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GENERALISED GAUSSIAN FIBONACCI NUMBERS

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In this paper, generalised Gaussian Fibonacci numbers are defined and, using the recurrence relation satisfied by them, we obtain a number of summation identities involving the products of combinations of Fibonacci, Pell and Chebyshev polynomials.

1. Introduction

Horadam [3], in 1963, and Berzsenyi [1], in 1977, defined complex Fibonacci numbers by following two different approaches. Horadam defined the complex Fibonacci sequence $\{F_n^*\}$ by writing

$$F_n^* = F_n + i\, F_{n+1}$$

where F_n is the n^{th} Fibonacci number. Berzsenyi defined them as a set of complex numbers at the Gaussian integers such that the Fibonacci recurrence relation is satisfied at any triple of adjacent points.

In 1981, Harman [2] also defined the complex Fibonacci numbers at the Gaussian integers, but used the direct analogy with the Fibonacci recurrence relation. These numbers include Horadam's complex Fibonacci numbers and they have a symmetry condition which is not satisfied by the numbers considered by Berzsenyi.

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The object of this article is not just to extend Harman's idea, but by doing so, to obtain a wealth of significant summation identities involving the products of combinations of Fibonacci numbers and polynomials, Pell numbers and polynomials, Chebyshev polynomials and sine functions. Our main result is in equations (5.1) and (5.2) which have the potential of providing many more identities than the ones mentioned in this paper.

2. Definition

Let (n,m), $n,m \in \mathbb{Z}$, *denote the set of Gaussian integers* $(n,m) = n + im$. *Let*

$$G : (n,m) \to C ,$$

where C *is the set of complex numbers, be a function defined as follows.* For fixed real numbers p_1, q_1, p_2 and q_2, define

(2.1) $G(0,0) = 0$, $G(1,0) = 1$, $G(0,1) = i$, $G(1,1) = p_2 + ip_1$

with the following conditions:

$$G(n+2,m) = p_1G(n+1,m) - q_1G(n,m) \quad \text{and} \tag{2.2}$$

$$G(n,m+2) = p_2G(n,m+1) - q_2G(n,m). \tag{2.3}$$

The conditions (2.2), (2.3) with the initial values (2.1) are sufficient to obtain a unique value for every Gaussian integer.

3. Expression for $G(n,m)$

Let U_n and V_n denote Lucas fundamental sequences [4] defined by the recurrence relations

$$\begin{cases} U_{n+2} = p_1U_{n+1} - q_1U_n , \text{ and} \\ V_{n+2} = p_2V_{n+1} - q_2V_n \end{cases} \tag{3.1}$$

with initial values

$U_0 = 0$, $U_1 = 1$ and $V_0 = 0$, $V_1 = 1$. Then

LEMMA 3.1.

$$\begin{cases} G(n,0) = U_n \text{ and} \\ G(0,m) = iV_m \end{cases} \tag{3.2}$$

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Proof. The proof is simple and therefore omitted.

The first few terms of $\{U_n\}$ and $\{V_n\}$ are

$$U_2 = p_1 \qquad U_3 = p_1^2 - q_1$$

$$U_4 = p_1^3 - 2p_1q_1 \qquad U_5 = p_1^4 - 3p_1^2q_1 + q_1^2$$

. . . .

with similar values for V_2, V_3, V_4 and V_5 when p_1 and q_1 are replaced respectively by p_2 and q_2.

THEOREM 3.2. *$G(n,m)$ is given by*

$$G(n,m) = U_nV_{m+1} + i\,U_{n+1}\,V_m \,. \tag{3.3}$$

Proof. The proof is by induction. Suppose (3.3) is true for all integers $0, 1, \ldots, n$ for the first number in the ordered pair (n,m) and for all interers $0, 1, \ldots, m$ for the second number. Now by (2.2)

$$G(n+1,m) = p_1G(n,m) - q_1\,G(n-1,m) \,. \tag{3.4}$$

Applying (3.3) to the right hand side of (3.4), we get

$$\begin{aligned} G(n+1,m) &= p_1(U_nV_{m+1} + iU_{n+1}V_m) - q_1(U_{n-1}V_{m+1} + iU_nV_m) \\ &= V_{m+1}(p_1U_n - q_1U_{n-1}) + iV_m(p_1U_{n+1} - q_1U_n) \,. \end{aligned}$$

Hence by (3.1), we have

$$G(n+1,m) = U_{n+1}V_{m+1} + iU_{n+2}V_m \,. \tag{3.5}$$

Similarly we can get

$$G(n,m+1) = U_nV_{m+2} + iU_{n+1}V_{m+1} \,. \tag{3.6}$$

(3.5) and (3.6) show that (3.3) is true for all non-negative integers.

4. Recurrence Relation for $G(n,m)$

THEOREM 4.1. *For fixed n,m $(n,m = 0,1,2,\ldots)$, the recurrence relation for $G(n,m)$ is given by the following:*

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$$(4.1)\quad G(n+2k+s,\ m+2k+s) = (p_2+ip_1)\sum_{j=1}^{k}(q_1q_2)^{k-j}\,U_{n+2j+s}\,V_{m+2j+s}$$

$$-\ (p_1q_2+ip_2q_1)\sum_{j=1}^{k}(q_1q_2)^{k-j}\,U_{n+2j+s-1}\,V_{m+2j+s-1}$$

$$+\ (q_1q_2)^k\,G(n+s,m+s)\ ,$$

where $s = 0,1$.

Proof. For the proof, we again resort to induction on k. First consider $G(n+2,m+2)$ and $G(n+3,m+3)$ that is (4.1) for $k=1$, $s = 0$ and $k = 1$, $s = 1$. By (3.3) we have,

$$\begin{aligned} G(n+2,m+2) &= U_{n+2}\,V_{m+3} + i\,U_{n+3}\,V_{m+2} \\ &= U_{n+2}(p_2V_{m+2} - q_2V_{m+1}) + i(p_1U_{n+2} - q_1U_{n+1})V_{m+2} \\ &= (p_2+ip_1)U_{n+2}V_{m+2}-q_2(p_1U_{n+1}-q_1U_n)V_{m+1}-iq_1U_{n+1}(p_2V_{m+1}-q_2V_m) \\ &= (p_2+ip_1)U_{n+2}V_{m+2}-(p_1q_2+ ip_2q_1)U_{n+1}V_{m+1}+q_1q_2(U_nV_{m+1}+iU_{n+1}V_m)\ . \end{aligned}$$

Hence, by (3.3) we get

$$(4.2)\quad G(n+2,m+2)= (p_2+ip_1)U_{n+2}V_{m+2}-(p_1q_2+ip_2q_1)U_{n+1}V_{m+1}+q_1q_2G(n,m).$$

Again, using (3.3), we have

$$\begin{aligned} G(n+3,m+3) &= U_{n+3}V_{m+4} + iU_{n+4}V_{m+3} \\ &= U_{n+3}(p_2V_{m+3} - q_2V_{m+2}) + i(p_1U_{n+3} - q_1U_{n+2})V_{m+3} \\ &= (p_2 + ip_1)U_{n+3}V_{m+3} - q_2(p_1U_{n+2} - q_1U_{n+1})V_{m+2} \\ &\qquad - iq_1U_{n+2}(p_2V_{m+2} - q_2V_{m+1}) \\ &= (p_2 + ip_1)U_{n+3}V_{m+3}- (p_1q_2+ip_2q_1)U_{n+2}V_{m+2}+ \\ &\qquad q_1q_2(U_{n+1}V_{m+2}+iU_{n+2}V_{m+1})\ . \end{aligned}$$

Thus, applying (3.3) again, we get

$$(4.3)\quad G(n+3,m+3)=(p_2+ip_1)U_{n+3}V_{m+3}-(p_1q_2+ip_2q_1)U_{n+2}V_{m+2}+q_1q_2G(n+1,m+1).$$

Now (4.2) and (4.3) show that (4.1) is true for $k = 1$ with $s = 0,1$. Suppose next that (4.1) is true for and up to some positive integer k. We will show that it is also true for $k+1$.

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First, let $s = 0$. Now, although n and m are assumed to be fixed in (4.2), it is easy to see that (4.2) is true for any positive integers n and m. Then replacing n and m by $n + 2k$ and $m + 2k$ respectively in (4.2), we get

$$(4.4)\quad G(n+2k+2, m+2k+2) = (p_2+ip_1)U_{n+2k+2}V_{m+2k+2} - (p_1q_2+ip_2q_1)U_{n+2k+1}V_{m+2k+1} + q_1q_2G(n+2k, m+2k) .$$

Hence from (4.1) with $s = 0$, (4.4) becomes

$$G(n+2k+2, m+2k+2) = (p_2+ip_1)U_{n+2k+2}V_{m+2k+2} - (p_1q_2+ip_2q_1)U_{n+2k+1}V_{m+2k+1}$$
$$+ q_1q_2[(p_2+ip_1)\sum_{j=1}^{k}(q_1q_2)^{k-j}U_{n+2j}V_{m+2j}$$
$$- (p_1q_2+ip_2q_1)\sum_{j=1}^{k}(q_1q_2)^{k-j}U_{n+2j-1}V_{m+2j-1} + (q_1q_2)^kG(n,m)].$$

Combining the first two terms on the right with those in the bracket, we get

$$(4.5)\quad G(n+2k+2, m+2k+2) = (p_2+ip_1)\sum_{j=1}^{k+1}(q_1q_2)^{k+1-j}U_{n+2j}V_{m+2j}$$
$$- (p_1q_2+ip_2q_1)\sum_{j=1}^{k+1}(q_1q_2)^{k+1-j}U_{n+2j-1}V_{m+2j-1}$$
$$+ (q_1q_2)^{k+1}G(n,m) .$$

Identity (4.5) shows that (4.1), with $s = 0$, holds if k is replaced by $k+1$. It can be shown similarly that (4.1)$'$, with $s = 1$, also holds if k is replaced by $k+1$. This completes the proof of Theorem 4.1.

5. Identities Involving Product Terms of $\{U_n\}$ and $\{V_n\}$

Making use of (3.3) in equation (4.1) and then equating the real and imaginary parts on both sides, we get

$$(5.1)\quad p_2\sum_{j=1}^{k}(q_1q_2)^{k-j}U_{n+2j+s}V_{m+2j+s} - p_1q_2\sum_{j=1}^{k}(q_1q_2)^{k-j}U_{n+2j+s-1}V_{m+2j+s-1}$$
$$= U_{n+2k+s}V_{m+2k+s+1} - (q_1q_2)^k\,U_{n+s}V_{m+s+1}$$

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and

$$(5.2)\quad p_1\sum_{j=1}^{k}(q_1q_2)^{k-j}U_{n+2j+s}V_{m+2j+s}-p_2q_1\sum_{j=1}^{k}(q_1q_2)^{k-j}\;U_{n+2j+s-1}V_{m+2j+s-1}$$

$$= U_{n+2k+s+1}V_{m+2k+s} - (q_1q_2)^k\;U_{n+s+1}V_{m+s}\;.$$

6. Special Cases

Case (A). Let $p_1 = p_2 = 2$ and $q_1 = q_2 = 1$. Then $\{U_n\} = \{V_n\}$ and each is the sequence of non-negative integers. Note that $U_n = n$ for each n. Equations (5.1) and (5.2) reduce to

$$(6.1)\qquad 2\sum_{j=1}^{k}(n+2j+s)(m+2j+s) - 2\sum_{j=1}^{k}(n+2j+s-1)(m+2j+s-1)$$

$$= (n+2k+s)(m+2k+s+1) - (n+s)(m+s+1)$$

$$= (n+2k+s+1)(m+2k+s) - (n+s+1)(m+s)$$

Case (B). Let $p_1 = p_2 = 1$ and $q_1 = q_2 = -1$. Then $\{U_n\} = \{V_n\} = \{F_n\}$, the Fibonacci number sequence. Thus (5.1) and (5.2) respectively reduce to

$$(6.2)\quad \sum_{j=1}^{k}F_{n+2j+s}F_{m+2j+s}+\sum_{j=1}^{k}F_{n+2j+s-1}F_{m+2j+s-1}=F_{n+2k+s}F_{m+2k+s+1}-F_{n+s}F_{m+s+1}$$

and

$$(6.3)\quad \sum_{j=1}^{k}F_{n+2j+s}F_{m+2j+s}+\sum_{j=1}^{k}F_{n+2j+s-1}F_{m+2j+s-1}= F_{n+2k+s+1}F_{m+2k+s} - F_{n+s+1}F_{m+s}\;.$$

Hence combining (6.2) and (6.3), we get

$$(6.4)\quad \sum_{j=1}^{2k}F_{n+s+j}F_{m+s+j}=F_{n+2k+s}F_{m+2k+s+1}-F_{n+s}F_{m+s+1}=F_{n+2k+s+1}F_{m+2k+s} - F_{n+s+1}F_{m+s}\;.$$

We observe that (6.4) is the identity unifying Harman's identities (3.8) and (3.9) in [2].

Case (C). Now let $p_1 = p_2 = 2$ and $q_1 = q_2 = -1$ so that $\{U_n\} = \{V_n\} = \{P_n\}$, where $\{P_n\}$ is the Pell number sequence.

Equations (5.1) and (5.2) then reduce to

$$(6.5)\qquad 2\sum_{j=1}^{2k} P_{n+s+j}P_{m+s+j} = P_{n+2k+s}P_{m+2k+s+1} - P_{n+s}P_{m+s+1}$$
$$= P_{n+2k+s+1}P_{m+2k+s} - P_{n+s+1}P_{m+s}$$

Case (D). Next, let $p_1 = 1$, $q_1 = -1$; $p_2 = p$ and $q_2 = q$ so that $\{U_n\} = \{F_n\}$ and $\{V_n\} = \{L_n\}$ is the Lucas Fundamental Sequence [4].

Then equations (5.1) and (5.2) reduce to

$$(6.6)\qquad p\sum_{j=1}^{k}(-q)^{k-j} F_{n+2j+s}L_{m+2j+s} - q\sum_{j=1}^{k}(-q)^{k-j}F_{n+2j+s-1}L_{m+2j+s-1}$$
$$= F_{n+2k+s}L_{m+2k+s+1} - (-q)^k F_{n+s}L_{m+s+1}$$

and

$$(6.7)\qquad \sum_{j=1}^{k}(-q)^{k-j} F_{n+2j+s}L_{m+2j+s} + p\sum_{j=1}^{k}(-q)^{k-j} F_{n+2j+s-1}L_{m+2j+s-1}$$
$$= F_{n+2k+s+1}L_{m+2k+s} - (-q)^k F_{n+s+1}L_{m+s}\ .$$

Solving (6.6) and (6.7) simultaneously, we get, provided $p^2 + q \neq 0$, $q \neq 0$

$$(6.8)\qquad \sum_{j=1}^{k}(-1)^j q^{-j} F_{n+2j+s-1}L_{m+2j+s-1}$$
$$= \frac{(-1)^k}{q^k(p^2+q)}\{pF_{n+2k+s+1}L_{m+2k+s} - p(-q)^kF_{n+s+1}L_{m+s} - F_{n+2k+s}L_{m+2k+s+1} + (-q)^k F_{n+s}L_{m+s+1}\}$$

$$(6.9)\qquad \sum_{j=1}^{k}(-1)^j q^{-j} F_{n+2j+s}L_{m+2j+s}$$
$$= \frac{(-1)^k}{q^k(p^2+q)}\{pF_{n+2k+s}L_{m+2k+s+1} - p(-q)^kF_{n+s}L_{m+s+1} + qF_{n+2k+s+1}L_{m+2k+s} - q(-q)^kF_{n+s+1}L_{m+s}\}$$

Case (E). Now let $p_1 = p_2 = p \neq 0$ and $q_1 = q_2 = q$. Then $\{U_n\} = \{V_n\}$ and each is the Lucas Fundamental Sequence $\{L_n\}$. Equations (5.1) and (5.2), after simplification, reduce to

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$$(6.10)\quad p\sum_{j=1}^{2k}(-1)^{j+1}q^{-j}L_{n+s+j}L_{m+s+j}=L_{n+s}L_{m+s+1}-q^{-2k}\;L_{n+2k+s}L_{m+2k+s+1}$$

$$=L_{n+s+1}L_{m+s}-q^{-2k}L_{n+2k+s+1}L_{m+2k+s}$$

Case (F). Finally, let $p_1 = 2,\ q_1 = -1$ and $p_2 = 1,\ q_2 = -1$. Then $\{U_n\} = \{P_n\}$ and $\{V_n\} = \{F_n\}$.

Equations (5.1) and (5.2), after simple calculations, give the following results.

$$(6.11)\quad \sum_{j=1}^{2k}P_{n+s+j}F_{m+s+j}=\frac{1}{3}[P_{n+2k+s}F_{m+2k+s+1}+P_{n+2k+s+1}F_{m+2k+s}-P_{n+s}F_{m+s+1} - P_{n+s+1}F_{m+s}]$$

$$(6.12)\quad \sum_{j=1}^{2k}(-1)^{j+1}P_{n+s+j}F_{m+s+j}=P_{n+2k+s}F_{m+2k+s+1}-P_{n+2k+s+1}F_{m+2k+s} -P_{n+s}F_{m+s+1}+P_{n+s+1}F_{m+s}\,.$$

7. Special Numerical Cases

It is worthwhile to note the above identities for particular values of m and n. These are listed below.

(A) $m = 0,\ n = 0$

$$\sum_{j=1}^{N}(-1)^j\,j^2=\frac{(-1)^N\,N(N+1)}{2}$$

$$\sum_{j=1}^{N}F_j^2=F_NF_{N+1}$$

$$\sum_{j=1}^{N}P_j^2=\frac{1}{2}\,P_NP_{N+1}$$

$$(7.1)\quad \sum_{j=1}^{k}(-1)^jq^{-j}F_{2j-1}L_{2j-1}=\frac{(-1)^k}{q^k(p^2+q)}\{p\,F_{2k+1}\,L_{2k}-F_{2k}\,L_{2k+1}\}$$

$$(7.2)\quad \sum_{j=1}^{k}(-1)^jq^{-j}F_{2j}\,L_{2j}=\frac{(-1)^k}{q^k(p^2+q)}\{pF_{2k+2}L_{2k+1}-F_{2k+1}L_{2k+2}\}$$

where in both the last identities $q \neq 0$ and $p^2+q \neq 0$.

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(7.3) $$\sum_{j=1}^{N} (-1)^j q^{-j} L_j^2 = \frac{(-1)^N}{p} q^{-N} L_N L_{N+1}$$

$$\sum_{j=1}^{N} P_j F_j = \frac{1}{3}[P_N F_{N+1} + P_{N+1} F_N]$$

$$\sum_{j=1}^{N} (-1)^{j+1} P_j F_j = (-1)^N [P_N F_{N+1} - P_{N+1} F_N]$$

(B) <u>$n = 1, m = 0$</u>

$$\sum_{j=1}^{N} F_j F_{j+1} = \begin{cases} F_{N+1}^2 - 1, & N \text{ even} \\ F_{N+1}^2, & N \text{ odd.} \end{cases}$$

$$\sum_{j=1}^{N} P_j P_{j+1} = \begin{cases} \frac{1}{2}[P_{N+1}^2 - 1], & N \text{ even} \\ \frac{1}{2} P_{N+1}^2, & N \text{ odd.} \end{cases}$$

(7.4) $$\sum_{j=1}^{N} (-1)^{j+1} q^{-j} L_j L_{j+1} = \begin{cases} \frac{1}{p}[1 - q^{-N} L_{N+1}^2], & N \text{ even} \\ \frac{q^{-N}}{p} L_{N+1}^2, & N \text{ odd} \end{cases}$$

where $p \neq 0$.

(7.5) $$\sum_{j=1}^{k} (-1)^j q^{-j} L_{2j-1} F_{2j} = \frac{(-1)^k}{q^k (p^2+q)} \{pL_{2k} F_{2k+2} - L_{2k+1} F_{2k+1} + (-1)^k q^k\}$$

(7.6) $$\sum_{j=1}^{k} (-1)^j q^{-j} L_{2j} F_{2j+1} = \frac{(-1)^k}{q^k (p^2+q)} \{pL_{2k+1} F_{2k+3} - L_{2k+2} F_{2k+2} - p(-q)^k\}$$

$$\sum_{j=1}^{N} F_j P_{j+1} = \frac{1}{3}[F_{N+1} P_{N+1} + F_N P_{N+2} - 1] .$$

$$\sum_{j=1}^{N} (-1)^{j+1} F_j P_{j+1} = (-1)^N [F_{N+1} P_{N+1} - F_N P_{N+2}] - 1 .$$

Similarly from the right hand sides of equations (6.4), (6.5), and (6.10), we get

$$F_N^2 - F_{N-1} F_{N+1} = (-1)^{N-1} ,$$

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$$P_N^2 - P_{N-1}P_{N+1} = (-1)^{N-1} ,$$

and

(7.7) $$L_N^2 - L_{N-1}L_{N+1} = q^{N-1} .$$

Remark 1. Many other identities may be obtained by other choices of n and m. For example, if $n = 3$ and $m = 0$, the right hand sides of equations (6.4), (6.5) and (6.10) provide the following identities:

$$F_{N+1}F_{N-1} - F_{N+2}F_{N-2} = 2(-1)^N ,$$

$$P_{N+1}P_{N-1} - P_{N+2}P_{N-2} = 5(-1)^N ,$$

and

(7.8) $$L_{N+1}L_{N-1} - L_{N+2}L_{N-2} = q^{N-2}(p^2-q) .$$

8. Products Involving Some Well Known Polynomials.

If we let p_1 or/and p_2 be functions of x, we get summations involving the products of some well known polynomials. We will deal only with those involving Fibonacci numbers and Chebyshev polynomials of the second kind.

Let $p_1 = 1$, $q_1 = -1$, $p_2 = 2x$ and $q_2 = 1$, with $x = \cos\theta$. Thus $\{U_n\} = \{F_n\}$ and $\{V_n\} = \{T_n(x)\}$, where $T_n(x) = \frac{\sin n\theta}{\sin\theta}$ is the nth Chebyshev polynomial of the second kind. Note that this is the same as special case (D) with $p = 2x$ and $q = 1$.

Equations (6.8) and (6.9) then give some summation formulae involving the Fibonacci numbers and Chebyshev polynomials of the second kind. Obviously, those with particular values of m and n are more interesting. We give a few of such formulae in the following.

Substituting $p = 2x$, $q = 1$, we find that equations (7.1), (7.2), (7.5) and (7.6) become

(8.1) $$\sum_{j=1}^{k}(-1)^{j+1}F_{2j-1}T_{2j-1}(x) = \frac{(-1)^k}{1+4x^2}\{F_{2k}T_{2k+1}(x)-2xF_{2k+1}T_{2k}(x)\}$$

(8.2) $$\sum_{j=1}^{k}(-1)^{j+1}F_{2j}T_{2j}(x) = \frac{(-1)^k}{1+4x^2}\{F_{2k+1}T_{2k+2}(x)-2xF_{2k+2}T_{2k+1}(x)\}$$

(8.3) $$\sum_{j=1}^{k}(-1)^{j+1}F_{2j}T_{2j-1}(x) = \frac{(-1)^k}{1+4x^2}\{F_{2k+1}T_{2k+1}(x)-2xF_{2k+2}T_{2k}(x)-(-1)^k\}$$

(8.4) $$\sum_{j=1}^{k}(-1)^{j+1}F_{2j+1}T_{2j}(x)= \frac{(-1)^k}{1+4x^2}\{F_{2k+2}T_{2k+2}(x)-2xF_{2k+3}T_{2k+1}(x)+(-1)^k 2x\}$$

By using the trigonometric form of the Chebyshev polynomial, we get the following interesting results.

THEOREM 8.1. *The following summation formulae involving the products of Fibonacci numbers and sine functions hold:*

(8.5) $$\sum_{j=1}^{k}(-1)^{j+1}F_{2j-1}\sin(2j-1)\theta= \frac{(-1)^{k+1}}{1+4\ \cos^2\theta}\{F_{2k-1}\sin(2k+1)\theta + F_{2k+1}\sin(2k-1)\theta\}$$

(8.6) $$\sum_{j=1}^{k}(-1)^{j+1}F_{2j}\sin\ 2j\theta= \frac{(-1)^{k+1}}{1+4\ \cos^2\theta}\{F_{2k}\sin(2k+2)\theta+F_{2k+2}\sin\ 2k\theta\}$$

(8.7) $$\sum_{j=1}^{k}(-1)^{j+1}F_{2j}\sin(2j-1)\theta= \frac{(-1)^{k+1}}{1+4\ \cos^2\theta}\{F_{2k}\sin(2k+1)\theta+F_{2k+2}\sin(2k-1)\theta + (-1)^k\sin\ \theta\}$$

(8.8) $$\sum_{j=1}^{k}(-1)^{j+1}F_{2j+1}\sin\ 2j\theta= \frac{(-1)^{k+1}}{1+4\ \cos^2\theta}\{F_{2k+1}\sin(2k+2)\theta+F_{2k+3}\sin 2k\theta - (-1)^k\sin\ 2\theta\}$$

Proof. We will prove (8.5). Then (8.6), (8.7) and (8.8) are proved similarly. Substituting $T_n(x) = \frac{\sin n\theta}{\sin\ \theta}$ and $x = \cos\ \theta$ in (8.1), we get

$$\sum_{j=1}^{k}(-1)^{j+1}F_{2j-1}\sin(2j-1)\theta = \frac{(-1)^k}{1+4\cos^2\theta}\{F_{2k}\sin(2k+1)\theta-2\ \cos\theta F_{2k+1}\sin 2k\theta\}$$

$$= \frac{(-1)^k}{1+4\cos^2\theta}\{F_{2k}\sin(2k+1)\theta-F_{2k+1}[\sin(2k-1)\theta+\sin(2k+1)\theta]$$

$$= \frac{(-1)^k}{1+4\cos^2\theta}\{-\sin(2k+1)\theta[F_{2k+1}-F_{2k}]-F_{2k+1}\sin(2k-1)\theta\}$$

$$= \frac{(-1)^{k+1}}{1+4\cos^2\theta}\{F_{2k-1}\sin(2k+1)\theta + F_{2k+1}\sin(2k-1)\theta\}.$$

Remark 2. By letting $p_1 = 2x$, $q_1 = -1$ and $p_1 = x$, $q_1 = -1$ we get respectively $\{U_n\} = \{F_n(x)\}$ and $\{U_n\} = \{P_n(x)\}$, the Fibonacci and and the Pell polynomials.

Remark 3. It is important to observe that by using appropriate values of p_1, q_1, p_2, q_2, m, n and s, we will be able to obtain various summation formulae involving the products of combinations of Fibonacci, Pell, and Chebyshev polynomials and sine functions.

Remark 4. It should be noted that the initial conditions in (2.1), are $G(0,0) = 0$, $G(1,0) = 1$ and those of $\{U_n\}$ and $\{V_n\}$ in (3.1) are also $U_0 = 0$, $U_1 = 1$ and $V_0 = 0$, $V_1 = 1$. If these are changed, possibly we will get new results involving some other sequences and polynomials. This is the topic of discussion of our next paper.

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