lambda_k^m}) = \otimes_{t \in \mathcal{T}_k^m} H^{j_t}((G_k^m)_t/(B_k^m)_t, \mathcal{O}_{-(\lambda_k^m)_t}),
$$

where $\mathcal{T}_k^m = \{(t_k, \ldots, t_{m-1}) \mid 1 \leq t_i \leq s_i\}$, $(G_k^m)_t$, and $(B_k^m)_t$ are the respective constituents of G_k^m and B_k^m corresponding to t, and $(\lambda_k^m)_t$ is the restriction of λ to $(G_k^m)_t$. Let w^t denote the element of the Weyl group $(W_k^m)_t$ of $(G_k^m)_t$ such that $w^t \cdot (\lambda_k^m)_t$ is dominant. We say that $t'' \in \mathcal{T}_k^{m+1}$ is an *immediate successor of* $t' \in \mathcal{T}_k^m$ if $t' = (t_k, \ldots, t_{m-1})$ and $t'' = (t_k, \ldots, t_{m-1}, t_m)$; we denote this relation by $t' \prec t''$. Künneth's formula implies that $j_t = \sum_{t \prec t'} j_{t'}$. Hence there is a finite collection of sequences $t^i = (t_k^i, t_{k+1}^i, \ldots)$, for $1 \leq i \leq l$ such that, for $t \in \mathcal{T}_k^m$, $j_t \neq 0$ if and only if $t = (t_k^i, \ldots, t_{m-1}^i)$ for some $1 \leq i \leq l$. Fix $m > k$ such that, for $t \in \mathcal{T}_k^m$ with $j_t \neq 0$, $j_{t'} \neq 0$ for exactly one immediate successor t' of t. In particular j_{t^i} stabilizes for $n > m$. Noting that $(\lambda_k^{m+1})_{t''}$ is dominant for every $t \prec t'' \neq t'$ we conclude that $\tau_n^{t_n^i}(w^{t^i}) = w^{t^i}$ for $n > m$. The last equation means that we have well-defined elements $w^{t^i} \in W^{t^i}$ if G is not of type B. If G is of type B, Corollary 4.4(i) ensures that $w^{t^i} \in W^{t^i}$ if we repeat the argument above with $k + 1$ in place of k. Furthermore, $\ell_B(w^{t^i}) = j_{t^i}$ and $w^{t^1}, \ldots, w^{t^l}$ define an element w of W_B of length j. The fact that $w \cdot \lambda \in \mathcal{P}_B^+$ follows from Proposition 3.6. The existence of an isomorphism $H^j(G/B, \mathcal{O}_{-\lambda}) \cong V_B(w \cdot \lambda)^*$ is obvious.

Conversely, assume that $w \in W_B$ satisfies $w \cdot \lambda \in \mathcal{P}_B^+$. We need to show that there exists k such that all maps between cohomology groups $H^j(G_m^n/B_m^n, \mathcal{O}_{-\lambda_m^n})$ corresponding to (4.4) with $n \geq m > k$ are nonzero. Assume that the support of w

is $t^1, \ldots, t^l$ and choose k so that the sequences $t^i_k, t^i_{k+1}, \ldots$ for $1 \leq i \leq l$ are distinct. The fact that $H^j(G^n_m, \mathcal{O}_{-\lambda^n_m}) \to H^j(G^{n-1}_m, \mathcal{O}_{-\lambda^{n-1}_m})$ is nonzero follows from Proposition 4.5, while the fact that $H^j(G^n_m, \mathcal{O}_{-\lambda^n_m}) \to H^j(G^n_{m-1}, \mathcal{O}_{-\lambda^n_{m-1}})$ is nonzero follows from Proposition 4.3 and Corollary 4.4. Finally, $H^j(G_{n+1}/B_{n+1}, \mathcal{O}_{-\lambda_{n+1}}) \to H^j(G_n/B_n, \mathcal{O}_{-\lambda_n})$ being the composition of

$$H^j(G^{n+1}_{n+1}, \mathcal{O}_{-\lambda^{n+1}_{n+1}}) \to H^j(G^{n+1}_n, \mathcal{O}_{-\lambda^{n+1}_n}) \quad \text{and} \quad H^j(G^{n+1}_n, \mathcal{O}_{-\lambda^{n+1}_n}) \to H^j(G^n_n, \mathcal{O}_{-\lambda^n_n})$$

is nonzero. ∎

Example 4.6 Let $G = SL(2^\infty)$. If B is the upper triangular Borel subgroup, then Theorem 4.2, together with the explicit description of W_B given above, implies that for each j there are line bundles $\mathcal{O}_\lambda$ with $H^j(G/B, \mathcal{O}_{-\lambda}) \neq 0$. If B is the interlacing Borel subgroup of $SL(2^\infty)$, then W_B is the trivial subgroup of W, and hence, by Theorem 4.2, $H^j(G/B, \mathcal{O}_{-\lambda}) = 0$ for all B-weights λ and all $j > 0$. Moreover, in this case $H^j(G/B, \mathcal{L}) = 0$ for all $j > 0$ and for any line bundle $\mathcal{L}$ on G/B, as it is easy to show that any $\mathcal{L}$ is G-equivariant, *i.e.,* $\mathcal{L} \simeq \mathcal{O}_{-\lambda}$ for some B-weight λ. This implies that the above two homogeneous ind-spaces are not isomorphic as ind-varieties, and in particular that the interlacing Borel subgroup is not conjugate to the upper triangular Borel subgroup by an automorphism of $SL(2^\infty)$.

Note that the group W_B which we use in Theorem 4.2 is different from the Weyl group W_F defined in [NRW] unless G is root reductive. (In fact W_F is a trivial group if G is diagonal but not root reductive.) Nevertheless, if $H^j(G/B, \mathcal{O}_{-\lambda}) \neq 0$ for some λ, the B-weight λ is cohomologically finite in the sense of [NRW]. A question we do not answer in this paper is whether the cohomological finiteness of λ is sufficient for $H^j(G/B, \mathcal{O}_{-\lambda})$ to be nonzero.

5 G/P and Projectivity

Let P be a parabolic subgroup of G and let $B \subset P$ be a Borel subgroup of G. It is easy to see that every finite-dimensional simple P-module admits a B-highest weight, *i.e.,* is the limit of the direct system of simple highest weight P_n-modules for an inverse system of G_n-weights λ_n. Setting $\lambda = \varprojlim \lambda_n$, we denote by M_λ the simple P-module with highest weight λ.

Proposition 5.1 Set $\mathcal{O}(M^*_\lambda) := \varprojlim(\mathcal{O}_{G_n/P_n}(M^*_\lambda))$, where $\mathcal{O}_{G_n/P_n}(M^*_\lambda)$ is the usual G_n-equivariant bundle on G_n/P_n with fibre M^*_λ. Then $H^j(G/P, \mathcal{O}(M^*_\lambda)) \neq 0$ if and only if $H^j(G/B, \mathcal{O}_{-\lambda}) \neq 0$, and in that case

$$H^j(G/P, \mathcal{O}(M^*_\lambda)) = H^j(G/B, \mathcal{O}_{-\lambda}) \cong V_B(w \cdot \lambda)^*,$$

where $w \in W_B$ *and* $w \cdot \lambda \in \mathcal{P}^+_B$.

Proof It is easy to see that $\mathcal{O}(M^*_\lambda) \cong pr_* \mathcal{O}_{-\lambda}$, $pr: G/B \to G/P$ being the natural submersion. Moreover, the fibre of pr equals $P/B = \varinjlim P_n/B_n$, hence the classical

Bott–Borel–Weil theorem implies $R^i pr_* \mathcal{O}_{G/P} = 0$ for $i > 0$ and $pr_* \mathcal{O}_{G/P} = \mathcal{O}_{G/B}$. This is sufficient to conclude that

$$H^j(G/P, \mathcal{O}(M_\lambda^*)) = H^j(G/B, \mathcal{O}_{-\lambda})$$

for any $j \in \mathbb{Z}_{\geq 0}$. The isomorphism $H^j(G/B, \mathcal{O}_{-\lambda}) \cong V_B(w \cdot \lambda)^*$ is established in Theorem 4.2. ∎

We conclude this paper by discussing the projectivity of the ind-varieties G/B and G/P. Recall that an ind-variety X is projective, *i.e.,* admits an embedding in the projective ind-space $\mathbb{P}^\infty$, if and only if it admits a very ample line bundle $\mathcal{L}$. An explicit criterion for the projectivity of G/B (and, more generally, of G/P) when G is root reductive is proved in [DPW].

For diagonal ind-groups we have the following corollary.

Corollary 5.2 *Let G a diagonal ind-group and let B be a Borel subgroup of G. Then if G/B is projective, G is necessarily root reductive.*

Proof If $\iota \colon G/B \to \mathbb{P}^\infty$ is a closed immersion, then $\mathcal{L} := \iota^*(\mathcal{O}_{\mathbb{P}^\infty}(1))$ is a very ample line bundle on G/B. In other words, $\mathcal{L}_{|G_n/B_n}$ is very ample for each n. Since G_n is simply-connected for each n, $\mathcal{L}_{|G_n/B_n} \cong (\mathcal{O}_{G_n/B_n})_{-\lambda_n}$ for some strictly dominant weight λ_n of G_n. The weights λ_n form an inverse system and hence define a strictly dominant weight $\lambda = \varprojlim \lambda_n$ of G. By Proposition 2.3(iii) G is root reductive. ∎

The following example shows that G/P may be projective even if G is not root reductive.

Example 5.3 Let $G = SL(2^\infty)$ and let P_n be the stabilizer of the span of the first i standard basis vectors in $\mathbb{C}^{2^n}$. Then $\varinjlim P_n$ is a well-defined maximal parabolic subgroup of G, and it is easy to see that $\overline{G/P}$ is isomorphic to the ind-Grassmannian of i-dimensional subspaces of $\mathbb{C}^\infty$. The latter is clearly projective.

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