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Similarity Classification of Cowen-Douglas Operators

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Abstract. Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the collection of bounded linear operators on $\mathcal{H}$. An operator A in $\mathcal{L}(\mathcal{H})$ is said to be strongly irreducible, if $\mathcal{A}'(T)$, the commutant of A, has no non-trivial idempotent. An operator A in $\mathcal{L}(\mathcal{H})$ is said to be a Cowen-Douglas operator, if there exists Ω, a connected open subset of C, and n, a positive integer, such that

(a) $\Omega \subset \sigma(A) = \{z \in C \mid A - z \text{ not invertible}\}$;
(b) $\operatorname{ran}(A - z) = \mathcal{H}$, for z in Ω;
(c) $\bigvee_{z\in\Omega} \ker(A - z) = \mathcal{H}$ and
(d) $\dim\ker(A - z) = n$ for z in Ω.

In the paper, we give a similarity classification of strongly irreducible Cowen-Douglas operators by using the K_0-group of the commutant algebra as an invariant.

0 Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the collection of bounded linear operators on $\mathcal{H}$. A basic problem in operator theory is to determine when two operators A and B in $\mathcal{L}(\mathcal{H})$ are similar, that is when there exists an invertible operator X on $\mathcal{H}$ satisfying $A = X^{-1}BX$. In a real sense the problem has no general solution but one restricts attention to special classes of operators. An important approach to this problem is via spectral theory in which one attempts to synthesize operators from elementary 'local operators', where 'local' refers to the spectrum. For example, an operator on a finite dimensional space can be obtained as the direct sum of scalar operators plus nilpotent Jordan blocks on generalized eigenspaces, where the scalars are just the eigenvalues which with their multiplicity determine the operator up to similarity. On infinite dimensional spaces direct sum must be replaced by a continuous direct sum or direct integral but the result is essentially the same (*cf.* [Na-Fo, Ap-Fi-He-Vo, Da-He, Ka]).

Conventional spectral theory attempts to extend such representations to as large a class of operators as possible. However, there exist operators which cannot be synthesized in this sense from local operators. One example is the backward shift T_z^* on l^2 defined by

$$T_z^*(\alpha_0, \alpha_1, \alpha_2, \dots) = (\alpha_1, \alpha_2, \alpha_3, \dots)$$

for $(\alpha_0, \alpha_1, \alpha_2, \dots)$ in l^2. Since

$$T_z^*(1, \lambda, \lambda^2, \dots) = \lambda(1, \lambda, \lambda^2, \dots)$$

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and $(1, \lambda, \lambda^2, \dots)$ is in l^2 for $|\lambda| < 1$, we see that the open unit disc D consists of eigenvalues for T_z^*. Such behavior is quite different from that for the finite dimensional case. Moreover, it can easily be shown that one cannot express l^2 as $\mathcal{M} \oplus \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are invariant for T_z^*. Thus one cannot study T_z^* using conventional spectral theory.

However, we probably know as much about this operator and its adjoint as we do about any single operator. In the functional representation, the adjoint T_z acts as multiplication by z on the Hardy space $H^2(\partial D)$ and an enormous literature exists on it (*cf.* [Do, He]).

M. J. Cowen and R. G. Douglas introduced in [Co-Do] a class of operators related to complex geometry now referred to as Cowen-Douglas operators. The Cowen-Douglas operators play an important role in studying the structure of non self-adjoint operators (*cf.* [He2, Ji-Wa]).

Definition 0.1 For Ω a connected open subset of C and n a positive integer, let $\mathcal{B}_n(\Omega)$ denote the operators T in $\mathcal{L}(\mathcal{H})$ which satisfy

(a) $\Omega \subset \sigma(T) = \{z \in C \mid T - z \text{ not invertible}\}$;
(b) $\operatorname{ran}(T - z) := \{(T - z)x \mid x \in \mathcal{H}\} = \mathcal{H}$ for z in Ω;
(c) $\bigvee_{z\in\Omega} \ker(T - z) = \mathcal{H}$; and
(d) $\dim\ker(T - z) = n$ for z in Ω.

We call an operator in $\mathcal{B}_n(\Omega)$ a Cowen-Douglas operator with index n. The collection $\mathcal{B}_n(\Omega)$ is void unless $\mathcal{H}$ is infinite dimensional. Condition (a) and (b) ensure that Ω is contained in the point spectrum of T and that $T - z$ is right invertible for z in Ω. It is easily seen that $T_z^* \in \mathcal{B}_1(\Omega)$.

If Ω_0 is an open subset of Ω, then $\mathcal{B}_n(\Omega) \subset \mathcal{B}_n(\Omega_0)$ because $\bigvee_{z\in\Omega_0} \ker(T - z) = \bigvee_{z\in\Omega} \ker(T-z)$, (*cf.* [Co-Do, Corollary 1.13]). There would seem to be some advantage in choosing Ω as large as possible. One kind of hypothesis implying that is the assumption that the closure of Ω is a spectral set for T. This means that $\|r(T)\| = \|r\|_\infty$ for each rational function with poles outside $\bar{\Omega}$, where $\|r\|_\infty$ denotes the supremum norm on $\bar{\Omega}$.

Now $\mathcal{B}_n(\Omega)$ is an especially rich class of operators containing the adjoint of many subnormal, hyponormal and weighted unilateral shift operators. D. A. Herrero, C. L. Jiang and Z. Y. Wang showed that almost all quasitriangular operators with connected spectrum can be approximated by Cowen-Douglas operator in the norm topology (*cf.* [He1], [Ji-Wa]).

Let T be in $\mathcal{B}_n(\Omega)$. Since $T - z$ is right invertible for z in Ω and $\operatorname{ran}(T - z)^k = \mathcal{H}$ for each positive integer k, it follows that $\dim\ker(T - z)^k = nk$, for z in Ω.

Now the generalized eigenspace $\ker(T - z)^k$ is invariant for T and hence Cowen and Douglas defined an operator

$$N_z^{(k)} = (T - z)|_{\ker(T-z)^{k+1}}$$

with $N_z = N_z^{(n)}$, the local operator associated to T at z in N_z. In [Co-Do], M. J. Cowen and R. G. Douglas obtained a remarkable result.

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***Theorem* CD1** *Operators T and $\tilde{T}$ in $\mathcal{B}_n(\Omega)$ are unitary equivalent if and only if N_z is unitarily equivalent to $\tilde{N}_z$ for each z in Ω.*

Call operators S and T unitarily equivalent, if there exists a unitary operator U satisfying $S = U^*TU$.

In the paper, we attempt to determine when two operators A and B in $\mathcal{B}_n(\Omega)$ are similar.

To express our results more carefully we need to introduce the following definitions, notation and theorems.

An operator T in $\mathcal{L}(\mathcal{H})$ is called strongly irreducible, if there is no non-trivial idempotent operator in $\mathcal{A}'(T)$, where $\mathcal{A}'(T)$ denotes the commutant of T, *i.e.*, $\mathcal{A}'(T) = \{B \in \mathcal{L}(\mathcal{H}), TB = BT\}$ (*cf.* [Co], [Gi], [Ji-Wa], [Ji1]).

From the definition, strongly irreducibility is invariant under similarity. In what follows, $A \in (\mathrm{SI})$ means A is a strongly irreducible operator.

Definition 0.2 Let $T \in \mathcal{L}(\mathcal{H})$. T is said to have finite (SI) decomposition if there exist $(P_1, P_2, \dots, P_n)$, a family of idempotents in $\mathcal{A}'(T)$, satisfying

1. $P_iP_j = \delta_{ij}P_i$ for $1 \le i, j \le n < +\infty$, where $\delta_{ij} = \begin{cases} 0 & i \ne j, \\ 1 & i = j; \end{cases}$
2. $\sum_{i=1}^n P_i = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$;
3. $T|_{P_i\mathcal{H}} \in (\mathrm{SI})$ for $i = 1, 2, \dots, n$.

We call $P = (P_1, P_2, \dots, P_n)$ a unit finite (SI) decomposition of T.

It is clear that an operator T in $\mathcal{L}(\mathcal{H})$ has finite (SI) decomposition if and only if T can be written as the direct sum of finitely many strongly irreducible operators. C. L. Jiang and Z. Y. Wang showed that every Cowen-Douglas operator can be written as the direct sum of finitely many strongly irreducible Cowen-Douglas operators (*cf.* [Ji-Wa, Chapter 3]).

Hence it is very important that one determine when two operators A and B in $\mathcal{B}_n(\Omega) \cap (\mathrm{SI})$ are similar.

Definition 0.3 Let $T \in \mathcal{L}(\mathcal{H})$ have finite (SI) decomposition and $P = \{P_i\}_{i=1}^n$ and $Q = \{Q_i\}_{i=1}^m$ be two unit finite (SI) decompositions of T. We say T has unique (SI) decomposition up to similarity if the following conditions are satisfied:

1. $m = n$.
2. There exists an invertible operator X in $\mathcal{A}'(T)$ and a permutation $\pi \in S_n$ such that $XQ_{\pi(i)}X^{-1} = P_i$ for $1 \le i \le n$.

For any unital Banach algebra $\mathcal{A}$, its K_0-group $K_0(\mathcal{A})$ is defined through an Abelian semi-group $V(\mathcal{A})$. Here we briefly recall the definition of $V(\mathcal{A})$. Two idempotent elements p and q in $M_\infty(\mathcal{A})$, the collection of all finite matrices with entries from $\mathcal{A}$, are said to be equivalent if there are u and v in $M_\infty(\mathcal{A})$ such that $uv = p$ and $vu = q$. The equivalence class containing p is denoted by $[p]$ and the set of all these classes is $V(\mathcal{A})$. $V(\mathcal{A})$ is an Abelian semigroup with the addition defined by

$$[p] + [q] = [\mathrm{diag}(p, q)], \quad \text{where}$$

$\operatorname{diag}(p,q)$ is the matrix $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ (*cf.* [Bl]). If two unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic (in short $\approx$), then $V(\mathcal{A}) \approx V(\mathcal{B})$, in particular, $V(M_n(\mathcal{A})) \approx V(M_n(\mathcal{B}))$.

***Theorem* CFJ1** ([Ca-Fa-Ji]) *Let $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{H}^{(n)}$ denote the orthogonal direct sum of n copies of $\mathcal{H}$ and $A^{(n)}$ the operator $\oplus_1^n A$ acting on $\mathcal{H}^{(n)}$. Then the following statements are equivalent:*

1. *T is similar to (in short $\sim$) $\sum_{i=1}^k \oplus A_i^{(n_i)}$ with respect to the decomposition $\mathcal{H} = \sum_{i=1}^k \oplus \mathcal{H}_i^{(n_i)}$ and $T^{(n)}$ has a unique* (SI) *decomposition up to similarity for each positive integer n, where $1 \le k, n_i < +\infty$; $A_i \in$* (SI) *and $A_i \not\sim A_j$ for $i \ne j$.*
2. *$V(\mathcal{A}'(T)) \approx N^{(k)}$, where $N = (0, 1, 2, \dots)$.*

Theorem CFJ1 gives a method of calculating the K_0-group. In Section 4, we characterize the K_0-group of a class of Banach algebras by using Theorem CFJ1.

For an operator T in $\mathcal{L}(\mathcal{H})$, $\mathcal{M}(\mathcal{A}'(T))$ denotes the set of maximal ideals of $\mathcal{A}'(T)$ and $\operatorname{rad} \mathcal{A}'(T)$ the Jacobson radical of $\mathcal{A}'(T)$.

The paper is organized as follows. In Section 1, we discuss certain properties of operators in $\mathcal{B}_1(\Omega)$ and show that $V(\mathcal{A}'(A)) \approx N$ for A in $\mathcal{B}_1(\Omega)$. In the Section 2, we show that $\mathcal{A}'(A)/\operatorname{rad} \mathcal{A}'(A)$ is commutative and $V(\mathcal{A}'(A)) \approx N$ for A in $\mathcal{B}_n(\Omega) \cap$ (SI) by using the result of Section 1. In Section 3, we discuss the structure of the commutant of Cowen-Douglas operators. Using the results of Section 2 and Section 3, we obtain the main result of the paper.

Theorem 4.4 *Two strongly irreducible Cowen-Douglas operators A and B are similar if and only if there is a group isomorphism $\alpha\colon K_0(\mathcal{A}'(A)) \to K_0(\mathcal{A}'(B))$ that satisfies*

1. *$\alpha(V(\mathcal{A}'(A))) = V(\mathcal{A}'(B))$;*
2. *$\alpha[I_{\mathcal{A}'(A)}] = [I_{\mathcal{A}'(B)}]$;*
3. *there exist two non-zero idempotent operators,*

$$p \in \mathcal{M}_\infty(\mathcal{A}'(A)) \quad \text{and} \quad q \in \mathcal{M}_\infty(\mathcal{A}'(B)), \quad \text{satisfying } \alpha[p] = [q],$$

where p and q are equivalent in $\mathcal{M}_\infty(\mathcal{A}'(A \oplus B))$.

To show the above results, we need to introduce the notation of Hermitian holomorphic vector bundle. Let Λ be a manifold with a complex structure and n be a positive integer. A rank n holomorphic vector bundle over Λ consists of a manifold E with a complex structure together with a holomorphic map π from E onto Λ such that each fibre $E_z = \pi^{-1}(z)$ is isomorphic to C^n and such that for each z_0 in Λ there exists a neighborhood Δ of λ_0 and holomorphic functions $e_1(z), \dots, e_n(z)$ from Δ to E whose values form a basis for E_z at each z in Δ. The functions $e_1, \dots, e_n$ are said to be frame for E on Δ. The bundle is said to be trivial if Δ can be taken to be all of Λ.

For T an operator in $\mathcal{B}_n(\Omega)$, the mapping $z \to \ker(T - z)$ defines a rank n holomorphic vector. Let (E_T, π) denote the sub-bundle of the trivial bundle $\Omega \times \mathcal{H}$ defined by

$$E_T = \{(z, x) \in \Omega \times \mathcal{H} \mid x \in \ker(T - z) \text{ and } \pi(z, x) = z\}.$$

That E_T is a complex bundle over Ω is due to Šubin (*cf.* [Su]).

Since all holomorphic bundles over Ω are trivial as holomorphic bundles by Grauert's Theorem *cf.* [Gr] and the fact that all such bundles over Ω are topologically trivial.

A Hermitian holomorphic vector bundle E over Λ is a holomorphic vector bundle such that each fibre E_z is an inner product space. Obviously, E_T is a Hermitian holomorphic vector bundle over Ω for T in $\mathcal{B}_n(\Omega)$ *cf.* [Co-Do, Corollary 1.12].

For $\mathcal{H}$ a separable complex Hilbert space and n a positive integer, let $G_r(n, \mathcal{H})$ denote the Grassmann manifold, the set of all n-dimensional subspaces of $\mathcal{H}$.

For Λ an open connected subspace of C^k, we shall say that a map $f\colon \Lambda \to G_r(n, \mathcal{H})$ is holomorphic at λ_0 in Λ if there exists a neighborhood of λ_0 and n holomorphic $\mathcal{H}$-valued functions $e_1(z), \dots, e_n(z)$ on Δ such that $f(z) = \bigvee_{j=1}^{k}\{e_j(z)\}$ for z in Δ. If $f\colon \Lambda \to G_r(n, \mathcal{H})$ is a holomorphic map, then a natural n-dimensional Hermitian holomorphic vector E_f is induced over Λ, *i.e.,*

$$E_f = \{(x, z) \in \mathcal{H} \times \Lambda \mid x \in f(z)\}$$

and

$$\Phi\colon E_f \to \Lambda$$

where

$$\Phi(x, z) = z \in \Lambda.$$

Now, given two holomorphic maps f and $g\colon \Lambda \to G_r(n, \mathcal{H})$, we have two bundles E_f and E_g over Λ. If there exists a unitary U on $\mathcal{H}$ such that $g = Uf$, then f and g are said to be congruent. If there exists a holomorphic isometric bundle map from $E_f|_\Delta$ onto $E_g|_\Delta$ for some open subset Δ in Λ, then E_f and E_g are said to be locally equivalent.

Recalling that a bundle map Φ from E to E is a holomorphic map such that $\Phi(z) = \Phi|_{E_z}$ is a linear endomorphism on the fibre E_z over z in Ω and we shall say that Φ is bounded if $sup_{z\in\Omega}\|\Phi(z)\| < \infty$. We denoted the collection of bounded bundle endomorphisms on E by $H^\infty_{\mathcal{L}(E)}(\Omega)$.

Rigidity Theorem [Co-Do, Theorem 2.2] *Let Λ be an open connected subset of C^k and f and g be holomorphic maps from Λ to $G_r(n, \mathcal{H})$ such that $\bigvee_{z\in\Lambda} f(z) = \bigvee_{z\in\Lambda} g(z) = \mathcal{H}$. Then f and g are congruent if and only if E_f and E_g are locally equivalent Hermitian holomorphic vector bundle over Λ.*

The Rigidity Theorem plays a very important role in this paper.

Important in the study of an operator T is an explicit characterization of its commutant $\mathcal{A}'(T)$. Cowen and Douglas showed that the commutant of an operator in $\mathcal{B}_1(\Omega)$ can always be identified with a subalgebra of $H^\infty(\Omega)$ of bounded holomorphic functions on Ω and for T in $\mathcal{B}_n(\Omega)$ there is a contractive monomorphism Γ_T from $\mathcal{A}'(T)$ into $H^\infty_{\mathcal{L}(E_T)}(\Omega)$, where $\Gamma_T X = X|_{\ker(T-z)}$ for X in $\mathcal{A}'(T)$ and z in Ω, or $(\Gamma_T X)(z) = X|_{\ker(T-z)} := X(z)$ for z in Ω.

For T in $\mathcal{B}_n(\Omega)$, let $(e_1(z), \dots, e_n(z))$ be a holomorphic frame of E_T. Fix z_0 in Ω. Let $\mathcal{H}_1 = \ker(T - z_0), \mathcal{H}_2 = \ker(T - z_0)^2 \ominus \ker(T - z_0), \dots, \mathcal{H}_m =$

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$\ker(T - z_0)^m \ominus \ker(T - z_0)^{m-1}, \dots$. Cowen and Douglas obtained the following results:

***Theorem* CD2**

1. $\sum_{k=1}^{m} \oplus \mathcal{H}_k = \bigvee \{e_j^{(k)}(z_0)\}_{j=1}^{n}{}_{k=1}^{m-1}$
2. $\sum_{k=1}^{\infty} \oplus \mathcal{H}_k = \mathcal{H}$;
3. $\{e_j^{(k)}(z_0)\}_{j=1}^{n}{}_{k=1}^{m-1}$ *form a base of* $\ker(T - z_0)^m$, $m = 1, 2, \dots$.

where $e_j^{(k)}(z_0)$ *denotes the k-th differential quotient of* $e_j(z)$ *at* $z = z_0$.

Theorem CD2 is used in Section 2.

1 The Cowen-Douglas Operators with Index 1

Main result of the Section is the following:

Theorem 1.1 *Let* $A \in \mathcal{B}_1(\Omega)$. *Then* $V(\mathcal{A}'(A)) \approx N$ *and* $K_0(\mathcal{A}'(A)) \approx Z$, *where* Z *denotes the group of integers.*

By Theorem 1.1 and Theorem CFJ1, we obtain immediately the following:

Corollary 1.2 *Let* $A \in \mathcal{B}_1(\Omega)$ *and* n *be a positive integer. Then* $A^{(n)}$ *has unique strongly irreducible decomposition up to similarity.*

In order to prove Theorem 1.1, we need the following auxiliary results.

***Theorem* CFJ2** *Let* P *be an idempotent operator in* $\mathcal{A}'(T_{z^*}^{(n)})$. *Then the following holds* (cf. [Ca-Fa-Ji, Theorem 2.1]).

1. *Set* $A = T_{z^*}^{(n)}|_{\operatorname{ran} P}$ *and* $d = \dim\ker A$. *Then there exists a unitary operator* U *such that*

$$UP^*U^* = \begin{bmatrix} I_{H^2(\partial D)^{(n-d)}} & R_{12} \\ 0 & 0 \end{bmatrix} \begin{matrix} H^2(\partial D)^{(n-d)} \\ H^2(\partial D)^{(d)} \end{matrix}$$

2. *A is unitarily equivalent to* $T_{z^*}^{(d)}$ *(denoted by* $A \cong T_{z^*}^{(d)}$*), where* $I_{H^2(\partial D)^{(n-d)}}$ *denotes the identity operator on* $H^2(\partial D)^{(n-d)}$.

Lemma 1.3 *Let* $e(z)$ *be a holomorphic vector-value function from* D *to* $H^2(\partial D)$ *satisfying* $(T_{z^*} - z)e(z) = 0$ *for* z *in* D, *and let*

$$e_k(z) = (0, \dots, 0, \overset{(k)}{e(z)}, 0, \dots, 0) \in H^2(\partial D)^{(n)}, \quad k = 1, 2, \dots, n.$$

If the following system of equations

$$\begin{cases} f_1(z) = a_{11}(z)e_1(z) + \cdots + a_{1n}(z)e_n(z) \\ \quad\vdots \\ f_m(z) = a_{m1}(z)e_1(z) + \cdots + a_{mn}(z)e_n(z) \end{cases}$$

satisfy

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(1) *for each z in D, $\{f_1(z),\dots,f_m(z)\}$ is linearly independent;*
(2) *each $a_{ij}(z)$ is a holomorphic function on D, $i=1,2,\dots,m$, $j=1,2,\dots,n$;*
(3) *for $\mathcal{M}=\bigvee_{z\in D}\{f_1(z),\dots,f_m(z)\}$, $T^{(n)}_{z^*}|_{\mathcal{M}}\in\mathcal{B}_m(D)$ and there exists an $\mathcal{N}$, the invariant subspace of $T^{(n)}_{z^*}$ such that $\mathcal{M}+\mathcal{N}=H^2(\partial D)^{(n)}$ and $\mathcal{M}\cap\mathcal{N}=\{0\}$.*

Then there exist $\{g_{ij}(z)\}_{i=1}^{m}{}_{j=1}^{n}$, a family of holomorphic functions on D such that

$$\begin{bmatrix} u_1(z)\\ \vdots\\ u_m(z)\end{bmatrix}=\begin{bmatrix} g_{11}(z) & \cdots & g_{1m}(z)\\ \vdots & & \vdots\\ g_{m1}(z) & \cdots & g_{mm}(z)\end{bmatrix}\begin{bmatrix} f_1(z)\\ \vdots\\ f_m(z)\end{bmatrix}$$

is an orthogonal system for z in D.

Proof Since $\mathcal{M}$ and $\mathcal{N}$ are invariant subspace of $T^{(n)}_{z^*}$, $\mathcal{M}^\perp$ and $\mathcal{N}^\perp$ are invariant subspaces of $T^{(n)}_{z}$. Hence there exists an idempotent P in $\mathcal{A}'(T^{(n)}_z)$ such that ran $p=\mathcal{N}^\perp$. By Theorem CFJ2, there exists a unitary operator U such that

$$UPU^*=\begin{bmatrix} I_{H^2(\partial D)^{(n-m)}} & R_{12}\\ 0 & 0\end{bmatrix}\begin{matrix} H^2(\partial D)^{(n-m)}\\ H^2(\partial D)^{(m)}\end{matrix} \tag{1.1}$$

So

$$UP^*U^*=\begin{bmatrix} I_{H^2(\partial D)^{(n-m)}} & 0\\ R_{12}^* & 0\end{bmatrix}\begin{matrix} H^2(\partial D)^{(n-m)}\\ H^2(\partial D)^{(m)}\end{matrix}$$

and

$$U(I-P^*)U^*=\begin{bmatrix} 0 & 0\\ -R_{12}^* & I_{H^2(\partial D)^{(m)}}\end{bmatrix}\begin{matrix} H^2(\partial D)^{(n-m)}\\ H^2(\partial D)^{(m)}\end{matrix} \tag{1.2}$$

Note that $(I-P^*)\in\mathcal{A}'(T^{(n)}_{z^*})$ and $\operatorname{ran}(I-P^*)=\mathcal{M}$.

The equality (1.2) shows that $\operatorname{ran}(U(I-P^*)U^*)=H^2(\partial D)^{(m)}$. Set

$$u_j(z)=U^*e_j(z);\ j=1,2,\dots,m \text{ and } z\in D$$

Then $u_j(z)_{j=1}^m$ are an orthogonal system for each z in D and

$$\bigvee_{z\in D} u_1(z),u_2(z),\dots,u_m(z)=\mathcal{M}.$$

Set $T_1=T^{(n)}_{z^*}|_{\mathcal{M}}$. A simple computation shows that

$$\bigvee(u_1(z),u_2(z),\dots,u_m(z))=\ker(T_1-z) \text{ for } z\in D$$

Thus we can find $\{g_{ij}(z)\}_{i,j=1}^m$ satisfying the required properties of Lemma 1.3. ∎

Lemma 1.4 *Let $T = T_{z^*}^{(n)} \in \mathcal{L}(H^2(\partial D)^{(n)})$. Then the mapping Γ_T determines an isometric isomorphism from $\mathcal{A}'(T)$ onto $M_n(H^\infty(D))$, where $(\Gamma_T X)(z) = X(z) = X|_{\ker(T-z)}$ for $X \in \mathcal{A}'(T)$ and z in D.*

Proof Since T_z is a subnormal and $\sigma(T_z) = D$, $\bar{D}$ is a spectral set for T_z. Hence $\bar{D}$ is a spectral set for T_{z^*}. By [Co-Do, Proposition 1.27], $\Gamma_{T_{z^*}}$ is an isometric isomorphism from $\mathcal{A}'(T_{z^*})$ onto $H^\infty(D)$. So Γ_T is an isometric isomorphism from $\mathcal{A}'(T)$ onto $M_n(H^\infty(D))$. ■

Lemma 1.5 *Let $A \in \mathcal{L}(H^2(\partial D))$ and $A \in \mathcal{B}_1(\Omega)$ and let $T = A^{(n)}$ and P be an idempotent in $\mathcal{A}'(T)$. If $A_1 = T|_{PH^2(\partial D)^{(n)}} \in \mathcal{B}_d(\Omega)$, then $A_1 \cong A^{(d)}$.*

Proof Without loss of generality, we may assume that $D \subset \Omega$. Let $v(z)$ be a holomorphic frame of $\ker(A - z)$ and set

$$v_k(z) = (0, \dots, 0, \overset{(k)}{v(z)}, 0, \dots, 0), k = 1, 2, \dots, n.$$

Suppose that $\{u_1(z), \dots, u_d(z)\}$ is a holomorphic frame of E_{A_1} on D. Then

$$\begin{aligned} u_1(z) &= f_{11}(z)v_1(z) + \cdots + f_{1n}(z)v_n(z) \\ &\vdots \\ u_d(z) &= f_{d1}(z)v_1(z) + \cdots + f_{dn}(z)v_n(z) \end{aligned}$$

Consider that $P(z) = (\Gamma_T P)(z) = P|_{\ker(T-z)}$, z in D. Then

$$P(z) = \begin{bmatrix} c_{11}(z) & \cdots & c_{1n}(z) \\ \vdots & & \vdots \\ c_{n1}(z) & \cdots & c_{nn}(z) \end{bmatrix} \begin{matrix} \ker(A-z) \\ \vdots \\ \ker(A-z) \end{matrix} .$$

It is easily seen that each $c_{ij}(z)$ is a holomorphic function on $\bar{D}$ and $P^2(z) = P(z)$ for z in D, furthermore ran $p(z) = \bigvee\{u_1(z), \dots, u_d(z)\}$ for z in $\bar{D}$. Let $X(z)$ be a holomorphic invertible operator from $\ker(T - z)$ onto $\ker(T_{z^*}^{(n)} - z)$ for z in D such that $X(z)v_j(z) = e_j(z)$.

Then $\tilde{P}(z) := X(z)P(z)X^{-1}(z)$. Furthermore $\tilde{P}(z)$ is an idempotent element of $M_n(H^\infty(D))$ and

$$\operatorname{ran} \tilde{P}(z) = \bigvee\{\tilde{u}_1(z), \dots, \tilde{u}_d(z)\},$$

where

$$\begin{aligned} \tilde{u}_1(z) &= f_{11}(z)e_1(z) + \cdots + f_{1n}(z)e_n(z) \\ &\vdots \\ \tilde{u}_d(z) &= f_{d1}(z)e_1(z) + \cdots + f_{dn}(z)e_n(z) \end{aligned} \tag{1.3}$$

where $(e_1(z),\dots,e_n(z))$ is given in Lemma 1.3.

Obviously, $\{\tilde{u}_1(z),\dots,\tilde{u}_d(z)\}$ is linearly independent for each z in D. By Lemma 1.3, there exist $\{k_{ij}(z)\}_{i=1}^{d}{}_{j=1}^{d}$ such that

$$(1.4)\qquad \begin{aligned} \tilde{g}_1(z) &= k_{11}(z)\tilde{u}_1(z)+\cdots+k_{1d}(z)\tilde{u}_d(z)\\ &\vdots\\ \tilde{g}_d(z) &= k_{d1}(z)\tilde{u}_1(z)+\cdots+k_{dk}(z)\tilde{u}_d(z)\end{aligned}$$

and

$$\langle \tilde{g}_k(z),\tilde{g}_{k'}(z)\rangle = 0,\, k\neq k' \text{ and } z \text{ in } D.$$

Set

$$(1.5)\qquad \begin{aligned} g_1(z) &= k_{11}(z)u_1(z)+\cdots+k_{1d}(z)u_d(z)\\ &\vdots\\ g_d(z) &= k_{d1}(z)u_1(z)+\cdots+k_{dd}(z)u_d(z)\end{aligned}$$

By (1.3), (1.4) and (1.5), we have

$$(1.6)\qquad \begin{aligned} \tilde{g}_1(z) &= \tilde{f}_{11}(z)e_1(z)+\cdots+\tilde{f}_{1n}(z)e_n(z)\\ &\vdots\\ \tilde{g}_d(z) &= \tilde{f}_{d1}(z)e_1(z)+\cdots+\tilde{f}_{dn}(z)e_n(z)\end{aligned}$$

and

$$(1.7)\qquad \begin{aligned} g_1(z) &= \tilde{f}_{11}(z)v_1(z)+\cdots+\tilde{f}_{1n}(z)v_n(z)\\ &\vdots\\ g_d(z) &= \tilde{f}_{d1}(z)v_1(z)+\cdots+\tilde{f}_{dn}(z)v_n(z)\end{aligned}$$

Since $\langle \tilde{g}_k(z),\tilde{g}_{k'}(z)\rangle = 0$ for $k\neq k'$, $\sum_{i=1}^{n} f_{ki}f_{k'i}=0$ for $k\neq k'$. So we can deduce that $\langle g_k(z),g_{k'}(z)\rangle = 0$ for $k\neq k'$. Obviously, $(g_1(z),\dots,g_d(z))$ is a holomorphic frame of E_{A_1} on D.

Set $\mathcal{M}=\bigvee_{z\in D}(g_1(z),\dots,g_d(z))$. Then by (1.3) and (1.4)

$$\mathcal{M}=\bigvee_{z\in D}(\tilde{u}_1(z),\dots,\tilde{u}_d(z)) = \operatorname{ran}\tilde{P}(z).$$

By Lemma 1.4 and Theorem CFJ2, we find an idempotent operator $\tilde{P}$ in $\mathcal{A}'(T_{z^*}^{(n)})$ and a unitary operator U such that $\tilde{P}|_{\ker(T_{z^*}^{(n)}-z)} = \tilde{P}(z)$ and $U(T_Z^{*(n)}|_{\mathcal{M}})U^* =$

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$T^{(d)}_{z^*}$. This shows that there exists a holomorphic isometric bundle map $U(z)$ from $\bigvee_{z\in D}(\tilde{g}_1(z),\dots,\tilde{g}_d(z))$ to $\bigvee_{z\in D}(e_1(z),\dots,e_d(z))$ and

$$U(z)(\tilde{g}_1(z),\dots,\tilde{g}_d(z)) = (e_1(z),\dots,e_d(z))\begin{bmatrix} Q_{11}(z) & \cdots & Q_{1d}(z)\\ \vdots & & \vdots \\ Q_{d1}(z) & \cdots & Q_{dd}(z)\end{bmatrix} \tag{1.8}$$

where each $Q_{ij}(z)$ is holomorphic function on D.

For each $Y(z) = \alpha_1(z)\tilde{g}_1(z) + \cdots + \alpha_d(z)\tilde{g}_d(z) \in \mathcal{M}$ and using (1.7) and (1.8), we have

$$\begin{aligned} \|Y(z)\| &= \Big(\sum_{k=1}^{n} |\alpha_1(z)\tilde{f}_{1k}(z) + \cdots + \alpha_d(z)\tilde{f}_{dk}(z)|^2\Big)^{\frac{1}{2}}\|e(z)\| \\ &= \Big(\sum_{k=1}^{d} |\alpha_1(z)Q_{1k}(z) + \cdots + \alpha_d(z)Q_{dk}(z)|^2\Big)^{\frac{1}{2}}\|e(z)\|. \end{aligned} \tag{1.9}$$

Define a map $U'(z)$ from $\ker(A_1 - z)$ onto $\ker(A^{(d)} - z)$ below.

$$U'(z)(g_1(z),\dots,g_d(z)) = (v_1(z),\dots,v_d(z))\begin{bmatrix} Q_{11}(z) & \cdots & Q_{1d}(z)\\ \vdots & & \vdots \\ Q_{d1}(z) & \cdots & Q_{dd}(z)\end{bmatrix} \text{ for } z\in D.$$

Noting that $\|v_i(z)\| = \|v(z)\|$ and using (1.9), we can deduce that $U'(z)$ is a holomorphic isometric bundle map from $\ker(A_1 - z)$ onto $\ker(A^{(d)} - z)$. By the Rigidity Theorem we have $A_1 \cong A^{(d)}$. Now the proof of Lemma 1.5 is completed. ■

Proof of Theorem 1.1 By Theorem CFJ1, we need only to show that if $A_1 = A^{(n)}|_{P\mathcal{H}^{(n)}}$ is strongly irreducible for each positive integer n and idempotent P in $\mathcal{A}'(A^{(n)})$, then $A_1 \sim A$. It is a straightforward conclusion of Lemma 1.5. ■

Proposition 1.6 *Let $A \in \mathcal{B}_1(\Omega)$ and $T = A^{(n)} \in \mathcal{L}(\mathcal{H}^{(n)})$. And let $P_1,\dots,P_k$ be idempotent operators in $\mathcal{A}'(T)$ and* dim ran $p_i(z) = l_i$, $i = 1,2,\dots,k$. *If $P_iP_j = \delta_{ij}P_i$ and $\sum_{i=1}^{k} P_i = I_{\mathcal{H}^{(n)}}$, then there exists an invertible operator X in $\mathcal{A}'(T)$ such that*

$$XP_jX^{-1} = (0_{\mathcal{H}^{(l_1)}},\dots,0_{\mathcal{H}^{(l_{j-1})}}, I_{\mathcal{H}^{(l_j)}},\dots,0_{\mathcal{H}^{(l_k)}})$$

where $0_{\mathcal{H}^{(k)}}$ denotes 0 operator on $\mathcal{H}^{(k)}$.

Proof It is easily seen that $^{(n)}$ = ran $p_1 \dotplus$ ran $p_2 \dotplus \cdots \dotplus$ ran p_k, where $\dotplus$ denotes direct sum. By Lemma 1.5, there exists a unitary operator $u_i \in (p_i{}^{(n)}, {}^{(l_i)})$ for each a_i $(= t|_{p_i{}^{(n)}})$ such that

$$u_ia_iU_i^* = A^{(l_i)}.$$

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Let $X = U_1 \dotplus U_2 \dotplus \cdots \dotplus U_k$. Then $X \in \mathcal{A}'(T)$ is invertible and

$$XTX^{-1} = A^{(l_1)} \oplus A^{(l_2)} \oplus \cdots \oplus A^{(l_k)} = A^{(n)}.$$

A simple computation shows that

$$XP_jX^{-1} = (0_{\mathcal{H}^{(l_1)}}, \ldots, I_{\mathcal{H}^{(l_j)}}, \ldots, 0_{\mathcal{H}^{(l_k)}}) \ j = 1, 2, \ldots, k$$

The proof of Proposition 1.6 is complete. ∎

Similar to the proof of Lemma 1.3, we have

Proposition 1.7 *Let $\{P_k(z)\}_{k=1}^m$ be a family of holomorphic idempotent elements in $M_n(H^\infty(D))$ such that $\sum_{k=1}^m P_k(z) = I_n$ and $P_i(z)P_j(z) = \delta_{ij}P_j(z)$ for z in Ω. Then there exists a holomorphic invertible element $X(z)$ in $M_n(H^\infty(\Omega))$ such that $X^{-1}(z)P_j(z)X(z) = I_{C^{k_j}} \oplus 0_{n-k_j}$ and $X(z)|_{\operatorname{ran} p_j(z)}$ is a holomorphic isometric bundle map from* ran $p_j(z)$ *onto* $\ker(T_{z^*}^{(k_j)} - z)$, *for $j = 1, 2, \ldots, m$, where $k_j =$* rank $P_j(z)$.

Proposition 1.8 (Kato Theorem [Ka]) *Let Ω be a bounded connected open subset of C and* inter $\bar{\Omega} =$ inter Ω. *Assume that $\{P_k(z)\}_{k=1}^m$ are a family of holomorphic idempotents in $M_n(H^\infty(\Omega))$ such that $\sum_{k=1}^m P_k(z) = I_n$ and $P_i(z)P_j(z) = \delta_{ij}P_i(z)$ for z in Ω and $1 \leq i, j \leq m$. Then fixing a z_0 in Ω, there exists a holomorphic invertible element $X(z)$ in $M_n(H^\infty(\Omega))$ such that*

$$X(z)P_k(z)X^{-1}(z) = P_k(z_0) \text{ and } X(z_0) = I_n, \ k = 1, 2, \ldots, m.$$

Proof Let $L_a^2(\Omega^*)$ denote the subspace of $L^2(\Omega^*)$ consisting of those functions that are analytic. The Bergman operator for Ω^* is the operator $B_z f = zf$ defined on $L_a^2(\Omega^*)$, where $\Omega^* = \{z \mid \bar{z} \in \Omega\}$. It is not difficult to show that $B_z^* \in \mathcal{B}_1(\Omega)$ *cf.* [Fo-Ji]. By the Yoshino Theorem [Con], we can show that $\mathcal{A}'(B_z) \approx \mathcal{A}'(B_z^*) \approx H^\infty(\Omega)$. Since B_z is subnormal, $\bar{\Omega}$ is a spectral set for B_z^*. Set $A = B_z^*$ and $T = A^{(n)}$. Similar to the proof of Lemma 1.3, we can deduce that Γ_T is an isometric isomorphism from $\mathcal{A}'(T)$ onto $M_n(H^\infty(\Omega))$. So $\{P_k(z)\}_{k=1}^m$ can be viewed as idempotent elements in $M_n(H^\infty(\Omega))$. For fixed z_0 in Ω, it is not difficult to see that we can find an invertible X_1 matrix in $M_n(C)$ such that

$$X_1P_k(z_0)X_1^{-1} = \big(0_{\ker(A-z_0)^{(l_1)}}, \ldots, \overset{(k)}{I_{\ker(A-z_0)^{(l_k)}}}, \ldots, 0_{\ker(A-z_0)^{(l_m)}}\big) \ k = 1, 2, \ldots, m$$

where $l_k = \dim \operatorname{ran} p_k(z_0)$ and $l_1 + l_2 + \cdots + l_m = n$.

Obviously, $X_1P_k(z)X_1^{-1} \in M_n(H^\infty(\Omega))$. By Proposition 1.6, we can find an invertible element $X_2(z)$ in $M_n(H^\infty(\Omega))$ such that

$$X_2(z)X_1P_k(z)X_1^{-1}X_2^{-1}(z) = \big(0_{\ker(A-z)^{(l_1)}}, \ldots, \overset{(k)}{I_{\ker(A-z)^{(l_k)}}}, \ldots, 0_{\ker(A-z)^{(l_m)}}\big)$$
$$k = 1, 2, \ldots, m$$

It is easily seen that $X_2(z_0) = I_{\ker(T-z_0)}$. This completes the proof of Proposition 1.8. ∎

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2 The Cowen-Douglas Operators with Index n

In the following sections we assume always $\bigvee_{z\in\Omega}\ker(T-z)=\mathcal{H}$ for each T in $\mathcal{B}_n(\Omega)$, where $\mathcal{H}$ is a complex separable infinite dimensional Hilbert space.

Definition 2.1 Let $A\in\mathcal{B}_n(\Omega)$ and $B\in\mathcal{A}'(A)$. If $\sigma(B(z))$ is disconnected at $z=z_0\in\Omega$, then there exists a positive number δ such that $\sigma(B(z))$ is disconnected for $z\in\{z\,;\,|z-z_0|<\delta\}\overset{\Delta}{=}D(z_0,\delta)$. Hence,we can find a positive number ε such that $\sigma(B(z))\cap\bar{D}(\lambda(z_0),\varepsilon)=\lambda(z_0)$, $z\in D(z_0,\delta)$, where $\lambda(z_0)$ is an eigenvalue of $B(z_0)$. Let

$$P(z)=\int_{\partial D(\lambda(z_0),\varepsilon)}(B(z)-\lambda)^{-1}\,d\lambda$$

Then $P(z)$ is said to be a holomorphic idempotent element defined on $D(\lambda(z_0),\varepsilon)$ induced by $\mathcal{A}'(A)$. If each holomorphic idempotent element defined on connected open set Φ induced by $\mathcal{A}'(A)$ with $\dim\ker(A|_{\bigvee_{z\in\Phi}P(z)}-z_0)<n$, then call n minimal index of A, or A is said to have minimal index n.

Example 2.2 Let $T\in\mathcal{B}_1(\Omega)$. Then 1 is the minimal index of T.

Example 2.3 Let $f=z(z-\frac{1}{8})$ and let T_f denote the analytic Teoplitz operator. Then $T_{f^*}\in\mathcal{B}_2(\Omega)\cap(SI)$, where $0\in\Omega$ *cf.* [Ji-Wa, Chapter 3]. But 2 is not the minimal index of T_{f^*}, and we can find a connected open set Ω_1 such that $T_{f^*}\in\mathcal{B}_1(\Omega_1)$.

Example 2.4 Let A_1 and A_2 be in $\mathcal{B}_n(\Omega)$ and $T=A_1\oplus A_2$. If n is the minimal index of A_1 and A_2, then $2n$ is the minimal index of T.

By the argument of Section 3 in [Co-Do], we have

Proposition 2.5 *Let $T\in\mathcal{B}_n(\Omega)\cap(SI)$. Then there exists a positive integer $m\le n$ and a connected open set Ω_1 such that $T\in\mathcal{B}_m(\Omega_1)$ and m is the minimal index of T.*

Lemma 2.6 *Let $A\in\mathcal{B}_n(\Omega)$ and n be the minimal index of A. And let $P(z)$ be a holomorphic idempotent element defined on open set Φ induced by $\mathcal{A}'(A)$and* ran $p(z)=k<n$, $z\in\Phi$. *Set $\mathcal{H}_1=\bigvee_{z\in\Phi}$ ran $p(z)$. Then $A|_{\mathcal{H}_1}\in\mathcal{B}_k(\Omega)$.*

Proof Let P be an orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_1$ and $A_1=A|_{\mathcal{H}_1}, A_2=(A^*|_{\mathcal{H}_1^\perp})^*$. Then

$$A=\begin{bmatrix}A_1 & A_{12}\\ 0 & A_2\end{bmatrix}\begin{matrix}\mathcal{H}_1\\ \mathcal{H}_1^\perp\end{matrix}.$$

By [He1, Corollary 1.2], $\sigma_P(A^*)=\varnothing$, where $\sigma_P(A^*)$ denotes the point spectrum of A^*. So $\dim\mathcal{H}_1^\perp=+\infty$. Since $A-z$ is right invertible for z in Ω, we can find an operator

$$B=\begin{bmatrix}B_1 & B_{12}\\ 0 & B_2\end{bmatrix}\begin{matrix}\mathcal{H}_1\\ \mathcal{H}_1^\perp\end{matrix}$$

such that

$$(A-z)B = \begin{bmatrix} A_1 - z & A_{12} \\ 0 & A_2 - z \end{bmatrix} \begin{bmatrix} B_1 & B_{12} \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{bmatrix}.$$

Hence $(A_2 - z)B_2 = I_{\mathcal{H}_1^\perp}$. This shows that $A_2 - z$ is right invertible for z in Ω. It is easily seen that $A_2 \in \mathcal{B}_m(\Omega)$, where $m < n$. Let π be the canonical map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denotes the Calkin algebra. Then

$$\pi(B)\pi(A-z) = \begin{bmatrix} \pi(I_{\mathcal{H}_1}) & 0 \\ 0 & \pi(I_{\mathcal{H}_1^\perp}) \end{bmatrix}.$$

This shows that $\operatorname{ran}(A_1 - z)$ is closed for z in Ω. Let $(e_1(z), \dots, e_k(z))$ be a holomorphic frame of $\operatorname{ran} p(z)$. Then $(A_1 - z)e_j(z) = ze_j(z)$, for $1 \le j \le k$. Note that $\mathcal{H}_1 = \bigvee_{z\in\Phi} \operatorname{ran} p(z)$. Thus $\operatorname{ran}(A_1 - z) = \mathcal{H}_1$, for $z \in \Phi$. This shows that $A_1 \in calB_k(\Omega)$ and hence we complete the proof of Lemma 2.6. ■

Proposition 2.7 *Let $A \in \mathcal{B}_n(\Omega)$. Let $(e_1(z), \dots, e_n(z))$ be a holomorphic frame of E_A on Ω and*

$$\mathcal{M} = \bigvee_{z\in\Omega} \{e_1(z), \dots, e_k(z)\}, k < n.$$

If $A|_{\mathcal{M}} \in \mathcal{B}_k(\Omega)$, then for each z in Ω, $e_j^{(m)}(z) \notin \mathcal{M}$ for $j > k$ and positive integer m.

Proof By Lemma 2.6, $A_1 = A|_{\mathcal{M}} \in \mathcal{B}_k(\Omega)$. So $\dim\ker(A_1 - z)^{m+1} = (m+1)k$. By Theorem CD2, $\ker(A_1 - z)^{m+1} = \bigvee\{e_i^{(t)}(z)\}_{i=1}^{k}{}_{t=1}^{m}$ Assume that $e_j^{(m)}(z) \in \mathcal{M}$ for $j > k$. Since $\mathcal{M} \in \operatorname{Lat} A$ and $e_j^{(m)}(z) \in \ker(A-z)^{m+1}$, $e_j^{(m)}(z) \in \ker(A_1 - z)^{m+1}$, where $\operatorname{Lat} A$ denotes the set of all invariant subspaces of A. Since $\{e_j^{(t)}(z)\}_{j=1}^{n}{}_{t=1}^{m}$ are linealyr independent, $\dim\ker(A_1 - z)^{(m+1)} > (m+1)k$. This contradicts A_1 in $\mathcal{B}_k(\Omega)$. The proof of Proposition 2.7 is complete. ■

Theorem 2.8 *Let $A \in \mathcal{B}_n(\Omega) \cap (\mathrm{SI})$ and n be the minimal index of A. Then $\sigma(\mathcal{B}(z))$ is connected for each B in $\mathcal{A}'(A)$ and z in Ω. Moreover, $\mathcal{A}'(T)/\operatorname{rad}\mathcal{A}'(T)$ is commutative for each T in $\mathcal{B}_n(\Omega) \cap (\mathrm{SI})$.*

Proof Without loss of generality, we may assume that $\bar{D} \subseteq \Omega$. If Theorem 2.8 is not true, then we may assume that there exists an operator B in $\mathcal{A}'(A)$ such that $\sigma(B(0)) = \{\lambda_1, \lambda_2\}$ and $\lambda_1 \neq \lambda_2$. Since $B(z)$ is holomorphic for z in Ω, we can find a positive ε such that

$$\sigma(B(z)) = \{\lambda_1(z), \lambda_2(z)\} \text{ for } z \text{ in } D(0,\varepsilon) := \{z, |z| < \varepsilon\}$$

and $\lambda_1(z) \neq \lambda_2(z)$; $\lambda_1(0) = \lambda_1$ and $\lambda_2(0) = \lambda_2$.

Since $B(z)$ is holomorphic on Ω, we can find a positive ε_1 such that

$$D(\lambda_1, \varepsilon_1) \cap \sigma(B(z)) = \bar{D}(\lambda_1, \varepsilon_1) \cap \sigma(B(z)) = \{\lambda_1(z)\}$$

and

$$D(\lambda_2, \varepsilon_1) \cap \sigma(B(z)) = \bar{D}(\lambda_2, \varepsilon_1) \cap \sigma(B(z)) = \{\lambda_2(z)\}$$

Set

$$P(z) = \int_{\partial D(\lambda_1, \varepsilon_1)} (B(z) - \lambda)^{-1} \, d\lambda, z \in D(0, \varepsilon).$$

Then

$$I - P(z) = \int_{\partial D(\lambda_2, \varepsilon_1)} (B(z) - \lambda)^{-1} \, d\lambda, z \in D(0, \varepsilon).$$

Obviously, $P(z)$ is holomorphic and idempotent and $\dim \operatorname{ran} p(z) = k < n$, hence $\dim \operatorname{ran}(I - P(z)) = n - k$ for z in $D(0, \varepsilon)$. Since n is the minimal index of A, $\mathcal{M} = \bigvee_{z \in D(\lambda, \varepsilon_2)} \operatorname{ran} p(z) \subsetneq \mathcal{H}$. By Lemma 2.6 we may assume that $A_1 = A|_{\mathcal{M}} \in \mathcal{B}_k(\Omega)$ and let $(e_1(z), \dots, e_k(z))$ be a holomorphic frame of E_{A_1} and $(e_{k+1}(z), \dots, e_n(z))$ be a holomorphic frame of $E_2 := \{(x, z), x \in (I - P(z)) \ker(A - z),\ z \in D(0, \varepsilon)\}$. Then $(e_1(z), \dots, e_k(z), e_{k+1}(z), \dots, e_n(z))$ is a holomorphic frame of E_A on $D(0, \varepsilon)$.

Set

$$A_1 = A|_{\mathcal{M}};\ A_2 = (A^*|_{\mathcal{M}^\perp})^*,$$
$$B_1 = B|_{\mathcal{M}};\ B_2 = (B^*|_{\mathcal{M}^\perp})^*.$$

Note that $\mathcal{M} \in \operatorname{Lat} A \cap \operatorname{Lat} B$ and

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^\perp \end{matrix}, \ B = \begin{bmatrix} B_1 & B_{12} \\ 0 & B_2 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^\perp \end{matrix}.$$

Since $AB = BA$, $B_1 A_1 = A_1 B_1$. By Lemma 2.6, $A_1 \in \mathcal{B}_k(\Omega)$ and $A_2 \in \mathcal{B}_{n-k}(\Omega)$. Set $\mathcal{H}_1 = \ker A, \mathcal{H}_2 = \ker A^2 \ominus \ker A, \dots, \mathcal{H}_m = \ker A^m \ominus \ker A^{m-1}, \dots$. By Theorem CD2, $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_m = \bigvee \{e_i^{(j)}(0)\}_{i=1}^{n}{}_{j=1}^{m-1}$ and

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} & \cdots \\ 0 & 0 & A_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \end{matrix},$$

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} & \cdots \\ 0 & B_{22} & B_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \end{matrix}.$$

Set $\mathcal{L}_1 = \ker A_1, \mathcal{L}_2 = \ker A_1^2 \ominus \ker A_1, \dots, \mathcal{L}_m = \ker A_1^m \ominus \ker A_1^{m-1}, \dots$. Using Theorem CD2 again, we can deduce that $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m = \bigvee \{e_i^{(j)}(0)\}_{i=1}^{k}{}_{j=1}^{m-1}$ and

$$A_1 = \begin{bmatrix} 0 & A'_{12} & A'_{13} & \cdots \\ 0 & 0 & A'_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \vdots \end{matrix},$$

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$$B_1 = \begin{bmatrix} B'_{11} & B'_{12} & B'_{13} & \cdots \\ 0 & B'_{22} & B'_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \vdots \end{matrix}.$$

Since $B \in \mathcal{A}'(A)$ and $B_1 \in \mathcal{A}'(A_1)$, we can show that $B_{k+1\,k+1} \sim B_{kk}$ and $B'_{k+1\,k+1} \sim B'_{kk}$, $k = 1, 2, \ldots$. Since $B'_{11} = B|_{\bigvee\{e_1(0),\ldots,e_k(0)\}}$, $\sigma(B'_{11}) = \{\lambda_1\}$ and $\sigma(B'_{kk}) = \{\lambda_1\}, k = 1, 2, \ldots$. Set

$$(\bar{B}_1)_m = \begin{bmatrix} B'_{11} & \cdots & B'_{1m} \\ & \ddots & \\ 0 & & B'_{mm} \end{bmatrix} \begin{matrix} \mathcal{L}_1 \\ \vdots \\ \mathcal{L}_m \end{matrix}.$$

Then $\sigma((\bar{B}_1)_m) = \{\lambda_1\}$. Since $B_{11} = B|_{kerA}, \sigma(B_{11}) = \{\lambda_1, \lambda_2\}$ and then $\sigma(B_{kk}) = \{\lambda_1, \lambda_2\}$. Set

$$\bar{B}_m = \begin{bmatrix} B_{11} & \cdots & B_{1m} \\ & \ddots & \\ 0 & & B_{mm} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_m \end{matrix}.$$

Then $\sigma(\bar{B}_m) = \{\lambda_1, \lambda_2\}$. Set

$$\bar{P}_m = \int_{\partial D(\lambda_1,\varepsilon_1)} (B_m - \lambda)^{-1}\, d\lambda$$

$$P_m = \bar{P}_m \oplus 0_{\sum_{k>m} \oplus \mathcal{H}_k}.$$

Then P_m is an idempotent and $P_m A_m = A_m P_m$, where

$$A_m = \begin{bmatrix} 0 & A_{12} & \cdots & A_{1m} \\ & \ddots & \ddots & \vdots \\ & & \ddots & A_{m-1\,m} \\ & & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{m-1} \\ \mathcal{H}_m \end{matrix} \oplus 0_{\sum_{k>m} \oplus \mathcal{H}_k}.$$

Set $\mathcal{N} = \bigvee_{m=1}^{\infty} \operatorname{ran} p_m$. Then $\mathcal{N} \in \operatorname{Lat} A \cap \operatorname{Lat} B$ and $\mathcal{N} \neq \{0\}$.

Claim 1 $\mathcal{N} = \mathcal{M} = \bigvee_{z \in D(0,\varepsilon)} \{e_1(z), \ldots, e_k(z)\} = \bigvee_{m=1}^{\infty} \mathcal{L}_m$.

Since $\bigvee\{e_i^{(j)}(0)\}_{i=1}^{k}{}_{j=1}^{m-1} \subseteq \bigvee\{e_i^{(j)}(0)\}_{i=1}^{n}{}_{j=1}^{m-1}\ \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m \subseteq \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$. Since $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m \in \operatorname{Lat} B, \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m \in \operatorname{Lat} \bar{B}_m$. Note that

$$\ker(\bar{B}_1 - \lambda_1)^n = \ker(\bar{B}_1 - \lambda_1)^k = \ker A_1 = \bigvee\{e_1(0), \ldots, e_k(0)\}.$$

A simple computation shows that

$$\dim \ker(\bar{B}_m - \lambda_1)^{mn} = mk.$$

For each x in $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_m$, we have $(\bar{B}_m - \lambda_1)^{mn}x = ((\bar{B}_1)_m - \lambda_1)^{mn}x = 0$. This shows that $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_m \subseteq \ker(\bar{B}_m - \lambda_1)^{mn} = \operatorname{ran} \bar{P}_m$. Since $\dim(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_m) = mk$, $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_m = \operatorname{ran} \bar{P}_m$. So $\mathcal{N} = \mathcal{M}$.

Set $\mathcal{K} = \bigvee_{z \in D(0,\varepsilon)} \{e_{k+1}(z), \dots, e_n(z)\}$. Then $\mathcal{K} \in \operatorname{Lat} A \cap \operatorname{Lat} B$.

Set

$$\bar{Q}_m = \int_{\partial D(\lambda_2,\varepsilon)} (\bar{B}_m - \lambda)^{-1}\, d\lambda$$

and

$$Q_m = \bar{Q}_m \oplus 0_{\sum_{k>m} \oplus \mathcal{H}_k}.$$

Similarly, we can obtain that $\mathcal{K} = \bigvee_{m=1}^{\infty} \operatorname{ran} Q_m$.

In order to complete the proof of the lemma, we need to introduce the following two results.

***Lemma* CD3** (Lemma 2.4, [C-D]) *If $f \colon \Omega \to Gr(n, \mathcal{H})$ is a holomorphic curve and $\gamma_1, \gamma_2, \dots, \gamma_n$ are holomorphic cross-sections of the vector bundle E_f defined over Ω such that $\gamma_1(z_0), \dots, \gamma_n(z_0)$ is an orthonormal basis for $f(z_0)$, then there exist holomorphic cross-sections $\bar{\gamma}_1, \dots, \bar{\gamma}_n$ of E_f defined on some open set Δ about z_0 such that $\bar{\gamma}_i(z_0) = \gamma_i(z_0)$ for $i = 1, 2, \dots$ and*

$$\langle \gamma_i^{(k)}(z_0),\ \bar{\gamma}_j(z_0)\rangle = 0 \ \text{ for } 1 \le i, j \le n \text{ and } k = 1, 2, \dots$$

***Lemma* JW** (Lemma 5.7,[Ji-Wa*)] Let $T \in \mathcal{B}_n(\Omega)$. Then $(T^*|_{\ker(B-z_0)^\perp})^*$ is similar to B, where $z_0 \in \Omega$.*

Notice that $(e_1(z), \dots, e_k(z), \dots, e_n(z))$ is a holomorphic frame of E_A, we may use Schmidt orthogonalizing procedure to obtain an orthogonal bases of $\ker A$. Without loss of generality, we may assume that $(e_1(0), \dots, e_k(0), \dots, e_n(0))$ is an orthonomal basis for $\ker(A)$.

Notice that $\lambda_1(z) \ne \lambda_2(z), z \in D(0, \varepsilon_1)$, imitating the argument of the Claim 1, we can obtain the following conclusion by using Lemma CD2 and Lemma JW.

Claim 2 For each $\mathcal{H}_i$, there exist two subspace ${E_1}^i$ and ${E_2}^i$ of $\mathcal{H}_i$ satisfying

a. $\mathcal{H}_i = {E_1}^i + {E_2}^i$ and ${E_1}^i \cap {E_2}^i = 0$,
b. $\sigma(B_{ii}|_{{E_1}^i}) = \lambda_1(0), \sigma(B_{ii}|_{{E_2}^i}) = \lambda_2(0)$,
c. $\bigvee_{i=1}^{m} {E_j}^i \in \operatorname{Lat} B$, $j = 1, 2$,
d. $\bigvee_{i=1}^{\infty} {E_1}^i = \mathcal{M}$, and $\bigvee_{i=1}^{\infty} {E_2}^i = \mathcal{K}$.
e. Let $(g_1^i, \dots, g_k^i)$ and $(g_{k+1}'^i, \dots, g_n'^i)$ be orthogonal basis of E_1^i and E_2^i respectively.

By the Schmidt orthogonalizing procedure we can obtain an orthogonal basis $(g_1^i, \dots, g_k^i, g_{k+1}^i, \dots, g_n^i)$ of $\mathcal{H}_i$, $i = 1, 2, \dots$. Using Properties (a) and (c) and a simple computation, then we have

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} & \cdots \\ 0 & 0 & A_{21} & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \end{matrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} & \cdots \\ 0 & B_{22} & \cdots \\ \vdots & \ddots & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \end{matrix}$$

where

$$A_{ij} = \begin{bmatrix} a_{11}^{ij} & a_{12}^{ij} \\ 0 & a_{22}^{ij} \end{bmatrix}, \quad B_{ij} = \begin{bmatrix} b_{11}^{ij} & b_{12}^{ij} \\ 0 & b_{22}^{ij} \end{bmatrix}.$$

Now, we have the folowing:

Claim 3 $\mathcal{K} + \mathcal{M} = \mathcal{H}$ and $\mathcal{K} \cap \mathcal{M} = \{0\}$.
By Claim 2, we may assume that $\bigvee_{j=1}^{n} e_j^{(m)}(0) = \mathcal{H}_{m+1}$ Set

$$A(d) = \begin{bmatrix} 0 & A_{12} & 0 & \cdots \\ 0 & 0 & A_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \end{matrix}, \quad B(d) = \begin{bmatrix} B_{11} & 0 & \cdots \\ 0 & B_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \end{matrix}.$$

Then $A(d) \in \mathcal{B}_n(\Omega_1)$ (*cf.* [Ji-Li, Proposition 2.1]) and $0 \in \Omega_1 \subset \Omega$. Since $AB = BA, A(d)B(d) = B(d)A(d)$. Since $\sigma(B_{11}) = \{\lambda_1, \lambda_2\}$ and $B_{k+1\,k+1} \sim B_{kk}$, $\sigma(B(d)) = \{\lambda_1, \lambda_2\}$ and $(B(d) - \lambda_1)^k (B(d) - \lambda_2)^{n-k} = 0$. Set

$$P = \int_{\partial D(\lambda_1, \varepsilon_1)} (B(d) - \lambda)^{-1} \, d\lambda.$$

Then P is idempotent. In order to verify Claim 3, we need to verify the following:

Claim 4 ran $p = \mathcal{M}$.
Set

$$\bar{B}_m(d) = \begin{bmatrix} B_{11} & \cdots & 0 \\ & \ddots & \\ 0 & & B_{mm} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_m \end{matrix};$$

$$\bar{P}_m(d) = \int_{\partial D(\lambda_1, \varepsilon_1)} (\bar{B}_m(d) - \lambda)^{-1} \, d\lambda;$$

and

$$P_m(d) = \bar{P}_m(d) \oplus 0_{\sum_{k>m} \oplus \mathcal{H}_k}.$$

Then $P_m(d)_{m=1}^{\infty}$ are uniformly bounded. By the Banach-Alaoglu Theorem, we have $P_m(d) \xrightarrow{WOT} P$. It is easily seen that

$$\operatorname{ran} \bar{P}_1(d) = \operatorname{ran} \bar{P}_1 = \bigvee \{e_1(0), \dots, e_k(0)\}.$$

Note that from $\bigvee \{e_j^{(i)}(0)\}_{j=1}^{k}{}_{i=0}^{m} \in \operatorname{Lat} B$, $\bigvee \{e_j^{(m)}(0)\}_{j=1}^{n} = \mathcal{H}_{m+1}$ and property (e) we can deduce that ran $\bar{P}_2 \subseteq$ ran $\bar{P}_2(d)$. Note that dim ran $p_2 = 2k =$ dim ran $p_2(d)$, so ran $\bar{P}_2 =$ ran $p_2(d)$. By the inductive method, we can show that ran $\bar{P}_m(d) =$ ran $\bar{P}_m$ and then ran $p = \mathcal{M}$. So Claim 4 is true. Similarly, we can show that $\operatorname{ran}(I - P) = \mathcal{K}$ and hence Claim 3 is true. Since n is the minimal index of A, $\mathcal{M}$ and $\mathcal{K}$ are non-trivial subspace of $\mathcal{H}$, but it is a contradiction with strong irreducibility of A.

Thus $\sigma(B(z))$ is connected for each B in $\mathcal{A}'(A)$ and z in Ω and we obtain the first part of the Theorem.

For T in $\mathcal{B}_n(\Omega)\cap(\mathrm{SI})$ we can find a positive integer $m \leq n$ and a connected open set Ω_1 such that $T \in \mathcal{B}_m(\Omega_1)$ and m is the minimal index of T following from Proposition 2.5. Let A and B in $\mathcal{A}'(T)$. Then $AB-BA \in \mathcal{A}'(T)$. By the first part of Theorem 2.8, we can show that

$$\begin{aligned}\sigma\big((AB-BA)(z)\big) &= \sigma\big((A(z)B(z)-B(z)A(z)\big)\\ &= \{0\},\ z \in \Omega_1\end{aligned}$$

Hence

$$\big((A(z)B(z)-B(z)A(z)\big)^m = 0 \text{ and } z \text{ in } \Omega_1.$$

Since $\bigvee_{z\in\Omega_1} \ker(T-z) = \mathcal{H}$, $(AB-BA)^m = 0$. So $\mathcal{A}'(T)/\operatorname{rad}\mathcal{A}'(T)$ is commutative [Au], completing the proof of Theorem 2.8. ■

By Proposition 3.10 we have the following:

Lemma 2.9 *Let $A \in \mathcal{B}_n(\Omega) \cap (\mathrm{SI})$ and $T = A^{(l)}$ and let $(P_1, \dots, P_m)$ be a unit* (SI) *decomposition of T. Then $m = l$ and $T|_{P_i\mathcal{H}^{(l)}} \in \mathcal{B}_n(\Omega)$.*

Lemma 2.10 *Let $A \in \mathcal{B}_n(\Omega) \cap (\mathrm{SI})$ and $T = A^{(l)}$. Let P be an idempotent operator in $\mathcal{A}'(T)$ satisfying $T|_{P\mathcal{H}^{(l)}} \in (\mathrm{SI})$. Then $A_1 = T|_{P\mathcal{H}^{(l)}}$ is similar to A.*

Proof Without loss of generality, we may assume that $\Omega = D$ and n is the minimal index of A, we will show Lemma 2.10 only for case of $n = 2$. Now $T = A \oplus A$. Note that P is an idempotent in $\mathcal{A}'(T)$. By Theorem 2.8 and Lemma 3.4 (see Section 3). We can find an idempotent P_1 in $\mathcal{A}'(T)$ and B in $\operatorname{rad}\mathcal{A}'(T)$ such that $P(z) = P_1(z)+B(z)$, where

$$P_1(z) = \begin{bmatrix} f_{11}(z) & f_{12}(z) \\ f_{21}(z) & f_{22}(z)\end{bmatrix} \begin{matrix}\ker(A-z)\\ \ker(A-z)\end{matrix}, \quad B(z) = \begin{bmatrix} B_{11}(z) & B_{12}(z) \\ B_{21}(z) & B_{22}(z)\end{bmatrix} \begin{matrix}\ker(A-z)\\ \ker(A-z)\end{matrix},$$

where scalar function $f_{ij}(z) \in H^\infty(D)$ and $B_{ij}(z) \in \operatorname{rad}(\mathcal{A}'(A))$. Let $G = -I_{\mathcal{H}^{(2)}} + (2P_1 + B)$. Since $B \in \operatorname{rad}\mathcal{A}'(T)$, G is an invertible operator in $\mathcal{A}'(T)$ and $PG = GP_1$. This shows that $G^{-1}PG = P_1 \in \mathcal{A}'(T)$. Without loss of generality, we now assume that $P = P_1$. That is,

$$P(z) = \begin{bmatrix} f_{11}(z) & f_{12}(z) \\ f_{21}(z) & f_{22}(z)\end{bmatrix} \begin{matrix}\ker(A-z)\\ \ker(A-z)\end{matrix}.$$

Also set

$$P'(z) = \begin{bmatrix} f_{11}(z) & f_{12}(z) \\ f_{21}(z) & f_{22}(z)\end{bmatrix} \begin{matrix}\ker(T_{z^*}-z)\\ \ker(T_{z^*}-z)\end{matrix}.$$

By $T|_{P\mathcal{H}^{(2)}} \in (SI)$ and Lemma 2.9, we can show that $A_1 \in \mathcal{B}_n(D)$ and $tr(P'(z)) = 1$ for each z in D. By Proposition 1.7, we can find a holomorphic invertible element $X(z)$ in $M_2(H^\infty(D))$ such that

$$X(z)P'(z)X^{-1}(z) = \begin{bmatrix} I_C & 0 \\ 0 & 0 \end{bmatrix}$$

$$X(z)(I - P'(z))X^{-1}(z) = \begin{bmatrix} 0 & 0 \\ 0 & I_C \end{bmatrix}$$

and $X(z)|_{\operatorname{ran} p'(z)}$ and $X(z)|_{(I-\operatorname{ran} p'(z))}$ are holomorphic isometric bundle maps from $\operatorname{ran} p'(z)$ and $\operatorname{ran}(I - P'(z))$ onto $\ker(T_{z^*} - z)$, respectively. Set

$$X(z) = \begin{bmatrix} u_{11}(z) & u_{12}(z) \\ u_{21}(z) & u_{22}(z) \end{bmatrix}$$

and

$$\bar{X}(z) = \begin{bmatrix} u_{11}(z)I_{\ker(A-z)} & u_{12}(z)I_{\ker(A-z)} \\ u_{21}(z)I_{\ker(A-z)} & u_{22}(z)I_{\ker(A-z)} \end{bmatrix}.$$

Then

$$\bar{X}(z)P(z)\bar{X}^{-1}(z) = \begin{bmatrix} I_{\ker(A-z)} & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $\bar{X}(z)\ker(T - z) = \ker(T - z)$. Now we claim that $\bar{G}(z) = \bar{X}(z)|_{\operatorname{ran} p(z)}$ is a holomorphic isometric bundle map from $\operatorname{ran} p(z)$ onto $\ker(A - z)$.

Note that $G(z) = X(z)|_{\operatorname{ran} p'(z)}$ is a holomorphic isometric bundle map from $\operatorname{ran} p'(z)$ onto $\ker(T_{z^*} - z)$. Let $e(z)$ be a holomorphic frame of $\ker(T_{z^*} - z)$ and let $t_1(z) = e(z) \oplus 0$ and $t_2(z) = 0 \oplus e(z)$. Then $(t_1(z), t_2(z))$ is a holomorphic frame of $\ker(T_{z^*}^{(2)} - z)$. Let $\bar{A}_1 = T_{z^*}^{(2)}|_{P'(z)H^2(D)^{(2)}}$ and let $l(z)$ be a holomorphic frame of $\ker(\bar{A}_1 - z)$. Then

$$l(z) = \alpha(z)t_1(z) + \beta(z)t_2(z),$$

where $\alpha(z)$ and $\beta(z)$ are analytic functions on D.

Since $G(z)$ is a holomorphic isometry, we can find a holomorphic function $C(z)$ on D such that $G(z)l(z) = C(z)e(z)$ and

$$\begin{aligned} \|l(z)\|^2 &= \left(|\alpha(z)|^2 + |\beta(z)|^2\right)\|e(z)| \\ &= |C(z)|^2\|e(z)\|, \ z \in D \end{aligned}$$

Let $(S_1(z), \dots, S_n(z))$ be a holomorphic frame of $\ker(A - z)$ and let $v_j(z) = S_j(z) \oplus 0$ and $u_j(z) = 0 \oplus S_j(z)$, $j = 1, 2, \dots, m$. Then $(v_1(z), \dots, v_n(z), u_1(z), \dots, u_n(z))$ is a holomorphic frame of $\ker(T - z)$.

Set $f_j(z) = \alpha(z)v_j(z) + \beta(z)u_j(z)$, $j = 1, 2, \dots, n$. Then $(f_1(z), \dots, f_n(z))$ is a holomorphic frame of $\ker(A_1 - z)$ and set $\bar{G}(z)f_j(z) = C(z)v_j(z)$.

Let $k_1(z), \dots, k_n(z)$ be analytic functions on D and let

$$\begin{aligned} g(z) &= k_1(z)f_1(z) + \cdots + k_n(z)f_n(z) \\ &= k_1(z)(\alpha(z)v_1(z) + \beta(z)u_1(z)) + \cdots + k_n(z)(\alpha(z)v_n(z) + \beta(z)u_n(z)). \end{aligned}$$

Then

$$\bar{G}(z)g(z) = C(z)(k_1(z)v_1(z) + k_2(z)v_2(z) + \cdots + k_n(z)v_n(z)) \triangleq g'(z).$$

Note that $\langle v_i(z), v_j(z)\rangle = \langle u_i(z), u_j(z)\rangle = \langle S_i(z), S_j(z)\rangle, z \in D$. So

$$\langle g(z), g(z)\rangle = \sum_{i=1}^{n} |k_i(z)|^2(|\alpha(z)|^2 + |\beta(z)|^2)\|S_i(z)\|^2 + \sum_{i,j=1}^{n} k_i(z)\bar{k_j}(z)(|\alpha(z)|^2 + |\beta(z)|^2)\langle S_i, S_j\rangle;$$

also

$$\langle g'(z), g'(z)\rangle = \sum_{i=1}^{n} |k_i(z)|^2|C(z)|^2\|S_i(z)\|^2 + \sum_{i,j=1}^{n} k_i(z)\bar{k_j}(z)|C(z)|^2\langle S_i, S_j\rangle.$$

This shows that $\|\bar{G}(z)g(z)\| = \|g(z)\|$, and then our claim is verified.

Similarly, we can deduce $\bar{X}(z)|_{\operatorname{ran}(I-P(z))}$ is a holomorphic isometric bundle map from $\operatorname{ran}(I - P(z))$ onto $\ker(A - z)$. By the Rigidity Theorem, we can find two isometric operators $U_1 \in \mathcal{L}(P\mathcal{H}^{(2)}, \mathcal{H} \oplus 0)$ and $U_2 \in \mathcal{L}((I - P)\mathcal{H}^{(2)}, 0 \oplus \mathcal{H})$, such that $X = U_1 + U_2 \in \mathcal{A}'(T)$ and $XPX^{-1} = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & 0 \end{bmatrix}$. By [Ca-Fo-Ji, Lemma 1.12], we have $A_1 {\sim} A$. This completes the proof of the lemma. ∎

Using Theorem CFJ1 and Lemma 2.10, we can obtain immediately the following:

Theorem 2.11 *Let $A \in \mathcal{B}_n(\Omega) \cap$ (SI) and $T = A^{(l)}$. Then T has unique (SI) decomposition up to similarity and*

$$V(\mathcal{A}'(A)) \approx N,\ K_0(\mathcal{A}'(A)) \approx Z.$$

3 The Commutant of Cowen-Douglas Operators

In this section, we assume always that $T = \oplus_{k=1}^{n} T_k$, where $T_k \in \mathcal{B}_{n_k}(\Omega_k) \cap$ (SI) and $\bigvee_{z\in\Omega_k} \ker(T_k - z) = \mathcal{H}_k$. By basic knowledge of operator theory, we can deduce the following properties.

(3.1) $\mathcal{A}'(T) = \{(S_{ij})_{n\times n} \mid S_{ij} \in \ker\tau_{T_i,T_j}, 1 \le i, j \le n\}$ is a unital Banach algebra, where τ_{T_i,T_j} is the Rosenblum operator defined by $\tau_{T_i,T_j}(C) = T_iC - CT_j$ for $C \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$.

(3.2) $\ker\tau_{T_i,T_j}$ is a linear space and $\ker\tau_{T_i,T_i} = \mathcal{A}'(T_i)$ is a unital Banach algebra.

(3.3) Let $e_{\mathcal{A}'(T)}$ denote the unit of $\mathcal{A}'(T)$. Then $e_{\mathcal{A}'(T)} = e_{\mathcal{A}'(T_1)} \oplus \cdots \oplus e_{\mathcal{A}'(T_n)}$.

(3.4) If $S_{ij} \in \ker\tau_{T_i,T_j}$ and $S_{jk} \in \ker\tau_{T_j,T_k}$, then $S_{ij}S_{jk} \in \ker\tau_{T_i,T_k}$. Particularly, if $S_{ij} \in \ker\tau_{T_i,T_j}$ and $S_{ji} \in \ker\tau_{T_j,T_i}$, then $S_{ij}S_{ji} \in \mathcal{A}'(T_i)$.

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(3.5) If $S = (S_{ij})_{n\times n} \in \mathcal{A}'(T)$, then

$$S(i,j) \triangleq \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & S_{ij} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \in \mathcal{A}'(T).$$

By property 3.5, we can define a canonical map Φ_{ij} from $\mathcal{A}'(T)$ onto $\ker \tau_{T_i,T_j}$ by $\Phi_{ij}(S) = S_{ij}$ for $S = (S_{ij})_{n\times n}$ in $\mathcal{A}'(T)$.

(3.6) Φ_{ij} is a linear map and $\Phi_{ii}(S) \in \mathcal{A}'(T_i)$ for S in $\mathcal{A}'(T)$. Throughout this paper, an ideal $\mathcal{J}$ means a proper two-sided ideal.

(3.7) Let $\mathcal{J}$ be an ideal of $\mathcal{A}'(T)$. Define

$$J_{ij} = \left\{ S_{ij} \;\middle|\; S_{ij} \in \ker \tau_{T_i,T_j} \text{ and } \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & S_{ij} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \in \mathcal{J} \right\}$$

Then

(3.7.1) J_{ii} is an ideal of $\mathcal{A}'(T_i)$ or $J_{ii} = \mathcal{A}'(T_i)$;

(3.7.2) J_{ij} is a subspace of $\ker \tau_{T_i,T_j}$;

(3.7.3) $S(i,j) \in \mathcal{J}$ for $S = (S_{ij})_{n\times n} \in \mathcal{J}$.

By property (3.7), we can define a canonical map from $\ker \tau_{T_i,T_j}$ onto $\ker \tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J})$ by $S_{ij} \to [S_{ij}]_{\mathcal{J}}$, where $\ker \tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J})$ is the quotient space of $\ker \tau_{T_i,T_j}$ by subspace $\Phi_{ij}(\mathcal{J})$. If $\mathcal{J}$ is closed, then

$$\mathcal{A}'(T)/\mathcal{J} = \{([S_{ij}]_{\mathcal{J}})_{n\times n} \mid S_{ij} \in \ker \tau_{T_i,T_j}\}$$

is a unital Banach algebra. It is easy to see the canonical map $\Phi_{\mathcal{J}}$ from $\mathcal{A}'(T)$ onto $\mathcal{A}'(T)/\mathcal{J}$ is:

$$\Phi_{\mathcal{J}}((S_{ij})_{n\times n}) = ([S_{ij}]_{\mathcal{J}})_{n\times n}.$$

Let $\mathcal{J}$ be a closed ideal of $\mathcal{A}'(T)$. If $([S_{ij}]_{\mathcal{J}})_{n\times n} = \Phi_{\mathcal{J}}(S) \in \mathcal{A}'(T)/\mathcal{J}$, then

$$\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & [S_{ij}]_{\mathcal{J}} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} = \Phi_{\mathcal{J}}(S(i,j)) \in \mathcal{A}'(T)/\mathcal{J}.$$

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Lemma 3.1 (Lifting Lemma) *Let $T = \oplus_{k=1}^{2} T_k$ and $\mathcal{J}_1$ be an ideal of $\mathcal{A}'(T_1)$. Then there exists an ideal $\mathcal{J}$ of $\mathcal{A}'(T)$ such that $\Phi_{11}(\mathcal{J}) = \mathcal{J}_1$, and if there exists another ideal $\mathcal{J}'$ of $\mathcal{A}'(T)$ such that $\Phi_{11}(\mathcal{J}') = \mathcal{J}_1$, then $\mathcal{J} \subseteq \mathcal{J}'$, where Φ_{11} is defined by property* (3.5).

Proof Let

$$\chi = \left\{ \begin{bmatrix} R_1 & R_2A_{12} \\ A_{21}R_3 & B_{21}R_4B_{12} \end{bmatrix} \middle| \; R_i \in \mathcal{J}_1, i = 1,2,3,4, B_{ij}, \right.$$

$$\left. \text{and } A_{ij} \text{in } \ker \tau_{T_i,T_j}, i, j = 1,2 \right\}$$

and set

$$\mathcal{J} = \{x_1 + x_2 + \cdots + x_n;\ 1 \le n < \infty,\ x_i \in \chi\}.$$

Claim $\mathcal{J}$ is an ideal of $\mathcal{A}'(T)$. It is clear that $\mathcal{J}$ is an additive group.
Set

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \in \mathcal{A}'(T)$$

and

$$X = \begin{bmatrix} R_1 & R_2A_{12} \\ A_{21}R_3 & B_{21}R_4B_{12} \end{bmatrix} \in \chi.$$

Then

$$WX = \begin{bmatrix} W_{11}R_1 & (W_{11}R_2)A_{12} \\ W_{21}R_1 & W_{21}R_2A_{12} \end{bmatrix} + \begin{bmatrix} (W_{12}A_{21})R_3 & (W_{12}B_{21}R_4)B_{12} \\ (W_{22}A_{21})R_3 & (W_{22}B_{21})R_4B_{12} \end{bmatrix}.$$

Note that

$$\begin{bmatrix} W_{11}R_1 & (W_{11}R_2)A_{12} \\ W_{21}R_1 & W_{21}R_2A_{12} \end{bmatrix}, \begin{bmatrix} (W_{12}A_{21})R_3 & (W_{12}B_{21}R_4)B_{12} \\ (W_{22}A_{21})R_3 & (W_{22}B_{21})R_4B_{12} \end{bmatrix} \in \chi.$$

Hence $WX \in \mathcal{J}$. Similarly, we can deduce that $XW \in \mathcal{J}$. Furthermore, we can show that WX and XW are in $\mathcal{J}$ for W in $\mathcal{A}'(T)$ and X in $\mathcal{J}$. Since $\Phi_{11}(X) \in \mathcal{J}_1$ for each $X \in \mathcal{J}$, $e_{\mathcal{A}'(T)} \notin \mathcal{J}$. Thus $\mathcal{J}$ is a proper ideal of $\mathcal{A}'(T)$ and $\Phi_{11}(\mathcal{J}) = \mathcal{J}_1$. Suppose that there exists another ideal $\mathcal{J}'$ of $\mathcal{A}'(T)$ such that $\Phi_{11}(\mathcal{J}') = \mathcal{J}_1$. By property (3.4) and property (3.7), we can deduce that $\mathcal{J} \subseteq \mathcal{J}'$. ∎

Corollary 3.2 *Let $T = \oplus_{k=1}^{n} T_k$ and $\mathcal{J}_1$ be an ideal of $\mathcal{A}'(T_1)$. Then there exists an ideal $\mathcal{J}$ of $\mathcal{A}'(T)$ such that $\Phi_{11}(\mathcal{J}) = \mathcal{J}_1$, and if there exists another ideal $\mathcal{J}'$ of $\mathcal{A}'(T)$ such that $\Phi_{11}(\mathcal{J}') = \mathcal{J}_1$, then $\mathcal{J} \subseteq \mathcal{J}'$.*

Corollary 3.3 *Let $T = \oplus_{k=1}^{n} T_k$ and let $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$. Then $\Phi_{kk}(\mathcal{J}) = \mathcal{A}'(T_k)$ or $\Phi_{kk}(\mathcal{J}) \in \mathcal{M}(\mathcal{A}'(T_k))$ for $k = 1, 2, \ldots, n$.*

Lemma 3.4 *Let $T = \oplus_{k=1}^{n} T_k$ and $S = (S_{ij})_{n\times n} \in \mathcal{A}'(T)$. If for each $R_{ji} \in \ker \tau_{T_j,T_i}$ such that $R_{ji}S_{ij} = 0$, then $S(i, j) \in \mathrm{rad}(\mathcal{A}'(T))$.*

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Proof If

$$R = \begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & & \vdots \\ R_{n1} & \cdots & R_{nn} \end{bmatrix},$$

then

$$RS(i,j) = \begin{bmatrix} 0 & \cdots & R_{1i}S_{ij} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & R_{ji}S_{ij} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & R_{ni}S_{ij} & \cdots & 0 \end{bmatrix}.$$

Since $R_{ji}S_{ij} = 0$, $(RS(i,j))^n = 0$. This shows that $S(i,j) \in \operatorname{rad} \mathcal{A}'(T)$. ■

Corollary 3.5 *Let $T = \oplus_{k=1}^{n} T_k$. Then $\Phi_{kk}(\operatorname{rad} \mathcal{A}'(T)) = \operatorname{rad}(\mathcal{A}'(T_k))$ for $k = 1, 2, \dots, n$.*

Corollary 3.6 *Let $T = \oplus_{k=1}^{n} T_k$ and $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$ and $S_{ij} \in \ker \tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J})$. If $S_{ij}r_{ji} = 0$ for each $r_{ji} \in \ker \tau_{T_j,T_i}/\Phi_{ji}(\mathcal{J})$, then $S_{ij} = 0$.*

Theorem 3.7 *Let $T = \oplus_{k=1}^{n} T_k$. Then for each $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$, there exists a positive integer $l_{\mathcal{J}} \leq n$ such that $\mathcal{A}'(T)/\mathcal{J} \approx M_{l_{\mathcal{J}}}(C)$. Furthermore, if $T_k \sim T_1$ for $k = 1, 2, \dots, n$, then $\mathcal{A}'(T)/\mathcal{J} \approx M_n(C)$ for every $\mathcal{J}$ in $\mathcal{M}(\mathcal{A}'(T))$.*

Proof By Corollary 3.3, $\Phi_{kk}(\mathcal{J}) \in \mathcal{M}(\mathcal{A}'(T_k))$ or $\Phi_{kk}(\mathcal{J}) = \mathcal{A}'(T_k)$ for $k = 1, 2, \dots, n$. By Theorem 2.8, $\mathcal{A}'(T_k)/\operatorname{rad} \mathcal{A}'(T_k)$ is commutative for $k = 1, 2, \dots, n$. So $\mathcal{A}'(T_k)/\Phi_{kk}(\mathcal{J}) \approx C$ or $\mathcal{A}'(T_k)/\Phi_{kk}(\mathcal{J}) = \{0\}$ for $k = 1, 2, \dots, n$. Without loss of generality, we may assume that there exists an integer $l_{\mathcal{J}} \leq n$ such that

$$\mathcal{A}'(T_k)/\Phi_{kk}(\mathcal{J}) \approx C,\ k = 1, 2, \dots, l_{\mathcal{J}}$$

and

$$\mathcal{A}'(T_k)/\Phi_{kk}(\mathcal{J}) = \{0\},\ k = l_{\mathcal{J}} + 1, \dots, n.$$

Now

$$\mathcal{A}'(T)/\mathcal{J} = \{([S_{ij}]_{\mathcal{J}})_{n\times n}; S_{ij} \in \ker \tau_{T_i,T_j} \text{ and } [S_{kk}]_{\mathcal{J}} = 0, \text{ for } l_{\mathcal{J}} < i \leq n\}.$$

By property (3.4), $0 = [S_{ij}R_{ji}]_{\mathcal{J}} = [S_{ij}]_{\mathcal{J}}[R_{ji}]_{\mathcal{J}} \in \mathcal{A}'(T)/\Phi_{ii}(\mathcal{J})$ for arbitrary $S_{ij} \in \ker \tau_{T_i,T_j}$, $R_{ji} \in \ker \tau_{T_j,T_i}$, $l_{\mathcal{J}} < i \leq n$. By Corollary 3.6, $[S_{ij}]_{\mathcal{J}} = [R_{ji}]_{\mathcal{J}} = 0$ for $l_{\mathcal{J}} < i \leq n$. Hence $l_{\mathcal{J}} \geq 1$ and

$$\mathcal{A}'(T)/\mathcal{J} = \left\{ \begin{bmatrix} ([S_{ij}]_{\mathcal{J}})_{l_{\mathcal{J}}\times l_{\mathcal{J}}} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n} \;\middle|\; S_{ij} \in \ker \tau_{T_i,T_j} \right\}.$$

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Claim 1 For $1 \le i, j, k \le l_{\mathcal{J}}$, if $\ker \tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J}) \neq \{0\}$ and $\ker \tau_{T_j,T_k}/\Phi_{jk}(\mathcal{J}) \neq \{0\}$, then $\ker \tau_{T_i,T_k}/\Phi_{ik}(\mathcal{J}) \neq \{0\}$.

Note that $\mathcal{A}'(T_i)/\Phi_{ii}(\mathcal{J}) \approx C$ and $\mathcal{A}'(T_j)/\Phi_{jj}(\mathcal{J}) \approx C$ for $1 \le i, j \le l_{\mathcal{J}}$. If $\ker \tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J}) \neq \{0\}$, by Corollary 3.6, there exists $S_{ij} \in \ker \tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J})$ and $S_{ji} \in \ker \tau_{T_j,T_i}/\Phi_{ji}(\mathcal{J})$ such that $S_{ij}S_{ji} = [e_{\mathcal{A}'(T_i)}]_{\mathcal{J}} = 1$. Similarly, there exists $S_{jk} \in \ker \tau_{T_j,T_k}/\Phi_{jk}(\mathcal{J})$ and $S_{kj} \in \ker \tau_{T_k,T_j}/\Phi_{kj}(\mathcal{J})$ such that $S_{jk}S_{kj} = [e_{\mathcal{A}'(T_j)}]_{\mathcal{J}} = 1$. So $S_{ij}S_{jk} \neq 0$, and then $\ker \tau_{T_i,T_k}/\Phi_{ik}(\mathcal{J}) \neq \{0\}$.

Claim 2 For $1 \le i \le l_{\mathcal{J}}$, $\Phi_{1i} \neq \ker \tau_{T_1,T_i}$.

Otherwise, we may assume that $1 \le j_0 < l_{\mathcal{J}}$ such that

$$\ker \tau_{T_1,T_i}/\Phi_{1i}(\mathcal{J}) \neq \{0\},\ 1 \le i \le j_0$$

and

$$\ker \tau_{T_1,T_j}/\Phi_{1j}(\mathcal{J}) = \{0\},\ j_0 < j \le l_{\mathcal{J}}.$$

By Claim 1, we can deduce that

$$\ker \tau_{T_j,T_i}/\Phi_{ji}(\mathcal{J}) = \{0\},\ 1 \le i \le j_0,\ j_0 < j \le l_{\mathcal{J}}.$$

By Corollary 3.6, we can deduce that

$$\ker \tau_{T_j,T_i}/\Phi_{ji}(\mathcal{J}) = \{0\},\ 1 \le i \le j_0,\ j_0 < j \le l_{\mathcal{J}}.$$

Therefore

$$\mathcal{A}'(T)/\mathcal{J} = \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}_{n\times n} \;\middle|\; \begin{array}{l} \Delta = \operatorname{diag}(([S_{ij}]_{\mathcal{J}})_{j_0\times j_0},\ ([S_{ij}]_{\mathcal{J}})_{(l_{\mathcal{J}}-j_0)\times(l_{\mathcal{J}}-j_0)}), \\ S_{ij} \in \ker \tau_{T_i,T_j} \end{array} \right\}.$$

It contradicts $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$.

Claim 3 $\mathcal{A}'(T)/\mathcal{J} \approx M_{l_{\mathcal{J}}}(C)$.

For $1 \le i \le l_{\mathcal{J}}$, let $e_{ii} = [e_{\mathcal{A}'(T_i)}]_{\mathcal{J}} = 1$. By Claim 1 and Claim 2, there exist $e_{1i} \in \ker \tau_{T_1,T_i}/\Phi_{1i}(\mathcal{J})$ and $e_{i1} \in \ker \tau_{T_i,T_1}/\Phi_{i1}(\mathcal{J})$ such that $e_{1i}e_{i1} = e_{11}$ for $1 \le i \le l_{\mathcal{J}}$. Since $\mathcal{A}'(T_i)/\Phi_{ii}(\mathcal{J}) \approx C$, $e_{i1}e_{1i} = e_{ii}$. Let $e_{ij} = e_{i1}e_{1j}$, then

$$e_{ij}e_{ji} = e_{i1}e_{1j}e_{j1}e_{1i} = e_{ii},$$

$$e_{ji}e_{ij} = e_{j1}e_{1i}e_{i1}e_{1j} = e_{jj}.$$

For arbitrary $S_{ij} \in \ker \tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J})$, $(1 \le i, j \le l_{\mathcal{J}})$, there exists $\lambda_{ij} \in C$ such that $\lambda_{ij} = S_{ij}e_{ji}$.

Note that

$$\begin{aligned} S_{ij} - \lambda_{ij}e_{ij} &= (S_{ij} - \lambda_{ij}e_{ij})e_{jj} = (S_{ij} - \lambda_{ij}e_{ij})e_{ji}e_{ij} \\ &= (S_{ij}e_{ji} - \lambda_{ij}e_{ii})e_{ij} = (\lambda_{ij} - \lambda_{ij})e_{ii}e_{ij} \\ &= 0 \end{aligned}$$

So $S_{ij} = \lambda_{ij}e_{ij}$. Therefore $\mathcal{A}'(T)/\mathcal{J} \approx M_{l_{\mathcal{J}}}(C)$.

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Now we complete the first part of Theorem 3.7.

If $T_k \sim T_1$ for $k = 1, 2, \dots, n$. It is not difficult to show that $\mathcal{A}'(T_k)/\Phi_{kk}(\mathcal{J}) \approx C$ for $k = 1, 2, \dots, n$ following from property (3.4) and property (3.5) and Corollary 3.3. Repeating the proof of the first part, we can deduce that $\mathcal{A}'(T)/\mathcal{J} \approx M_n(C)$ for every $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$. Now we complete the proof of Theorem 3.7. ■

Theorem 3.7 implies the following properties:

(3.8) If $\mathcal{A}'(T_i)/\Phi_{ii}(\mathcal{J}) \neq \{0\}$ and $\mathcal{A}'(T_j)/\Phi_{jj}(\mathcal{J}) \neq \{0\}$ for $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$, then

$$\ker \tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J}) \approx \ker \tau_{T_j,T_i}/\Phi_{ji}(\mathcal{J}) \approx C.$$

(3.9) If $\mathcal{A}'(T_i)/\Phi_{ii}(\mathcal{J}) = \{0\}$ for some $1 \leq i \leq n$ and $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$, then

$$\ker \tau_{T_i,T_k}/\Phi_{ik}(\mathcal{J}) = \ker \tau_{T_k,T_i}/\Phi_{ki}(\mathcal{J}) = \{0\} \text{ for } k = 1, 2, \dots, n.$$

Theorem 3.8 *Let $T = \oplus_{k=1}^n T_k$. Then for each $\mathcal{J}_1 \in \mathcal{M}(\mathcal{A}'(T_1))$, there exists a unique $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$ such that $\Phi_{11}(\mathcal{J}) = \mathcal{J}_1$.*

Proof Without loss of generality, we only discuss the case of $n = 2$. Now $T = T_1 \oplus T_2$. By Lemma 3.1, there exists an ideal $\mathcal{J}_0$ of $\mathcal{A}'(T)$ such that $\Phi_{11}(\mathcal{J}_0) = \mathcal{J}_1$. Set $\mathcal{J}' = \mathcal{J}_0 + \operatorname{rad} \mathcal{A}'(T)$, then $\mathcal{J}'$ is still an ideal of $\mathcal{A}'(T)$ and $\Phi_{11}(\mathcal{J}') = \mathcal{J}_1$. By the Corollary 3.5, $\operatorname{rad} \mathcal{A}'(T_2) \subseteq \Phi_{22}(\mathcal{J}')$. So we may assume that $\mathcal{J}_0 = \mathcal{J}'$, hence $\mathcal{A}'(T_1)/\Phi_{11}(\mathcal{J}_0) \approx C$ and $\mathcal{A}'(T_2)/\Phi_{22}(\mathcal{J}_0)$ are semisimple and commutative since $\mathcal{A}'(T_k)/\operatorname{rad} \mathcal{A}'(T_k)$ is commutative (see Theorem 2.8).

Note that

$$\mathcal{A}'(T)/\mathcal{J}_0 = \left\{ \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \;\middle|\; S_{ij} \in \ker \tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J}_0); 1 \leq i, j \leq 2 \right\}.$$

Let $e_{kk} = [e_{\mathcal{A}'(T_k)}]_{\mathcal{J}}$ for $k = 1, 2$. Note that $\mathcal{A}'(T_1)/\Phi_{11}(\mathcal{J}_0) \approx C$ and then $e_{11} = 1$.
Case 1: Suppose that there exist $e_{12} \in \ker \tau_{T_1,T_2}/\Phi_{12}(\mathcal{J}_0)$ and $e_{21} \in \ker \tau_{T_2,T_1}/\Phi_{21}(\mathcal{J}_0)$ such that $e_{12}e_{21} = e_{11} = 1$. Set $Q_1 = e_{21}e_{12}$ and $Q_2 = e_{22} - Q_1$. Then Q_1 and Q_2 are idempotents in $\mathcal{A}'(T_2)/\Phi_{22}(\mathcal{J}_0)$ and $Q_1Q_2 = Q_2Q_1 = 0$. Set

$$\mathcal{A}' = \left\{ \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & Q_1S_{22} \end{bmatrix} \;\middle|\; S_{ij} \in \ker \tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J}_0); 1 \leq i, j \leq 2 \right\}$$

$$\mathcal{A}'' = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & Q_2S_{22} \end{bmatrix} \;\middle|\; S_{22} \in \mathcal{A}'(T_2)/\Phi_{22}(\mathcal{J}_0) \right\}$$

Claim 1 $\mathcal{A}'(T)/\mathcal{J}_0 = \mathcal{A}' \oplus \mathcal{A}''$.

It is obvious that for $S = (S_{ij})_{2\times 2} \in \mathcal{A}'(T)/\mathcal{J}_0$,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & Q_1S_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Q_2S_{22} \end{bmatrix},$$

where
$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & Q_1S_{22} \end{bmatrix} \in \mathcal{A}' \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & Q_2S_{22} \end{bmatrix} \in \mathcal{A}''.$$
For
$$t = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & Q_1S_{22} \end{bmatrix} \in \mathcal{A}' \quad \text{and} \quad r = \begin{bmatrix} 0 & 0 \\ 0 & Q_2S_{22} \end{bmatrix} \in \mathcal{A}''$$
we have
$$tr = \begin{bmatrix} 0 & S_{12}Q_2S_{22} \\ 0 & 0 \end{bmatrix}, \quad rt = \begin{bmatrix} 0 & 0 \\ Q_2S_{22}S_{21} & 0 \end{bmatrix}.$$

To verify Claim 1, we need only to show that $S_{12}Q_2 = Q_2S_{21} = 0$. For arbitrary $S_{12} \in \ker \tau_{T_1,T_2}/\Phi_{12}(\mathcal{J}_0)$ and $S_{21} \in \ker \tau_{T_2,T_1}/\Phi_{21}(\mathcal{J}_0)$, we can find a λ in C such that $S_{12}Q_2 = \lambda e_{12}$ by property (3.4).

So $(S_{12}Q_2 - \lambda e_{12})e_{21} = 0$, and $\sigma(e_{21}(S_{12}Q_2 - \lambda e_{12})) = \{0\}$.

Noting that $\mathcal{A}'(T_2)/\Phi_{22}(\mathcal{J}_0)$ is semi-simple and commutative,
$$e_{21}(S_{12}Q_2 - \lambda e_{12}) = 0.$$
That is $(e_{21}S_{12})Q_2 = \lambda Q_1$. This implies that $\lambda = 0$ and $S_{12}Q_2 = 0$.

Similarly, we can prove that $Q_2S_{21} = 0$. So $rt = tr = 0$. We verify Claim 1.

Claim 2 $\mathcal{A}' \approx M_2(C)$.

Set $\mathcal{A}_1 = \{S_{11} \mid S_{11} \in \mathcal{A}'(T_1)/\Phi_{11}(\mathcal{J}_0)\}$ and $\mathcal{A}_2' = \{Q_1S_{22} \,;\, S_{22} \in \mathcal{A}'(T_2)/\Phi_{22}(\mathcal{J}_0)\}$. Note that $\mathcal{A}_1 \approx C$. We will show that $\mathcal{A}_2 \approx C$. Define a map Φ from $\mathcal{A}_2$ to $\mathcal{A}_1$ by $\Phi(b) = e_{12}be_{21}$ for $b \in \mathcal{A}_2$. It is clear that Φ is a homomorphism. Since $\Phi(Q_1) = e_{11} = 1$, Φ is a surjective homomorphism. If $\Phi(b) = \Phi(b')$ for b and b' in $\mathcal{A}_2$, then $e_{21}\Phi(b)e_{12} = e_{21}\Phi(b')e_{12}$. Thus $Q_1bQ_1 = Q_1b'Q_1$, so $\mathcal{A}_2 \approx \mathcal{A}_1 \approx C$.

Similar to the proof of Theorem 3.7, we can deduce that $\mathcal{A}' \approx M_2(C)$.

Now we define a map π from $\mathcal{A}'(T)$ onto $\mathcal{A}'$ by
$$\pi((S_{ij})_{2\times 2}) = \begin{bmatrix} [S_{11}]_{\mathcal{J}_0} & [S_{12}]_{\mathcal{J}_0} \\ [S_{21}]_{\mathcal{J}_0} & Q_1[S_{22}]_{\mathcal{J}_0} \end{bmatrix}.$$
Then π is a homomorphism. Since $\mathcal{A}' \approx M_2(C)$, $\mathcal{J} = \ker \pi \in \mathcal{M}(\mathcal{A}'(T))$ and $\Phi_{11}(\mathcal{J}) = \mathcal{J}_1$.

Case 2: If we can not find $e_{12} \in \ker \tau_{T_1,T_2}/\Phi_{12}(\mathcal{J}_0)$ and $e_{21} \in \ker \tau_{T_2,T_1}/\Phi_{21}(\mathcal{J}_0)$ such that $e_{12}e_{21} = e_{11} = 1$. By $\mathcal{A}'(T_1)/\Phi_{11}(\mathcal{J}_0) \approx C$, we can deduce that $S_{12}S_{21} = 0$ for $S_{12} \in \ker \tau_{T_1,T_2}/\Phi_{12}(\mathcal{J}_0)$ and $S_{21} \in \ker \tau_{T_2,T_1}/\Phi_{21}(\mathcal{J}_0)$. By Lemma 3.4,
$$\ker \tau_{T_1,T_2}/\Phi_{12}(\mathcal{J}_0) = \{0\} \text{ and } \ker \tau_{T_2,T_1}/\Phi_{21}(\mathcal{J}_0) = \{0\}.$$
Thus
$$\mathcal{A}'(T)/\mathcal{J}_0 \approx \mathcal{A}'(T_1)/\Phi_{11}(\mathcal{J}_0) \oplus \mathcal{A}'(T_2)/\Phi_{22}(\mathcal{J}_0) \approx C \oplus \mathcal{A}'(T_2)/\Phi_{22}(\mathcal{J}_0).$$

Similar to the proof of Case 1, there exists a $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$ such that $\Phi_{11}(\mathcal{J}) = \mathcal{J}_1$.

Now we prove the uniqueness. Let $\mathcal{J}$ and $\mathcal{J}'$ be in $\mathcal{M}(\mathcal{A}'(T))$ such that $\Phi_{11}(\mathcal{J}) = \Phi_{11}(\mathcal{J}') = \mathcal{J}_1$. Set $\bar{\mathcal{J}} = \mathcal{J} + \mathcal{J}' = \{S + S' \,;\, S \in \mathcal{J}; S' \in \mathcal{J}'\}$. Then $\bar{\mathcal{J}}$ is an ideal of $\mathcal{A}'(T)$. Since $\Phi_{11}(\bar{\mathcal{J}}) = \mathcal{J}_1$, so $\mathcal{J} = \mathcal{J}' = \bar{\mathcal{J}}$. The proof of Theorem 3.8 is complete. ■

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Lemma 3.9 *Let* $A \in \mathcal{B}_m(\Omega) \cap (\mathrm{SI})$ *and* $B \in \mathcal{B}_n(\Omega) \cap (\mathrm{SI})$, *where* $m > n$. *And let* $X_i \in \ker \tau_{A,B}$ *and* $Y_i \in \ker \tau_{B,A}$, $i = 1, 2, \dots, k$. *Then*

$$\sum_{i=1}^{k} X_i Y_i \neq I.$$

Furthermore, set

$$\mathcal{J}_1 = \Big\{ \sum_{i=1}^{k} X_i Y_i, X_i \in \ker \tau_{A,B} \mid Y_i \in \ker \tau_{B,A}, i = 1, 2, \dots, k \text{ and } k = 1, 2, \dots \Big\}$$

and

$$\mathcal{J}_2 = \Big\{ \sum_{i=1}^{k} Y_i X_i, X_i \in \ker \tau_{A,B} \mid Y_i \in \ker \tau_{B,A}, i = 1, 2, \dots, k \text{ and } k = 1, 2, \dots \Big\}.$$

Then $\mathcal{J}_1 \subset \operatorname{rad} \mathcal{A}'(A)$ *and* $\mathcal{J}_2 \subset \operatorname{rad} \mathcal{A}'(B)$.

Proof Obviously, the second part of the Lemma implies the first part. Without loss of generality, we may assume that m is the minimal index of A. Note that $X_i Y_i \in \mathcal{A}'(A)$ and $m > n$. By Theorem 3.8, we can deduce that

$$\begin{aligned}\sigma((\Gamma_A X_i Y_i)(z)) &= \sigma(X_i(z) Y_i(z)) \\ &= \{0\};\ z \in \Omega,\ i = 1, 2, \dots, k.\end{aligned}$$

Since $\operatorname{tr}\big(\sum_{i=1}^{k} X_i(z) Y_i(z)\big) = \sum_{i=1}^{k} \operatorname{tr}(X_i(z) Y_i(z)) = 0$, $\sigma\big(\sum_{i=1}^{k} X_i(z) Y_i(z)\big) = \{0\}$.

Set $B(z) = \sum_{i=1}^{k} X_i(z) Y_i(z)$. By $\dim \ker(A - z) = m, z \in \Omega$, we have $B(z)^m = 0$. Since $\bigvee_{z \in \Omega} \ker(A - z) = \mathcal{H}$, $(\sum_{i=1}^{k} X_i Y_i)^m = 0$. This shows that $\mathcal{J}_1 \subseteq \operatorname{rad} \mathcal{A}'(A)$. For X in $\ker \tau_{A,B}$ and Y in $\ker \tau_{B,A}$, since $\sigma(XY) = \{0\}$, $\sigma(YX) = \{0\}$. Hence $\sigma(\sum_{i=1}^{k} Y_i(z) X_i(z)) = \{0\}$ and $(\sum_{i=1}^{k} Y_i(z) X_i(z))^n = 0$. This shows that $\mathcal{J}_2 \subseteq \operatorname{rad} \mathcal{A}'(B)$, completing the proof of Lemma 3.9. ■

Proposition 3.10 *Let* $T = A^{(l)}$ *and* $A \in \mathcal{B}_n(\Omega) \cap (\mathrm{SI})$. *And let* $\{P_1, \dots, P_m\}$ *be an* (SI) *decomposition of* T. *Then* $m = n$ *and* $A_i = A^{(l)}|_{P_i \mathcal{H}^{(l)}} \in \mathcal{B}_n(\Omega)$.

Proof First we show that $m \leq l$. By Theorem 2.8, $\mathcal{A}'(A) / \operatorname{rad} \mathcal{A}'(A)$ is commutative. By the Gelfand Theorem, there exists a continuous natural homomorphism φ from $\mathcal{A}'(A)$ into $C(\mathcal{M}(\mathcal{A}'(A)))$. So φ can induce a continuous homomorphism Ψ from $\mathcal{A}'(T)$ into $M_l(\mathcal{M}(\mathcal{A}'(A)))$ defined by

$$\Psi(S)(\mathcal{J}) = (\varphi(S_{ij})(\mathcal{J}))_{l \times l} \text{ for } S = (S_{ij})_{l \times l} \in \mathcal{A}'(T) \text{ and } \mathcal{J} \in \mathcal{M}(\mathcal{A}'(A)).$$

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Set $P_k = (P_{ij}^k)_{l\times l}$ for $k = 1, 2, \dots, m$. Then $\Psi(P_k)(\mathcal{J}) = (\varphi(P_{ij}^k)(\mathcal{J}))_{l\times l}$.

Set

$$\operatorname{tr}(\Psi(P_k)(\mathcal{J})) := \sum_{i=1}^{l} \varphi(P_{ii}^k)(\mathcal{J}).$$

Then $\operatorname{tr}(\cdot)$ defines a continuous function on $\mathcal{M}(\mathcal{A}'(A))$. Since $\mathcal{A}'(A)/\operatorname{rad}\mathcal{A}'(A))$ is commutative and $A \in (\mathrm{SI})$, $\mathcal{M}(\mathcal{A}'(A))$ is connected, by Proposition 1.17 of [Ji-Wa]. Since $\Psi(P_k)(\mathcal{J})$ is idempotent, $\operatorname{tr}(\Psi(P_k)(\mathcal{J})) \equiv n_k \geq 1$. Note that $\sum_{k=1}^{m} P_k = I$ and $P_k P_{k'} = \delta_{kk'} P_k$, we have $\sum_{k=1}^{m} \operatorname{tr}(\Psi(P_k)(\mathcal{J})) = l$. Hence

$$\sum_{k=1}^{m} \operatorname{tr}(\Psi(P_k)(\mathcal{J})) = \sum_{k=1}^{m} n_k = l.$$

So $m \leq l$.

Now we show that $A_i \in \mathcal{B}_n(\Omega)$. Otherwise, we may assume that $A_1 \in \mathcal{B}_k(\Omega)$ and $k < n$. Let $S = A \oplus A_1$. By Lemma 3.9 and the proof of Theorem 3.7, we can find an $\mathcal{J}_1 \in \mathcal{M}(\mathcal{A}'(S))$ such that $\mathcal{A}'(S)/\mathcal{J}_1 \approx C$. Let $T_1 = A \oplus T = A^{(l+1)}$. By Theorem 3.7, $\mathcal{A}'(T_1)/\mathcal{J} \approx M_{l+1}(C)$ for $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T_1))$. Note that $T_1 \sim A \oplus A_1 \oplus \cdots \oplus A_m$ and $m \leq l$. Repeating the proof of Theorem 3.7 and using Lemma 3.9, we can find $\mathcal{J}_2 \in \mathcal{M}(\mathcal{A}'(T_1))$ such that $\mathcal{A}'(T_1)/\mathcal{J}_2 \approx M_d(C)$ and $d < l + 1$ This contradicts $\mathcal{A}'(T_1)/\mathcal{J} \approx M_{l+1}(C)$ for $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T_1))$. Similarly, we can show that it is impossible for $k > n$. So $k = n$ and $m = l$. We complete the proof of Proposition 3.10. ■

Summarizing the above argument, we have

Theorem 3.11 *Let $T \in \mathcal{B}_n(\Omega)$. Then for $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$ there exists a natural number $l \leq n$ such that*

$$\mathcal{A}'(T)/\mathcal{J} \approx M_l(C).$$

4 The Similar Classification of Cowen-Douglas Operators

In order to obtain the main result, we need the following results.

Lemma 4.1 *Let $A \in \mathcal{B}_m(\Omega) \cap (\mathrm{SI})$ and $B \in \mathcal{B}_n(\Omega) \cap (\mathrm{SI})$ and $A \nsim B$. And let*

$$\mathcal{J} = \Big\{ \sum_{i=1}^{k} X_i Y_i \;\Big|\; X_i \in \ker \tau_{A,B},\ Y_i \in \ker \tau_{B,A},\ 1 \leq i \leq k,\ k = 1, 2, \dots \Big\}$$

Then $\mathcal{J}$ is a proper ideal of $\mathcal{A}'(A)$.

Proof We need only to show that $\sum_{i=1}^{k} X_i Y_i \neq I$ for $X_i \in \ker \tau_{A,B}$ and $Y_i \in \ker \tau_{B,A}$ and $k = 1, 2, \dots$. Otherwise, there exists a positive integer $k \geq 1$, $X_i \in \ker \tau_{A,B}$ and $Y_i \in \ker \tau_{B,A}$ such that $\sum_{i=1}^{k} X_i Y_i = I$. Set

$$P = \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix} \begin{bmatrix} X_1 & \cdots & X_k \end{bmatrix} = \begin{bmatrix} Y_1X_1 & \cdots & Y_1X_k \\ \vdots & & \vdots \\ Y_kX_1 & \cdots & Y_kX_k \end{bmatrix}$$

Then P is an idempotent operator in $\mathcal{A}'(B^{(k)})$. Let $T = A \oplus B^{(k)}$ and let

$$a = \begin{bmatrix} 0 & X_1 & \cdots & X_k \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ Y_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ Y_k & 0 & \cdots & 0 \end{bmatrix}.$$

Then a and b are in $\mathcal{A}'(T)$ and $ab = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $ba = \left[\begin{smallmatrix} 0 & 0 \\ 0 & P \end{smallmatrix}\right]$. By [Bl, Proposition 2.21], there exists an invertible operator X in $M_2(\mathcal{A}'(T)) = \mathcal{A}'(T \oplus T)$ such that

$$X\left(\begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) X^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

By [Ca-Fa-Ji, Lemma 1.9],

$$A \sim B^{(k)}|_R := B_1 \quad \text{where } R = \operatorname{ran} \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix}.$$

By $A \in (\mathrm{SI})$ and Lemma 2.10, $B_1 \sim B \sim A$. This contradicts $A \nsim B$. We complete the proof of Lemma 4.1. ■

Proposition 4.2 *Let $A \in \mathcal{B}_m(\Omega) \cap (\mathrm{SI})$ and $B \in \mathcal{B}_n(\Omega) \cap (\mathrm{SI})$ and $A \nsim B$. And let $T = A^{(n_1)} \oplus B^{(n_2)}$, where n_1 and n_2 are two fixed positive integers. Then the following holds:*

1. *There exists $\mathcal{J}_1$ and $\mathcal{J}_2$ in $\mathcal{M}(\mathcal{A}'(T))$ such that*

$$\mathcal{A}'(T)/\mathcal{J}_1 \approx M_{n_1}(C)$$

and

$$\mathcal{A}'(T)/\mathcal{J}_2 \approx M_{n_2}(C).$$

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2. $\min\{n_1, n_2\} = \min\{l_{\mathcal{J}}, \mathcal{A}'(T)/\mathcal{J} \approx M_{l_{\mathcal{J}}}(C), \mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))\}$.
3. *For each* $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T)), \mathcal{A}'(T)/\mathcal{J} \approx M_k(C)$, *where* $k \in \{n_1, n_2\}$ *if* $m \neq n$ *and* $k \in \{n_1, n_2, n_1 + n_2\}$ *if* $m = n$.

Proof Let $S_1 = \bigoplus_{i=1}^{n_1} B_i$ and $S_2 = \bigoplus_{i=n_1+1}^{n_1+n_2} B_i$, where

$$B_i = \begin{cases} A & \text{if } 1 \le i \le n_1, \\ B & \text{if } n_1 + 1 \le i \le n_1 + n_2. \end{cases}$$

Then $T = \bigoplus_{i=1}^{n_1+n_2} B_i$. Set

$$\varepsilon_1 = \Big\{ \sum_{i=1}^{k} X_i Y_i, X_i \in \ker \tau_{A,B} \mid Y_i \in \ker \tau_{B,A}, k = 1, 2, \dots \Big\}.$$

By Lemma 4.1, ε_1 is proper ideal of $\mathcal{A}'(B_1)$. So we can find an ε_2 in $\mathcal{M}(\mathcal{A}'(B_1))$ such that $\varepsilon_1 \subseteq \varepsilon_2$. By Theorem 3.8, there exists a unique $\mathcal{J}_1$ in $\mathcal{M}(\mathcal{A}'(T))$ such that $\Phi_{11}(\mathcal{J}_1) = \varepsilon_2$. By $B_1 = B_2 = \cdots = B_{n_1}$ and property (3.7), we have $\Phi_{ij}(\mathcal{J}_1) = \varepsilon_2, 1 \le i, j \le n_1$. Define a new subset in $\mathcal{A}'(T)$ by

$$\mathcal{J}_1' := \begin{bmatrix} \varepsilon_2 & \cdots & \varepsilon_2 & \ker \tau_{A,B} & \cdots & \ker \tau_{A,B} \\ \vdots & & \vdots & \vdots & & \vdots \\ \varepsilon_2 & \cdots & \varepsilon_2 & \ker \tau_{A,B} & \cdots & \ker \tau_{A,B} \\ \ker \tau_{B,A} & \cdots & \ker \tau_{B,A} & \mathcal{A}'(B) & \cdots & \mathcal{A}'(B) \\ \vdots & & \vdots & \vdots & & \vdots \\ \ker \tau_{B,A} & \cdots & \ker \tau_{B,A} & \mathcal{A}'(B) & \cdots & \mathcal{A}'(B) \end{bmatrix},$$

where ε_2 appears $n_1 \times n_1$ times and $\mathcal{A}'(B)$ appears $n_2 \times n_2$ times. By $\varepsilon_1 \subseteq \varepsilon_2, \mathcal{J}_1'$ is a proper ideal in $\mathcal{A}'(T)$ and $\Phi_{11}(\mathcal{J}_1') = \varepsilon_2$. It is easy to see that $\mathcal{J}_1 \subseteq \mathcal{J}_1'$, so $\mathcal{J}_1 \in \mathcal{M}(\mathcal{A}'(T))$ and $\mathcal{J}_1 = \mathcal{J}_1'$. Repeating the proof of Theorem 3.7, we can deduce that $\mathcal{A}'(T)/\mathcal{J}_1 \approx M_{n_1}(C)$. Similarly, we can find an $\mathcal{J}_2$ in $\mathcal{M}(\mathcal{A}'(T))$ such that $\mathcal{A}'(T_2)/\mathcal{J}_2 \approx M_{n_2}(C)$. This complete the proof of the part one.

Now, to show part (ii), we may assume that $n_1 = \min\{n_1, n_2\}$. We need only to verify that

$$n_1 \le \min\{l_{\mathcal{J}}; \mathcal{A}'(T)/\mathcal{J} \approx M_{l_{\mathcal{J}}}(C) \text{ for } \mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))\}.$$

Otherwise, there exists a $\mathcal{J}$ in $\mathcal{M}(\mathcal{A}'(T))$ such that $\mathcal{A}'(T)/\mathcal{J} \approx M_l(C)$ and $l < n_1$. By Properties (3.8) and (3.9) of Theorem 3.7, there exist i and j ($1 \le i, j \le n_1$ or $n_1 + 1 \le i, j \le n_1 + n_2$) such that $\Phi_{ii}(\mathcal{J}) = \mathcal{A}'(B_i)$, but $\Phi_{jj}(\mathcal{J}) \in \mathcal{M}(\mathcal{A}'(B_j))$. By $A = B_k, k = 1, 2, \dots, n_1$, $B = B_k, k = n_1 + 1, \dots, n_1 + n_2$, and property (3.7), we have $\Phi_{11}(\mathcal{J}) = \cdots = \Phi_{n_1 n_1}(\mathcal{J})$ and $\Phi_{n_1+1\, n_1+1}(\mathcal{J}) = \cdots = \Phi_{n_1+n_2\, n_1+n_2}(\mathcal{J})$. This is a contradiction.

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Proof of Part Three First, we consider the case of $m \neq n$. For $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$. It follows from the proof of part two, we can deduce that

$$\Phi_{11}(\mathcal{J}) = \Phi_{kk}(\mathcal{J}),\ k = 1, 2, \ldots, n_1$$

and

$$\Phi_{n_1+1\, n_1+1}(\mathcal{J}) = \Phi_{kk}(\mathcal{J}),\ k = n_1 + 1, \ldots, n_1 + n_2.$$

Then $\Phi_{11}(\mathcal{J}) \in \mathcal{M}(\mathcal{A}'(A))$ or $\Phi_{n_1+1\, n_1+1}(\mathcal{J}) = \mathcal{A}'(B)$. Without loss of generality, we may assume that $\Phi_{11}(\mathcal{J}) \in \mathcal{M}(\mathcal{A}'(A))$. Then we can claim that

$$\Phi_{n_1+1\, n_1+1}(\mathcal{J}) = \mathcal{A}'(B).$$

Otherwise, by Lemma 3.9, $XY \in \operatorname{rad} \mathcal{A}'(A)$ and $YX \in \operatorname{rad} \mathcal{A}'(B)$ for X in $\ker \tau_{A,B}$ and Y in $\ker \tau_{B,A}$. Set

$$\mathcal{J}_1 = \begin{bmatrix} \Phi_{11}(\mathcal{J}) & \cdots & \Phi_{11}(\mathcal{J}) & \ker \tau_{A,B} & \cdots & \ker \tau_{A,B} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Phi_{11}(\mathcal{J}) & \cdots & \Phi_{11}(\mathcal{J}) & \ker \tau_{A,B} & \cdots & \ker \tau_{A,B} \\ \ker \tau_{B,A} & \cdots & \ker \tau_{B,A} & \mathcal{A}'(B) & \cdots & \mathcal{A}'(B) \\ \vdots & & \vdots & \vdots & & \vdots \\ \ker \tau_{B,A} & \cdots & \ker \tau_{B,A} & \mathcal{A}'(B) & \cdots & \mathcal{A}'(B) \end{bmatrix},$$

where $\Phi_{11}(\mathcal{J})$ appears $n_1 \times n_1$ times and $\mathcal{A}'(B)$ appears $n_2 \times n_2$ times. Then $\mathcal{J}_1$ is a proper ideal of $\mathcal{A}'(T)$ and $\mathcal{J} \subset (\neq)\mathcal{J}_1$. It contradicts $\mathcal{J} \in \mathcal{M}(\mathcal{A}'(T))$. Repeating the proof of Theorem 3.7, we have $\mathcal{A}'(T)/\mathcal{J} \approx M_{n_1}(C)$.

If $\Phi_{n_1+1\, n_1+1}(\mathcal{J}) \in \mathcal{M}(\mathcal{A}'(B))$. Similarly we can show that $\mathcal{A}'(T)/\mathcal{J} \approx M_{n_2}(C)$.

Secondly, we consider the case of $m = n$. By part one, we can find $\mathcal{J}_1$ and $\mathcal{J}_2$ in $\mathcal{M}(\mathcal{A}'(T))$ such that $\mathcal{A}'(T)/\mathcal{J}_1 \approx M_{n_1}(C)$ and $\mathcal{A}'(T)/\mathcal{J}_2 \approx M_{n_2}(C)$. For arbitrary $\mathcal{J}$ in $\mathcal{A}'(T)$, we can not determine whether $\Phi_{n_1+1\, n_1+1}(\mathcal{J}) = \mathcal{A}'(B)$ when $\Phi_{11}(\mathcal{J}) \in \mathcal{M}(\mathcal{A}'(A))$. That means that $\mathcal{A}'(T)/\mathcal{J} \approx M_{n_1+n_2}(C)$ is possible. This completes the proof of Proposition 4.2. ■

By Theorem 3.8 and Proposition 4.2, we obtain immediately the following conclusion.

Theorem 4.3 *Let A and B be strongly irreducible Cowen-Douglas operators. Then $A \sim B$ if and only if*

$$\mathcal{A}'(A \oplus B)/\mathcal{J} \approx M_2(C)$$

for each $\mathcal{J}$ in $\mathcal{M}(\mathcal{A}'(A \oplus B))$.

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Theorem 4.4 *Two strongly irreducible Cowen-Douglas operators A and B are similar if and only if there is a group isomorphism* $\alpha\colon K_0(\mathcal{A}'(A)) \to K_0(\mathcal{A}'(B))$ *that satisfies*

1. $\alpha(V(\mathcal{A}'(A))) = V(\mathcal{A}'(B))$;
2. $\alpha[I_{\mathcal{A}'(A)}] = [I_{\mathcal{A}'(B)}]$;
3. *there exist two non-zero idempotent operators,*

$$p \in \mathcal{M}_\infty(\mathcal{A}'(A)) \quad \text{and} \quad q \in \mathcal{M}_\infty(\mathcal{A}'(B)), \quad \text{satisfying } \alpha[p] = [q],$$

where p and q are equivalent in $\mathcal{M}_\infty(\mathcal{A}'(A \oplus B))$.

Proof of the "if" part By Theorem 2.11, $V(\mathcal{A}'(A)) \approx V(\mathcal{A}'(T)) \approx N$ and $K_0(\mathcal{A}'(A)) \approx K_0(\mathcal{A}'(B)) \approx Z$. Suppose that $A \sim B$. Then there exists an invertible operator X such that $A = X^{-1}BX$. Note that for each P in $\mathcal{A}'(A)$, $(X^{-1}PX)^{(m)}$ is idempotent in $\mathcal{A}'(B^{(m)})$. Hence the map $\alpha\colon [p] \to [X^{-1}PX]$ defines a group isomorphism from $K_0(\mathcal{A}'(A))$ to $K_0(\mathcal{A}'(B))$ and α is as desired.

Proof of the "only if" part Suppose that α is a group isomorphism with properties 1, 2 and 3 given by Theorem 4.4. Then there exist two positive integers m and n and two non-zero idempotent operators p in $\mathcal{A}'(A^{(n)})$ and q in $\mathcal{A}'(B^{(m)})$ and two operators X in $\ker \tau_{A^{(n)},B^{(m)}}$ and Y in $\ker \tau_{B^{(m)},A^{(n)}}$ such that $XY = p$ and $YX = q$ and $\alpha[p] = [q]$. Let

$$T = A^{(n)} \oplus B^{(m)},\ a = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}^{(n)} \\ \mathcal{H}^{(m)} \end{matrix},\ b = \begin{bmatrix} 0 & 0 \\ Y & 0 \end{bmatrix} \begin{matrix} \mathcal{H}^{(n)} \\ \mathcal{H}^{(m)} \end{matrix}.$$

Then a and b are in $\mathcal{A}'(T)$ such that

$$ab = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}^{(n)} \\ \mathcal{H}^{(m)} \end{matrix},\ ba = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} \begin{matrix} \mathcal{H}^{(n)} \\ \mathcal{H}^{(m)} \end{matrix}.$$

By [Bl, Proposition 2.21], there exists an invertible operator G in $\mathcal{A}'(T \oplus T)$ such that

$$G\left(\begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) G^{-1} = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

By [Ca-Fa-Ji, Lemma 1.9],

$$A^{(n)}|_{\operatorname{ran} p} \sim B^{(m)}|_{\operatorname{ran} q}.$$

So $\operatorname{rank}(\Gamma_{A^{(n)}} p)(z) = \operatorname{rank}(\Gamma_{B^{(m)}} q)(z) = k$ for $z \in \Omega$, where we assume that $A \in \mathcal{B}_{n_1}(\Omega)$ and $B \in \mathcal{B}_{n_2}(\Omega)$.

By Theorem 2.11 and Theorem CFJ1, we have $A^{(n)}|\operatorname{ran} p \sim A^{(k)}$ and $B^{(m)}|\operatorname{ran} q \sim B^{(k)}$. This implies $A^{(k)} \sim B^{(k)}$.

Let $R = A^{(k)} \oplus B^{(k)}$. Then $R \sim A^{(2k)}$ and $\bigvee(\mathcal{A}'(R)) \approx N$. Using Theorem CFJ1 again, we have $A \sim B$. This complete the proof of Theorem 4.4. ■

Similar to the argument of Theorem 4.4, we have the following conclusions:

Theorem 4.5 *Two strongly irreducible Cowen-Douglas operators A and B are similar if and only if there exist two positive integers m and n and two operators X in* $\ker \tau_{A^{(n)},B^{(m)}}$ *and Y in* $\ker \tau_{B^{(m)},A^{(n)}}$ *such that XY is an idempotent operator in $\mathcal{A}'(A^{(n)})$.*

Theorem 4.6 *Two strongly irreducible Cowen-Douglas operators A and B are similar if and only if there exists a positive integer n such that*

$$A^{(n)} \sim B^{(n)}$$

We now calculate the K_0-group of a class unital Banach algebras.

Theorem 4.7 *Let Ω be a bounded connected open subset of C and* $\operatorname{inter} \bar{\Omega} = \operatorname{inter} \Omega$. *Then*

$$\bigvee(H^\infty(\Omega)) \approx N \text{ and } K_0(H^\infty(\Omega)) \approx Z.$$

Proof Let B_z be the Bergman operator on $L_a^2(\Omega^*)$ and let $A = B_z^*$. Then $K_0(\mathcal{A}'(A)) \approx H^\infty(\Omega)$ (cf. [Co-Do]). By Theorem 2.11, we can complete the proof of Theorem 4.7. ■

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