Can. J. Math., Vol. XXV, No. 5, 1973, pp. 986–999

CENTERS OF INFINITE BOUNDED SETS IN A NORMED SPACE

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Introduction. Čebyšev centers have been studied extensively. In this paper an alternate concept of center for infinite bounded point sets is introduced. Some of the results in this paper for this new type of center are similar to previous results for Čebyšev centers.

Throughout this paper, S will denote a normed linear space, and $||\cdot||$ will denote the norm on S. $N(S)$ will denote the origin in S, and K will denote a point set in S. If x is in S and r is a positive number, then $B(x, r)$ will denote the open norm ball centered at x with radius r, B will denote the open norm ball, $B(N(S), 1)$, and $\bar{B}$ will denote the closure of B.

1. Uniqueness of centers.

Definition 1.1. C_1 denotes the collection to which the interval, $[x, y]$, belongs if and only if x and y are two points of S such that $||x|| \leqq 1$ and $||y|| \leqq 1$. D_1 denotes the subcollection of C_1 to which $[x, y]$ belongs if and only if $||x|| = ||y|| = 1$. If $[x, y]$ is in C_1, then $I(x, y)$ denotes the interval $[p, q]$ of D_1 such that $[x, y] \subseteq [p, q]$, and $||p - x|| < ||p - y||$.

Definition 1.2. Suppose that C is a subcollection of C_1. The statement that C is uniformly convex means that if $h > 0$, then there is a number, d, in $(0, 1)$ such that if $[x, y]$ is in C, and if $||x - y|| \geqq h$, then $||\frac{1}{2}x + \frac{1}{2}y|| \leqq 1 - d$.

THEOREM 1.1. *Suppose that C is a subcollection of C_1. Then the following two statements are equivalent:*

(1) *C is uniformly convex;*

(2) *if $h > 0$, then there is a number, d, of $(0, 1)$ such that if $[x, y]$ is in C, and if $||x - y|| \geqq h$, then there is a point, w, of $[x, y]$ such that $||w|| \leqq 1 - d$.*

Proof. Clearly (2) follows from (1). Assume (2) is true and $h > 0$. Then there is a number, d, of $(0, 1)$ such that if $[x, y]$ is in C and $||x - y|| \geqq h$, then there is a point, p, of $[x, y]$ such that $||p|| \leqq 1 - d$. Hence there is a number, t, of $[\frac{1}{2}, 1]$ such that $\frac{1}{2}x + \frac{1}{2}y = tp + (1 - t)z$, where

$$z = \begin{cases} x, \text{ if } ||x - p|| \geqq \frac{1}{2}||x - y||; \\ y, \text{ if } ||x - p|| < \frac{1}{2}||x - y||. \end{cases}$$

Received May 25, 1972 and in revised form, October 24, 1972.

Then

$$\begin{aligned}||\tfrac{1}{2}x + \tfrac{1}{2}y|| &\leqq z||p|| + (1 - t)||z|| \\ &\leqq 1 - td \\ &\leqq 1 - \tfrac{1}{2}d.\end{aligned}$$

Hence C is uniformly convex.

THEOREM 1.2. *Suppose that C is a subcollection of C_1, that D is the subcollection of D_1 to which $[p, q]$ belongs if and only if there is an interval, $[x, y]$, of C such that $[p, q] = I(x, y)$, and that D is uniformly convex. Then C is uniformly convex.*

Proof. Suppose that $0 < h < 2$, and suppose that d is a number of $(0, 1)$ such that if $[u, v]$ is in D, and if $||u - v|| \geqq h$, then $||\tfrac{1}{2}u + \tfrac{1}{2}v|| \leqq 1 - d$. Suppose also that $[x, y]$ is in C, $||x - y|| \geqq h$, $m = \tfrac{1}{2}x + \tfrac{1}{2}y$, $I(x, y) = [p, q]$, and $M = \tfrac{1}{2}p + \tfrac{1}{2}q$. Then either m is in $[p, M]$ or m is in $(M, q]$.

Suppose that m is in $[p, M]$, $||m|| > 0$, and $t > 0$ is the number such that if $z = M + tm$, then $||z|| = 1$. Also, if $t_1 = ||m - p||/||M - p||$, then $m = t_1M + (1 - t_1)p$, and if $G = (1 + tt_1)m$, then G is in $[p, z]$. Thus:

$$\begin{aligned}1 - ||m|| &\geqq ||G|| - ||m|| \\ &\geqq t_1||z - M|| \\ &\geqq t_1(1 - ||M||) \\ &\geqq hd/2.\end{aligned}$$

Hence $||m|| \leqq 1 - hd/2$. A similar argument holds in case m is in $[M, q]$, and thus C is uniformly convex.

Note that if C has the following property, then D is uniformly convex in case C is uniformly convex: if $h > 0$, then there exists a number, k, in $(0, h)$ such that if $[p, q]$ is in D and $||p - q|| \geqq h$, then there exists an interval, $[x, y]$, of C such that $||x - y|| \geqq k$ and $I(x, y) = [p, q]$.

Definition 1.3. The statement that S is uniformly convex in every direction of K (u.c.e.d. K) means that if p and q are points in K, and if $h > 0$, then there is a number, d, of $(0, 1)$ such that if x and y are points in $\bar{B}$ such that $||x - y|| \geqq h$, and for some number, c, $x - y = c(p - q)$, then $||\tfrac{1}{2}x + \tfrac{1}{2}y|| < 1 - d$. The statement that S is u.c.e.d. means that S is u.c.e.d. S.

THEOREM 1.3. *Suppose that each of f and g is a norm on S, that f is u.c.e.d., and that F is a uniformly convex norm on E_2 such that if each of (a, b) and (c, d) is a point of E_2 such that $0 \leqq a \leqq c$, and $0 \leqq b \leqq d$, then $F(a, b) \leqq F(c, d)$. Let $|\cdot|$ be the function from S into E_1 such that if x is in S, $|x| = F(f(x), g(x))$. Then $|\cdot|$ is a u.c.e.d. norm on S.*

The concept of u.c.e.d. was used by A. L. Garkavi [**3**] to characterize the normed linear spaces in which no bounded point set has two Čebyšev centers. V. Zizler [**4**] has shown that if the conjugate space of S contains a total point sequence, then s is isomorphic to a u.c.e.d. space.

Definition 1.4. Suppose that M is an infinite bounded point set and that x is a point. The statement that the nonnegative number, $r(M, x)$, is the radius of M at x means that if $r > r(M, x)$, then $M - M \cap B(x, r)$ is finite; and if $0 < r < r(M, x)$, then $M - M \cap B(x, r)$ is infinite. The statement that $r(M, K)$ is the radius of M relative to K means that $r(M, K) = \inf \{k |$ for some p in K, $k = r(M, p)\}$. The statement that the point, p, is a K center of M means that p is in K and that $r(M, p) = r(M, K)$. The statement that the point, p, is a center for M means that p is an S center for M.

Definition 1.5. Suppose that M is a point set and that D is a subset of M. The statement that D is a principal subset of M means that $M - D$ is finite.

Definition 1.6. Suppose that M is an infinite point set and that p is a point. The statement that M is almost symmetric about p means that if $c > 0$, then there is a principal subset, R, of M such that if x is in R, then $2p - x$ is in R, or $R \cap B(2p - x, c)$ is infinite.

THEOREM 1.4. *Suppose that M is an infinite bounded point set which is almost symmetric about a point, p. Then p is a center of M.*

Proof. Clearly if $r(M, p) = 0$, then $r(M, S) = 0$, and p is a center of M.

Suppose that $r = r(M, p) > 0$, and $m = r(M, S) < r$, and $0 < k < r - m$. Let q be a point of S, and let R be a principal subset of M such that if x is in R, then $2p - x$ is in R or $R \cap B(2p - x, k)$ is infinite. Since $m + k < r$, $K = R - R \cap B(p, m + k)$ is infinite.

Let $K_1 = \{x \text{ in } K | 2p - x \text{ is in } R\}$, $K_2 = K - K_1$, and for each x in K_2, let $R(x) = B(2p - x, k) \cap R$. Also let

$$H_1(q) = \{x \text{ in } K_2 |\ ||x - q|| \geqq m + k/2\},$$

and let $H_2(q) = K_2 - H_1(q)$. Then one of K_1, $H_1(q)$, and $H_2(q)$ is infinite.

If K_1 is infinite, and if x is in K_1, then $||x - q|| + ||q - (2p - x)|| \geqq 2||x - p|| \geqq 2(\mathrm{m} + k)$, and it follows that $M - M \cap B(q, m + k)$ is infinite; thus, $r(M, q) \geqq m + k$.

Now if x is in K_2, and if y is in $R(x)$, then $||q - x|| + ||y - q|| \geqq ||x - y|| \geqq ||x - (2p - x)|| - ||2p - x - y|| \geqq 2m + k$. If $H_1(q)$ is infinite, then $r(M, q) \geqq m + k/2$. If x is in $H_2(q)$, and if y is in $R(x)$, then $||y - q|| \geqq m + k/2$, and since $R(x)$ is infinite, $r(M, q) \geqq m + k/2$.

So for each point, q, in S, $r(M, q) \geqq m + k/2$; thus, $r(M, S) \geqq m + k/2$. $r(M, S) = m$; thus, by contradiction, $r(M, p) = r(M, S)$. Hence p is a center of M.

THEOREM 1.5. *Suppose that K is convex. Then each two of the following seven statements are equivalent:*

(1) *S is u.c.e.d. K;*

(2) *no infinite bounded point set in S has two K centers;*

(3) *no countably infinite bounded point set in S has two K centers;*

(4) *if M is an infinite bounded point set in S which is almost symmetric about a point, p, in K, then p is the only K center of M;*

(5) *if p and q are points in K such that $||p - q|| < 2$, then $B(p, 1) \cap B(q, 1)$ has only one K center;*

(6) *if p and q are points in K such that $||p - q|| < 2$, then $r(B(p, 1) \cap B(q,1), K) < 1$;*

(7) *no infinite bounded convex point set has two K centers.*

Proof. Assume that M is an infinite bounded point set which has two K centers, p and q. Let $Q = \frac{1}{2}p + \frac{1}{2}q$. Clearly Q is also a K center of M, and $r(M, K) > 0$.

Let $h = ||p - q||/2r(M, K)$, and let d be a number in $(0, 1)$. Let $r = r(M, K) + dr(M, K)$, and let $r' = r(M, K)(1 - d)$. There is a point, t, in $B(p, r) \cap B(q, r) \cap M$ which is not in $B(q, r')$. Let $x = (p - t)/r$, and let $y = (q - t)/r$. x and y are in B, $x - y = 1/r(p - q)$, and $||x - y|| = ||p - q||/r > h$.

$$\begin{aligned} ||\tfrac{1}{2}x + \tfrac{1}{2}y|| &= ||Q - t||/r \\ &\geqq r'/r \\ &> 1 - d. \end{aligned}$$

Hence S is not u.c.e.d. K; thus, (1) implies (2).

Clearly (2) implies (3).

Assume (3), and assume that M is an infinite bounded point set which is almost symmetric about a point, p, in K. By Theorem 1.4, p is a center of M; thus, p is a K center of M. If $r(M, K) = 0$, then clearly p is the only K center of M.

Assume that $r(M, K) > 0$. For each positive integer, n, let R_n be a principal subset of M such that if x is in R_n, then $2p - x$ is in R_n or $R_n \cap B(2p - x, r(M, K)/n)$ is infinite. For each positive integer, n, $R_n - R_n \cap B(p, r(M, K) - r(M, K)/2n)$ is infinite; thus, there is a sequence, $x_1, x_2, \ldots$, of distinct points of M such that for each positive integer, n, x_n is in R_n, and $||x_n - p|| \geqq r(M, K) - r(M, K)/2n$. Let $y_1, y_2, \ldots$ be a sequence of points in M such that for each positive integer, n, $y_n = 2p - x_n$ if $2p - x_n$ is in M, and if $2p - x_n$ is not in M, then y_n is a point of $M \cap B(2p - x_n, r(M, K)/n)$ such that if $i < n$, then $y_i \neq y_n$. $\{y_1, y_2, \ldots\}$ is infinite.

Let $L = \{x_1, y_1, x_2, y_2, \ldots\}$. $L \subseteq M$; thus, $r(L, K) \leqq r(M, K)$. Suppose that $r < r(M, K)$. Let N be a positive integer such that if $n > N$, then $r(M, K) - r(M, K)/n > r$. For $n > N$,

$$\begin{aligned} ||x_n - y_n|| &\geqq ||x_n - (2p - x_n)|| - ||2p - x_n - y_n|| \\ &\geqq 2\left[r(M,K) - \frac{r(M,K)}{2n}\right] - \frac{r(M,K)}{n} \\ &\geqq 2\left[r(M,K) - \frac{r(M,K)}{n}\right] \\ &> 2r. \end{aligned}$$

https://doi.org/10.4153/CJM-1973-105-3 Published online by Cambridge University Press

So, if t is in K, either x_n or y_n is not in $B(t, r)$; thus, $L - L \cap B(t, r)$ is infinite. This implies that $r(L, K) \geqq r(M, K)$; thus, $r(L, K) = r(M, K)$. So if a point, x, is a K center for M, then x is a K center for L. L has only one K center; thus, M has only one K center.

Hence (3) implies (4).

Now assume (4) and assume that p and q are points in K such that $||p - q|| < 2$. Let $L = B(p, 1) \cap B(q, 1)$. L is symmetric about $\frac{1}{2}p + \frac{1}{2}q$, and $\frac{1}{2}p + \frac{1}{2}q$ is in K; thus, L has only one K center. Hence (4) implies (5).

Assume (5), and assume that p and q are points of K such that $||p - q|| < 2$. Let $M = B(p, 1) \cap B(q, 1)$. If $r(M, K) = 1$, then both p and q would be K centers of M. M has only one K center; thus, $r(M, K) < 1$. So (5) implies (6).

Now assume (1) is not true. Then there are points, u and v, in K and a positive number, h, such that $h \leqq 1$, $h \leqq ||u - v||$, and if d is a positive number, then there are points, x and y, in $\bar{B}$ such that $||x - y|| \geqq h$, for some number, c, $x - y = c(u - v)$, and $||\frac{1}{2}x + \frac{1}{2}y|| \geqq 1 - d$.

Let p and q be points in $[u, v]$ such that $||p - q|| = h$. Let $M = B(p, 1) \cap B(q, 1)$. p and q are in K; thus, $r(M, K) \leqq 1$.

Suppose that $d > 0$. Let x and y be points of $\bar{B}$ such that $||x - y|| \geqq h$, for some positive number, c, $p - q = c(x - y)$, and $||\frac{1}{2}x + \frac{1}{2}y|| \geqq 1 - d$. $h = ||p - q|| = c||x - y|| \geqq ch$; thus $1 - c \geqq 0$.

$$\begin{aligned}\left\|\frac{(x+p)+(y+q)}{2} - p\right\| &= \left\|\frac{x+y}{2} - \frac{p-q}{2}\right\| \\ &= \tfrac{1}{2}||x + y - c(x - y)|| \\ &\leqq \tfrac{1}{2}||(1 - c)x|| + \tfrac{1}{2}||(1 + c)y|| \\ &\leqq 1.\end{aligned}$$

Likewise, $||[(x + p) + (y + q)]/2 - q|| \leqq 1$; thus, $[(x + p) + (y + q)]/2$ is in $\bar{M}$. $||(x + p + y + q)/2 - (p + q)/2|| = ||\frac{1}{2}x + \frac{1}{2}y|| \geqq 1 - d$. Since M is convex, this implies that $r(M, (p + q)/2) \geqq 1 - d$; thus, $r(M, (p + q)/2) \geqq 1$. M is symmetric about $(p + q)/2$; thus, $(p + q)/2$ is a K center for M. $r(M, K) \leqq 1$, and $r(M, (p + q)/2) \geqq 1$; thus, $r(M, K) = 1$. This implies that (6) is not true; thus, (6) implies (1).

Clearly (2) implies (7). Assume (7), and assume that p and q are points of K such that $||p - q|| < 2$. Let $M = B(p, 1) \cap B(q, 1)$. M is symmetric about $\frac{1}{2}p + \frac{1}{2}q$; thus, M has a K center. M is infinite, bounded and convex; thus, M does not have two K centers. Hence (7) implies (5).

THEOREM 1.6. *The following two statements are equivalent.*

(1) *if M is an infinite bounded point set and L is the closed convex hull of M, then M does not have two L centers;*

(2) *S is u.c.e.d.*

Proof. Assume that (2) is not true. Then by Theorem 1.5, there is an infinite bounded point set, D, which has two centers, p and q. Let $M = D \cup \{p, q\}$, and let L be the closed convex hull of M. Then p and q are L centers of M. Hence (1) implies (2).

Now assume (2), and assume that M is an infinite bounded point set. Let L be the closed convex hull of M. Clearly S is u.c.e.d. L; thus, by Theorem 1.5, M does not have two L centers So (2) implies (1).

Definition 1.7. The statement that S has property H in every direction of $K(HK)$ means that if $h > 0$, and if p and q are two points of K, then there is a number, d, in $(0, 1)$ such that if $x \neq N(S)$ is a point such that for some number, c, $x = c(p - q)$, and $B(x, 1 - d)$ intersects B, then there is a point, t, such that $\|t\| < h$, and $B \cap B(x, 1 - \mathrm{d}) \subseteq B(t, 1 - d)$. If t is also required to be in the interval, $[N(S), hx/\|x\|]$, then S is said to have linear property H in every direction of $K(LHK)$. If S has HS (LHS), and if for each $h > 0$, there is a number, d, as above which does not depend on the direction, then S is said to have property H (linear property H).

THEOREM 1.7. *Suppose that S is strictly convex and that K is convex. Then each two of the following three statements are equivalent:*

(1) *S is u.c.e.d. K;*

(2) *S has LHK;*

(3) *S has HK.*

Proof. Assume (1), and assume that t is a point of unit norm such that S is uniformly convex in the t direction. Suppose that $h > 0$. Let d be a number in $(0, 1)$ such that if x and y are two points in $\bar{B}$ such that for some number, c, $x - y = ct$, and $\|x - y\| \geqq h$, then $\|\frac{1}{2}x + \frac{1}{2}y\| < 1 - d$. Let k be a number such that $B(kt, 1 - d)$ intersects B, $k \neq 0$.

Assume that $|k| < h$. Then $\|kt\| < h$, kt is in $[N(S)hkt/|k|]$, and $B \cap B(kt, 1 - d) \subseteq B(kt, 1 - d)$.

Now assume that $|k| \geqq h$. Let $q = (hk/2|k|)t$. $\|q\| < h$ and q is in $[N(s), (hk/|k|)t]$. Let x be a point in $B \cap B(kt, 1 - d)$, and let $y = x - 2q$. $1 - h/|k| \geqq 0$; thus,

$$\begin{aligned} \|y\| &= \left\| \frac{hkt}{|k|} - \frac{h}{|k|} x + \left(\frac{h}{|k|} - 1 \right) x \right\| \\ &\leqq \frac{h}{|k|} \|kt - x\| + \left(1 - \frac{h}{|k|} \right) \|x\| \\ &< 1. \end{aligned}$$

Hence y is in B. $x - y = 2q = (hk/|k|)t$, and $\|x - y\| = h$; thus, $\|\frac{1}{2}x + \frac{1}{2}y\| < 1 - d$. $\|\frac{1}{2}x + \frac{1}{2}y\| = \|x - q\|$; thus, x is in $B(q, 1 - d)$. So $B \cap B(kt, 1 - d) \subseteq B(q, 1 - d)$.

This implies that S has linear property H in the t direction; thus S has LHK. So (1) implies (2).

Clearly (2) implies (3). Assume (3), and let t be a point of unit norm such that S has property H in the t direction. Suppose that $h > 0$. Let d be a number in $(0, 1)$ such that if k is a number such that B intersects $B(kt, 1 - d)$, then there is a point, q, such that $||q|| < h/2$, and $B \cap B(kt, 1 - d) \subseteq B(q, 1 - d)$.

Suppose that x and y are points of $\bar{B}$ such that $||x - y|| \geqq h$, $||\frac{1}{2}x + \frac{1}{2}y|| = 1 - d$, and for some number, c, $x - y = ct$. $||x - (x - y/2)|| = ||-y - (x - y)/2|| = ||\frac{1}{2}x + \frac{1}{2}y||$; thus, x and $-y$ are in the closure of $B \cap B((x - y)/2, 1 - d)$.

Let p be a point such that $||p|| < h/2$, and $B \cap B((x - y)/2, 1 - d) \subseteq B(p, 1 - d)$. Then $||x - p|| = ||y + p|| = 1 - d$.

$$\left\|\frac{(x - p) + (y + p)}{2}\right\| = ||\tfrac{1}{2}x + \tfrac{1}{2}y|| = 1 - d.$$

So S is not strictly convex.

S is strictly convex; thus, if x and y are points in $\bar{B}$ such that $||x - y|| \geqq h$, and for some number, c, $x - y = ct$, then $||\frac{1}{2}x + \frac{1}{2}y|| \neq 1 - d$; thus, $||\frac{1}{2}x + \frac{1}{2}y|| < 1 - d$. So S is uniformly convex in the t direction. This implies that S is u.c.e.d. K; thus, (3) implies (1).

THEOREM 1.8. *Suppose that g is a norm on E_2 such that $g(1, 1) = g(1, -1) = g(0, 1) = 1$. Then (E_2, g) does not have linear property H.*

Proof. Let $h = g(1, 0)/2$, and let d be a number in $(0, \frac{1}{2})$. Let L denote the line which contains $(1, d)$ and $(0, 0)$. For each number, b, in $[0, 1]$, let L_b denote the line which contains $(1, 1)$ and $(b, 0)$, and let $k(b) = b/(1 - d + db)$. Then if $0 \leqq b < 1$, L intersects L_b at the point $(k(b), dk(b))$ and $g((k(b), dk(b)) - (b, 0)) < d$.

Suppose that x is in the interval from $(0, 0)$ to $(1, d)/g(1, d)$ such that $g(x) < h$. $h < 1$; thus, there is a number, b, such that $0 \leqq b < 1$, and $L \cap L_b = \{x\}$. $g((b, 0) - (1, 1)) = g(1 - b, 1) = 1$, $g(x - (1, 1)) = 1 - g((k(b), dk(b)) - (b, 0)) > 1 - d$, and $g((1, d) - (1, 1)) = g(0, 1 - d) = 1 - d$. So $(1, 1)$ is a limit point of $B \cap B((1, d), 1 - d)$, and $(1, 1)$ is not a limit point of $B(x, 1 - d)$. This implies that $B \cap B((1, d), 1 - d)$ is not a subset of $B(x, 1 - d)$; thus, (E_2, g) does not have linear property H.

COROLLARY 1.1. *Suppose that S has linear property H. Then S is strictly convex.*

Proof. Assume that S is not strictly convex. Let p and q be points such that $||p|| = ||q|| = ||\frac{1}{2}p + \frac{1}{2}q|| = 1$, and let L be the linear span of $\{p, q\}$. Then there is a norm, g, on E_2 such that $g(1, 1) = g(1, -1) = g(0, 1) = 1$, and such that L is congruent to (E_2, g). This implies that L does not have linear property H; thus, S does not have linear property H. So if S has linear property H, then S is strictly convex.

THEOREM 1.9. *Each two of the following three statements are equivalent:*

(1) *S is uniformly convex;*
(2) *S has linear property H;*
(3) *S has property H and is strictly convex.*

Proof. Assume (1), and assume that $h > 0$. Let d be a number in $(0,1)$ such that if x and y are points in $\bar{B}$ such that $||x - y|| \geqq h$, then $||\frac{1}{2}x + \frac{1}{2}y|| < 1 - d$. Let $p \neq N(S)$ be a point such that $B(p, 1 - d)$ intersects B. Then by the proof of Theorem 1.7, there is a point, q, in $[N(S), hp/||p||]$ such that $||q|| < h$, and $B \cap B(p, 1 - d) \subseteq B(q, 1 - d)$. This implies that S has linear property H; thus, (1) implies (2).

By Corollary 1.1, if S has linear property H, then S is strictly convex; thus, clearly (2) implies (3).

Assume (3), and assume that $h > 0$. Let d be a number in $(0, 1)$ such that if D is the intersection of a $1 - d$ norm ball with B, then there is a point, q, such that $||q|| < h/2$, and $D \subseteq B(q, 1 - d)$. By the proof of Theorem 1.7, this implies that if x and y are two points in $\bar{B}$ such that $||x - y|| \geqq h$, then $||\frac{1}{2}x + \frac{1}{2}y|| < 1 - d$; thus, S is uniformly convex. Hence (3) implies (1).

COROLLARY 1.2. *There is a reflexive Banach space which does not have property H.*

Proof. M. M. Day [**2**] has shown that there is a reflexive Banach space, S, which is strictly convex, but which is not uniformly convex. By Theorem 1.9, S does not have property H.

We note that there is a norm, g, on l_2 (equivalent to the usual norm) such that (l_2, g) is locally uniformly convex and not u.c.e.d. There is a norm, h, on c_0 such that (c_0, h) is u.c.e.d. but not locally uniformly convex.

THEOREM 1.10. *If M is an infinite bounded point set and p is a center of M, then p is a center of each principal subset of M.*

2. Existence of centers.

Definition 2.1. Suppose that M is an infinite bounded point set and that $r > r(M, K)$. Then $G(M, K, r)$ is the set to which p belongs if and only if p is a point of K, and $r(M, p) \leqq r$. If M has a K center, then $G(M, K)$ is the set of all K centers of M.

The following lemma is stated without proof.

LEMMA 2.1. *Suppose that M is an infinite bounded point set in S and that $r > r(M, K)$. Then:*
(1) *$G(M, K, r)$ is bounded;*
(2) *if K is closed, $G(M, K, r)$ is closed;*
(3) *if K is convex, $G(M, K, r)$ is convex;*
(4) *if $r_1 > r$, then $G(M, K, r) \subseteq G(M, K, r_1)$;*

(5) *if* $c = r - r(M, r)$ *and if* p *is in* $G(M, K, r - c/2)$, *then* $K \cap B(p, c/2) \subseteq G(M, K, r)$;

(6) *a point*, p, *is a* K *center for* M *if and only if* p *is in* $G(M, r, R)$ *for each* $R > r(M, K)$.

Definition 2.2. The statement that S is K center point complete means that each infinite bounded point set in S has a K center.

THEOREM 2.1. *Suppose that* K *is closed and compact. Then* S *is* K *center point complete.*

Proof. Suppose that M is an infinite bounded point set in S and that $r_n = r(M, K) + 1/n$ for each positive integer, n. If for some positive integer, n, $G(M, K, r_n)$ is finite, then clearly M has a K center.

Suppose that $G(M, K, r_n)$ is infinite for each positive integer, n. Then there is a sequence, $x_1, x_2, \ldots$ of distinct points such that x_n is in $G(M, K, r_n)$ for each positive integer, n. Since K is compact, $\{x_1, x_2, \ldots\}$ has a limit point, x. So x is a limit point of and thus in $G(M, K, r_n)$ for each positive integer, n. This implies that if $r > r(M, K)$, then x is in $G(M, K, r)$; thus, x is a K center of M.

THEOREM 2.2. *Suppose that* S *is reflexive and that* K *is closed and convex. Then* S *is* K *center point complete.*

Proof. Suppose that M is an infinite bounded point set. For each positive integer, n, let $G_n = G(M, K, r(M, K) + 1/n)$. $G_1, G_2, \ldots$ is a monotonic sequence of closed convex point sets, and S is reflexive; thus, there is a point, p, which is in each term of the sequence. This implies that if $r > r(M, K)$, then p is in $G(M, K, r)$; thus, p is a K center of M. So S is K center point complete.

THEOREM 2.3. *Suppose that* K *is a linear manifold in* S *and that* A *is a Hamel basis for* K. *Let* $\{a_1, a_2, \ldots, a_N\}$ *be a finite subset of* A *which does not span* K, *let* $L_0 = K$, *and let* L_n *be the linear span of* $A - \{a_1, \ldots, a_n\}$ *for* $n = 1, 2, \ldots, N$. *Suppose that* M *is an infinite bounded point set in* S *and that* $0 \leqq n \leqq N$. *Then there is a point*, p_n, *in* K *such that if* $r > r(M, K)$, *then* $G(M, K, r)$ *intersects* $L_n + p_n$.

Definition 2.3. Suppose that L is a linear manifold in S. The statement that L has property H in S means that if $h > 0$, there is a number, d, in $(0, 1)$ such that if p is in L and if $B(p, 1 - d)$ intersects B, then there is a point, q, in L such that $||q|| < h$, and $B \cap B(p, 1 - d) \subseteq B(q, 1 - d)$.

Definition 2.4. Suppose that K is a linear manifold in S. The statement that K almost has property H in S means that there is a linear manifold, L, of K and a finite point set, A, of K such that $L \cup A$ spans K, and L has property H in S.

THEOREM 2.4. *Suppose that* K *is a complete linear manifold in* S *and that* K *almost has property* H *in* S. *Then* S *is* K *center point complete.*

Proof. Let L be a linear manifold in K such that there is a finite point set, A, in K such that $L \cup A$ spans K, and L has property H in S. Let M be an infinite bounded point set in S. If $L = \{N(S)\}$, then clearly there is a point, p, in K such that if $r > r(M, K)$, then $L + p$ intersects $G(M, K, r)$. If $L \neq \{N(S)\}$, then by Theorem 2.3, there is a point, p, in K such that if $r > r(M, K)$, then $L + p$ intersects $G(M, K, r)$. Let $h_1, h_2, \ldots$ be a sequence of positive numbers such that $\prod_{i=1}^{\infty}(1 - h_i) \geqq \frac{1}{2}$, $\sum_{i=1}^{\infty} h_i$ exists, and $h_i < 1$ for $i = 1, 2, \ldots$ For each positive integer, n, let d_n be a number in $(0, 1)$ such that if t is in L, and $B(t, 1 - d_n)$ intersects B, then there is a point, q, in L such that $||q|| < h_n$, and $B \cap B(t, 1 - d_n) \subseteq B(q, 1 - d_n)$, let $r_n = r(M, K)/\prod_{i=n}^{\infty}(1 - d_i)$, and let p_n be a point in $L + p$ such that $r(M, p_n) < r_n$. Clearly, $d_n \leqq h_n$ for each n; thus,

$$\prod_{i=1}^{\infty} (1 - d_i) \geqq \prod_{i=1}^{\infty} (1 - h_i).$$

$r_{n+1} = r_n - r_n d$ for each n; thus $r_1, r_2, \ldots$ converges to $r(M, K)$.

Since L has property H in S, there is a sequence of points, $q_1, q_2, \ldots$, in $L + p$ such that for each positive integer, n,

$$||q_{n+1} - q_n|| < r_n h_n,\ B(q_n, r_n) \cap B(p_{n+1}, r_{n+1}) \subseteq B(q_{n+1}, r_{n+1})$$

and $r(M, q_{n+1}) < r_{n+1}$.

Suppose that $c > 0$. Let N be a positive integer such that if $m > n \geqq N$, then $\sum_{i=n}^{m-1} h_i < c/r_1$. Suppose that $m > n \geqq N$. Then

$$\begin{aligned} ||q_m - q_n|| &\leqq ||q_m - q_{m-1}|| + \ldots + ||q_{n+1} - q_n|| \\ &\leqq \sum_{i=n}^{m-1} r_i h_i \\ &< r_1 \sum_{i=n}^{m-1} h_i \\ &< c. \end{aligned}$$

So $q_1, q_2, \ldots$ is a Cauchy sequence in K. K is complete; thus, there is a point, q, in K such that $q_1, q_2, \ldots$ converges to q.

Suppose that $r > r(M, K)$. Then there is an integer, n, such that $|r_n - r(M, K)| < (r - r(M, K))/2$, and $||q_n - q|| < (r - r(M, K))/2$. This implies that $B(q_n, r_n) \subseteq B(q, r)$; thus, $r(M, q) < r$. This implies that $r(M, q) = r(M, K)$; thus, q is a K center of M. Hence S is K center point complete.

COROLLARY 2.1. *Suppose that S is complete and has property H. Then S is S center point complete.*

Proof. Since S has property H, S almost has property H in S; thus, by Theorem 2.4, S is S center point complete.

Definition 2.5. Suppose that I is a set. Then $m(I)$ denotes the set of all bounded functions from I into the numbers.

Definition 2.6. Suppose that I is a set, that $k > 0$, and that $x = \{x_i\}$ is a point in $m(I)$. Then $x(k)$ is the point of $m(I)$ defined by: if j is in I, then

$$x_j(k) = \begin{cases} x_j, \text{ if } |x_j| \leqq k; \\ k, \text{ if } k < x_j; \\ -k, \text{ if } x_j < -k. \end{cases}$$

The statement that the linear manifold, L, of $m(I)$ is a T-manifold means that if x is in L and if $k > 0$, then $x(k)$ is in L.

THEOREM 2.5. *Suppose that I is a set and that L is a T-manifold in $m(I)$. Then L has property H in $m(I)$.*

Proof. Suppose that $h > 0$, that $0 < d < k < \min\{1, h\}$, and that p is a point in L such that $B(p, 1 - d)$ intersects B. Let $q = p(k)$, and let x be a point in $m(I)$ which is in $B \cap B(p, 1 - d)$. Then q is in L, and $||q|| = \min\{||p||, k\} < h$. Also if j is in I,

$$|x_j - q_j| = \begin{cases} |x_j - p_j|, \text{ if } |p_j| \leqq k; \\ |x_j - k|, \text{ if } k < p_j; \\ |x_j + k|, \text{ if } p_j < -k. \end{cases}$$

If $q_j = p_j$, then $|x_j - q_j| = |x_j - p_j| \leqq ||x - p|| < 1 - d$. If $q_j = k$ and if $k \leqq x_j$, then $|x_j - q_j| = x_j - k < 1 - k < 1 - d$. If $q_j = k$ and if $k > x_j$, then $|x_j - q_j| = k - x_j < p_j - x_j \leqq ||x - p|| < 1 - d$. A similar argument holds in case $q_j = -k$; thus, $||x - q|| < 1 - d$. So L has property H in $m(I)$.

COROLLARY 2.2. *Each of m, c_0, c, and $c[0, 1]$ has property H, and c_0 has property H in m.*

THEOREM 2.6. *There is a Banach space, S, which almost has property H in S, but which does not have property H.*

Proof. Let $p = (1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \ldots) = (p_i)$, let S be the linear span of $c_0 \cup \{p\}$, and let $|| \cdot ||$ be the sup norm on S. Clearly S is complete. c_0 has property H in m, and S is a linear manifold of m which contains c_0; thus, c_0 has property H in S. So S almost has property H in S.

Suppose that $h < \frac{1}{8}$ and that d is in $(0, 1)$. If $d > h$, then clearly there is a point, x, in S such that if q is in $B(N(S), h)$, then $B(x, 1 - d) \cap B$ is not a subset of $B(q, 1 - d)$.

Suppose that $d \leqq h$ and that q is in $B(N(S), h)$. Then there is a number, k, and a point $x = (x_1, x_2, \ldots)$ in c_0 such that $q = kp + x$. $||q|| < h$; thus, $|k| < h$.

Let n be the positive integer such that $1/n > d$, and $1/n + 1 \leqq d$. Let N be a positive integer such that if $i \geqq N$, then $|x_i| < h/n$, and let m be a positive integer such that $m \geqq N$, and $p_m = 1/n$. Then $kp_m < hp_m < \frac{1}{8}n$; thus, $q_m = x_m + kp_m < h/n + \frac{1}{8}n < \frac{1}{4}n$. Hence $-q_m > -\frac{1}{4}n$.

Let b be a number such that $1 + \frac{1}{4}n - \frac{1}{2}n < b < 1$, and let c be a number such that $nd < c < 1$. Then $b - c/n < b - d < 1 - d$. Let $y = (y_1, y_2, \ldots)$ be the point such that $y_m = b - c/m$, and $y_i = 0$ if $i \neq m$, and let $p_0 = cp + y$. $||p_0|| = \sup \{c, b\}$; thus, $||p_0|| < 1$. $||p - p_0|| = \sup \{1 - c, b - c\ n\}$; thus, $||p - p_0|| < 1 - d$. So p_0 is in $B \cap B(p, 1 - d)$. $||p_0 - q|| \geqq cp_m + y_m - q_m - b - q_m \geqq b - \frac{1}{4}n$; thus, $||p_0 - q|| > 1 - \frac{1}{2}n \geqq 1 - d$. Hence p_0 is not in $B(q, 1 - d)$. So S does not have property H.

Definition 2.7. Suppose that M is a bounded point set in S and that x is a point in S. The statement that the nonnegative number, $R(M, x)$, is the Čebyšev radius of M at x means that if $r > R(M, x)$, then $M \subseteq B(x, r)$, and if $0 < r < R(M, x)$, then M is not a subset of $B(x, r)$. The statement that $R(M, K)$ is the Čebyšev K radius of M means that $R(M, K) = \inf \{R(M, x) | x$ is in $K\}$. The statement that the point, p, is a Čebyšev K center of M means that p is in K, and $R(M, K) = R(M, p)$.

Note that Theorem 1.10 is not true for Čebyšev centers.

THEOREM 2.7. *Suppose that M is an infinite bounded convex point set in S and that p is a point of S. Then $R(M, p) = r(M, p)$.*

Proof. Suppose that $r > R(M, p)$. Then $M \subseteq B(p, r)$; thus, $r(M, p) \leqq r$. Hence $r(M, p) \leqq R(M, p)$.

Suppose that $r > r(M, p)$. Then $M - M \cap B(p, r)$ is finite. Since M is convex, this implies that if $c > 0$, then $M \subseteq B(p, r + c)$; thus, $R(M, p) \leqq r + c$. So $R(M, p) \leqq r(M, p)$. Hence $R(M, p) = r(M, p)$.

COROLLARY 2.3. *Suppose that M is an infinite bounded convex point set in S and that p is a point of S. Then $r(M, K) = R(M, K)$, and p is a K center of M if and only if p is a Čebyšev K center of M.*

THEOREM 2.8. *Suppose that S is K center point complete. Then S is Čebyšev K center point complete.*

Proof. Suppose that x is a point of S. Let $x_1, x_2, \ldots$ denote a sequence of distinct points which converges to x. Let $M = \{x_1, x_2, \ldots\}$, and let $L = \{x\}$. M has a K center, p, and clearly p is also a Čebyšev K center of L. So each degenerate point set in S has a Čebyšev K center.

Suppose that L is a nondegenerate bounded point set in S. Let M denote the convex hull of L. M is infinite and bounded; thus, M has a K center, p. Since M is convex, p is also a Čebyšev K center of M. A norm ball contains M if and only if it contains L; thus, p is a Čebyšev K center of L.

Hence S is Čebyšev K center point complete.

THEOREM 2.9. *Suppose that S is a conjugate space and that M is an infinite bounded convex point set in S. Then M has a center.*

Proof. It is known that S is Čebyšev S center point complete [**3**]; thus, by Corollary 2.3, M has a center.

THEOREM 2.10. *There is a u.c.e.d. conjugate space isomorphic to m which contains an infinite, bounded, closed, and convex point set, M, which contains none of its centers.*

Proof. Let $||\cdot||$ denote the usual norm on m, and let g denote the norm on m defined by

$$g(x) = \left[||x||^2 + \sum_{i=1}^{\infty} \frac{x_i^{\,2}}{4^i}\right]^{\frac{1}{2}}, \text{ if } x = (x_i) \text{ is in } m.$$

Clearly g is equivalent to $||\cdot||$, and by Theorem 1.3, (m, g) is u.c.e.d. Let g_0 denote the norm on c_0 such that $g_0(x) = g(x)$ if x is in c_0. Then (m, g) is the second conjugate space of (c_0, g_0).

Let $M = \{x = (x_i) \text{ in } c_0 | 0 \leqq x_i \leqq 1 \text{ for } i = 1, 2, \ldots\}$, and let $p = (p_i)$ be a point in c_0. Let n be an integer such that $p_n < \frac{1}{4}$, and let $q = (q_i)$ be a point of c_0 such that $q_n = \frac{1}{2}$, and $q_i = p_i$ if $i \neq n$. Suppose that $x = (x_i)$ is in M. Let $y = (y_i)$ be the point of M such that $y_n = 1$ and $y_i = x_i$ if $i \neq m$. $||q - x||^2 \leqq ||p - y||^2$, and $(q_n - x_n)^2/4^n < (p_n - y_n)^2/4^n - 2/4^n$; thus, $g^2(q - x) < g^2(p - y) - \frac{1}{4}n$. This implies that $r(M, p) > r(M, q)$; thus, no point of c_0 is a center of M. Hence no point of M is a center of M.

COROLLARY 2.4. *There is a u.c.e.d. space isomorphic to c_0 which contains an infinite, bounded, closed, and convex point set which does not have a center.*

Proof. Let g_0 denote the norm and M denote the set defined in the proof of Theorem 2.10. Clearly g_0 is equivalent to the usual norm on c_0, and (c_0, g_0) is u.c.e.d. by Theorem 1.3. By the proof of Theorem 2.10, M has no center.

THEOREM 2.11. *There is an infinite, bounded, closed and convex point set, M, in c_0 which does not have an M center.*

Proof. Let $f = (f_i)$ denote a point in the conjugate space of c_0 such that

$$f_1 = f_2 = 1, \sum_{i=1}^{\infty} f_i = 3, \text{ and } f_i > 0 \text{ for } i = 1, 2, \ldots$$

Let $M = \{x \text{ in } c_0 | f(x) = 1 \text{ and } ||x|| \leqq 1\}$. Clearly M is infinite, bounded, closed, and convex. Let c denote the distance from M to the origin. Clearly $c > 0$.

Suppose that $x = (x_i)$ is in M and that $e > 0$. Let n be a positive integer such that $|x_n| > ||x|| - e/2$.

Case 1. Suppose that $n \geqq 3$. Let $y = (y_i)$ be the point such that if $x_n \geqq 0$, then $y_1 = f_n$, $y_2 = 1$, $y_n = -1$, and $y_i = 0$ otherwise, and if $x_n < 0$, then $y_1 = -f_n$, $y_2 = 1$, $y_n = 1$, and $y_i = 0$ otherwise. $||y|| = 1$, and $f(y) = 1$; thus, y is in M. $||x - y|| \geqq |x_n - y_n| = 1 + |x_n| > 1 + ||x|| - e$.

Case 2. Suppose that $n = 1$ or $n = 2$. Let $k = 1$ if $n = 2$, and let $k = 2$ if $n = 1$. $\sum_{i=3}^{\infty} f_i = 1$; thus, there is a positive integer, N, such that $\sum_{i=3}^{N} f_i > 1 - e/2$. Let $y = (y_i)$ be the point such that if $x_n \geqq 0$, $y_k = 1$, $y_n = -\sum_{i=3}^{N} f_i$, $y_i = 1$ for $i = 3, 4, \ldots, N$, and $y_i = 0$ for $i > N$, and if $x_n < 0$, $y_k = 1$,

$y_n = \sum_{i=3}^{N} f_i$, $y_i = -1$ for $i = 3, 4, \ldots, N$, and $y_i = 0$ for $i > N$. $||y|| = 1$, and $f(y) = 1$; thus, y is in M. $||x - y|| \geqq |x_n - y_n| > 1 + ||x|| - e$.

$r(M, x) > 1 + ||x|| - e$ since $||x - y|| > 1 + ||x|| - e$, and M is convex; thus, $r(M, x) \geqq 1 + ||x||$. Since $r(M, N(c_0)) \leqq 1$, $r(M, x) \leqq 1 + ||x||$; thus, $r(M, x) = 1 + ||x||$. Hence x is an M center for M if and only if x is a near point of M to $N(c_0)$. Since f is not regular, M has no near point to $N(c_0)$; thus, M has no M center.

COROLLARY 2.5. *There is an infinite, bounded, closed and convex point set, M, in m which has no M center.*

Proof. Clearly the point set, M, defined in the proof of Theorem 2.11 is such an example.

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