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OSCILLATION CRITERIA FOR SECOND ORDER SUPERLINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. This paper is concerned with the question of oscillation of the solutions of second order superlinear ordinary differential equations with alternating coefficients.

Consider the second order nonlinear ordinary differential equation

$$\text{(E)} \qquad x''(t) + a(t)f[x(t)] = 0,$$

where a is a continuous function on the interval $[t_0, \infty)$, $t_0 > 0$, and f is a continuous function on the real line $\mathbf{R}$, which is continuously differentiable, except possibly at 0, and satisfies

$$yf(y) > 0 \text{ and } f'(y) \geqq 0 \text{ for all } y \in \mathbf{R} - \{0\}.$$

It will be supposed that (E) is strongly superlinear in the sense that

$$\int^{\infty} \frac{dy}{f(y)} < \infty \quad \text{and} \quad \int^{-\infty} \frac{dy}{f(y)} < \infty.$$

We are concerned with the case where no restriction on the sign of the coefficient a is assumed. The special case where

$$f(y) = |y|^{\gamma} \operatorname{sgn} y, \quad y \in \mathbf{R} \ (\gamma > 1)$$

is of particular interest. The differential equation

$$(\mathrm{E}_0) \qquad x''(t) + a(t)|x(t)|^{\gamma} \operatorname{sgn} x(t) = 0 \quad (\gamma > 1)$$

is the prototype of (E).

Also, consider the differential equation with damped term

$$(\mathrm{E}') \qquad x''(t) + q(t)x'(t) + a(t)f[x(t)] = 0,$$

where q is a continuous function on the interval $[t_0, \infty)$.

Throughout the paper, we restrict our attention only to the solutions of the differential equation (E) or of the equation (E′) which exist on some ray $[T, \infty)$, where $T \geqq t_0$ may depend on the particular solution. Note that under quite general conditions there will always exist solutions of (E) which are continuable to an interval $[T, \infty)$, $T \geqq t_0$, even though there will

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also exist non-continuable solutions (cf. [**2**]). Such a solution is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise it is said to be *nonoscillatory*. Equation (E) or (E′) is called oscillatory if all its solutions are oscillatory.

It is of interest to discuss conditions on the alternating coefficient a which are sufficient for all solutions of (E) to be oscillatory. An interesting case is that of finding oscillation criteria for (E) which involve the average behavior of the integral of a. This problem has received the attention of many authors in recent years. Among numerous papers dealing with such averaging techniques in the oscillation of second order superlinear ordinary differential equations, we choose to refer to the papers by Butler [**1**, **3**], Butler and Erbe [**4**], Kamenev [**5**, **6**], Kwong and Wong [**8**], Onose [**9**], the present author [**10**, **11**, **12**] and Wong [**14**, **15**, **16**, **17**, **18**]. A fairly extensive list of the earlier works can be found, for instance, in the survey article by Wong [**16**] (cf., also, the papers by Wong [**14**, **17**]). In this paper, we deal with the possibility of averaging techniques for studying the oscillatory behavior of the superlinear differential equation (E) or, more generally, of the damped equation (E′).

Wong [**15**] established that the conditions

$$(\mathrm{A}_1) \qquad \liminf_{t\to\infty} \int_{t_0}^{t} a(s)ds > -\infty$$

and

$$(\mathrm{A}_2) \qquad \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} a(\tau)d\tau ds = \infty$$

are sufficient for the oscillation of (E_0). In [**10**] the author obtained an extension of this result to the general case of the differential equation (E). More precisely, the following oscillation result is proved in [**10**].

THEOREM A. *Equation* (E) *is oscillatory if* (A_1) *and* (A_2) *hold and*:
(F_1) *f is such that*

$$(*) \qquad \int^{\infty} \frac{\sqrt{f'(y)}}{f(y)}dy < \infty \quad \text{and} \quad \int^{-\infty} \frac{\sqrt{f'(y)}}{f(y)}dy < \infty$$

and

$$\min\left\{\inf_{y>0}\left(\left[\int_y^{\infty} \frac{\sqrt{f'(z)}}{f(z)}dz\right]^2 \Big/ \int_y^{\infty}\frac{dz}{f(z)}\right),\right.$$
$$\left.\inf_{y<0}\left(\left[\int_y^{-\infty} \frac{\sqrt{f'(z)}}{f(z)}dz\right]^2 \Big/ \int_y^{-\infty}\frac{dz}{f(z)}\right)\right\} > 0.$$

Recently, Wong [**18**] proved that, under the conditions (A_1) and

$$(A_3) \qquad \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} a(s)ds = \infty$$

for some integer $n > 2$,

the differential equation (E_0) is oscillatory. Extending this criterion of Wong, the author [**12**] presented, very recently, the following oscillation theorem for the general differential equation (E).

THEOREM B. *Equation* (E) *is oscillatory if* (A_1) *and* (A_3) *hold and*:

(F_2) f *satisfies* (*) *and*

$$\min\left\{\inf_{y>0} \sqrt{f'(y)} \int_y^{\infty} \frac{\sqrt{f'(z)}}{f(z)}dz, \right.$$

$$\left. \inf_{y<0} \sqrt{f'(y)} \int_y^{-\infty} \frac{\sqrt{f'(z)}}{f(z)}dz \right\} > 0.$$

It is remarkable that condition (A_3) has been introduced by Kamenev [**7**], who proved that this condition suffices for the oscillation in the linear case, i.e., in the case of the differential equation

$$x''(t) + a(t)x(t) = 0.$$

This criterion of Kamenev includes the classical oscillation result of Wintner (see [**13**]). Also, it must be noted that the conditions (F_1) and (F_2) are satisfied (cf. [**10**], [**12**]) in the special case where

$$f(y) = |y|^{\gamma} \operatorname{sgn} y, \quad y \in \mathbf{R}\ (\gamma > 1),$$

i.e., in the case of the differential equation (E_0). Moreover, it is noteworthy that condition (F_2) implies (F_1), as it is verified in [**10**], and hence from Theorems A and B it follows that: Equation (E) is oscillatory if (A_1) and

$$(A_4) \qquad \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} a(s)ds = \infty$$

for some integer $n \geqq 2$

hold, and (F_2) is satisfied. Condition (A_4) unifies (A_2) and (A_3). Finally, we note that the author [**12**] has presented an extension of Theorem B for the case of the damped differential equation (E′).

The main purpose of this paper is to extend and improve Theorems A and B. The results obtained will be extended to the more general case of the damped differential equation (E′). It must be noted that, by simple transformations, our results can easily be carried over to more general differential equations involving the term $(rx')'$ in place of the second derivative x'' of the unknown function x, where r is a positive continuous function on the interval $[t_0, \infty)$.

2. Oscillation criteria for (E). In this section, we shall give our main results. More precisely, we shall state and prove two oscillation criteria for the differential equation (E).

THEOREM 1. *Suppose that* (F_1) *holds and let* ρ *be a positive continuously differentiable function on the interval* $[t_0, \infty)$ *such that* ρ' *is nonnegative and decreasing on* $[t_0, \infty)$. *Equation* (E) *is oscillatory if*

$$(A_1[\rho]) \quad \liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)a(s)ds > -\infty$$

and

$$(A_2[\rho]) \quad \limsup_{t\to\infty} \left[\int_{t_0}^{t} \frac{ds}{\rho(s)}\right]^{-1} \int_{t_0}^{t} \frac{1}{\rho(s)} \int_{t_0}^{s} \rho(\tau)a(\tau)d\tau ds = \infty.$$

Proof. Suppose that the differential equation (E) possesses a nonoscillatory solution x on an interval $[T, \infty)$, $T \geqq t_0$. Without loss of generality, we shall assume that $x(t) \neq 0$ for all $t \geqq T$. Furthermore, we observe that the substitution $z = -x$ transforms (E) into the equation

$$z''(t) + a(t)\hat{f}[z(t)] = 0,$$

where $\hat{f}(y) = -f(-y)$, $y \in \mathbf{R}$. Since the function $\hat{f}$ is subject to the conditions posed on f, we can restrict our discussion to the case where the solution x is positive on $[T, \infty)$.

Let w be defined by

$$w(t) = \rho(t)\frac{x'(t)}{f[x(t)]}, \quad t \geqq T.$$

Then, for $t \geqq T$, we get

$$w'(t) = \rho'(t)\frac{x'(t)}{f[x(t)]} + \rho(t)\left[\frac{x''(t)}{f[x(t)]} - \left\{\frac{x'(t)}{f[x(t)]}\right\}^2 f'[x(t)]\right]$$

and consequently

$$\rho(t)a(t) = -w'(t) + \rho'(t)\frac{x'(t)}{f[x(t)]} - \frac{1}{\rho(t)}w^2(t)f'[x(t)] \tag{1}$$

for every $t \geqq T$.

So, for any $t \geqq T$

$$\int_{T}^{t} \rho(s)a(s)ds = -w(t) + w(T)$$

$$+ \int_T^t \rho'(s)\frac{x'(s)}{f[x(s)]}ds - \int_T^t \frac{1}{\rho(s)}w^2(s)f'[x(s)]ds.$$

But, by the Bonnet theorem, for a fixed $t \geqq T$ and for some $\xi \in [T, t]$

$$\int_T^t \rho'(s)\frac{x'(s)}{f[x(s)]}ds = \rho'(T)\int_T^{\xi}\frac{x'(s)}{f[x(s)]}ds$$
$$= \rho'(T)\int_{x(T)}^{x(\xi)}\frac{dy}{f(y)}$$

and hence, since $\rho'(T) \geqq 0$ and

$$\int_{x(T)}^{x(\xi)}\frac{dy}{f(y)} < \begin{cases} 0, & \text{if } x(\xi) < x(T) \\ \displaystyle\int_{x(T)}^{\infty}\frac{dy}{f(y)}, & \text{if } x(\xi) \geqq x(T), \end{cases}$$

we have

$$(2) \qquad \int_T^t \rho(s)\frac{x'(s)}{f[x(s)]}ds \leqq K \quad \text{for all } t \geqq T$$

with

$$K = \rho'(T)\int_{x(T)}^{\infty}\frac{dy}{f(y)}.$$

Therefore, by taking into account (2), we conclude that

$$(3) \qquad \int_T^t \rho(s)a(s)ds \leqq -w(t) + w(T) + K$$
$$- \int_T^t \frac{1}{\rho(s)}w^2(s)f'[x(s)]ds \quad \text{for } t \geqq T.$$

There are two cases to consider:

Case 1. The integral

$$\int_T^{\infty}\frac{1}{\rho(s)}w^2(s)f'[x(s)]ds$$

is finite. In this case, there exists a positive constant N so that

$$(4) \qquad \int_T^t \frac{1}{\rho(s)}w^2(s)f'[x(s)]ds \leqq N \quad \text{for every } t \geqq T.$$

For $t \geqq T$, we may use the Schwarz inequality to obtain

$$\left|\int_T^t \frac{x'(s)}{f[x(s)]}\sqrt{f'[x(s)]}ds\right|^2$$

$$= \left| \int_T^t \frac{1}{\sqrt{\rho(s)}} \left[\frac{1}{\sqrt{\rho(s)}} w(s) \sqrt{f'[x(s)]} \right] ds \right|^2$$

$$\leqq \left[\int_T^t \frac{ds}{\rho(s)} \right] \int_T^t \frac{1}{\rho(s)} w^2(s) f'[x(s)] ds.$$

So, in view of (4), we have

$$\left| \int_T^t \frac{x'(s)}{f[x(s)]} \sqrt{f'[x(s)]} ds \right|^2 \leqq N \int_{t_0}^t \frac{ds}{\rho(s)} \quad \text{for all } t \geqq T. \tag{5}$$

Next, we observe that, because of condition (F_1), one has

$$\int_{x(t)}^\infty \frac{dy}{f(y)} \leqq M \left[\int_{x(t)}^\infty \frac{\sqrt{f'(y)}}{f(y)} dy \right]^2, \quad t \geqq T, \tag{6}$$

where M is a positive constant. As in [**10**], we put

$$K_1 = \int_{x(T)}^\infty \frac{dy}{f(y)} > 0 \quad \text{and} \quad K_2 = \int_{x(T)}^\infty \frac{\sqrt{f'(y)}}{f(y)} dy > 0$$

and, by using (6), for every $t \geqq T$ we get

$$\begin{aligned}
\left| \int_T^t \frac{1}{\rho(s)} w(s) ds \right| &= \left| \int_T^t \frac{x'(s)}{f[x(s)]} ds \right| = \left| \int_{x(T)}^{x(t)} \frac{dy}{f(y)} \right| \\
&= \left| K_1 - \int_{x(t)}^\infty \frac{dy}{f(y)} \right| \leqq K_1 + \int_{x(t)}^\infty \frac{dy}{f(y)} \\
&\leqq K_1 + M \left[\int_{x(t)}^\infty \frac{\sqrt{f'(y)}}{f(y)} dy \right]^2 \\
&= K_1 + M \left[K_2 - \int_{x(T)}^{x(t)} \frac{\sqrt{f'(y)}}{f(y)} dy \right]^2 \\
&\leqq K_1 + M \left[K_2 + \left| \int_{x(T)}^{x(t)} \frac{\sqrt{f'(y)}}{f(y)} dy \right| \right]^2 \\
&= K_1 + M \left[K_2 + \left| \int_T^t \frac{x'(s)}{f[x(s)]} f'[x(s)] ds \right| \right]^2.
\end{aligned}$$

Thus, by (5), for any $t \geqq T$ we derive

$$\begin{aligned}
\left| \int_T^t \frac{1}{\rho(s)} w(s) ds \right| &\leqq K_1 + M \left[K_2 + \left\{ N \int_{t_0}^t \frac{ds}{\rho(s)} \right\}^{1/2} \right]^2 \\
&= (K_1 + MK_2^2) + 2MK_2\sqrt{N} \left[\int_{t_0}^t \frac{ds}{\rho(s)} \right]^{1/2}
\end{aligned}$$

$$+ MN \int_{t_0}^{t} \frac{ds}{\rho(s)}.$$

Since ρ is positive on $[t_0, \infty)$ and ρ' is nonnegative and bounded above on $[t_0, \infty)$, it follows that

$$\rho(t) \leqq \mu t \quad \text{for all large } t,$$

where $\mu > 0$ is a constant. This ensures that

$$(7) \qquad \int_{t_0}^{\infty} \frac{dt}{\rho(t)} = \infty.$$

So, we can choose a $T_0 > t_0$ so that

$$(8) \qquad \int_{t_0}^{T_0} \frac{ds}{\rho(s)} \geqq 1.$$

Hence, in view of (8), we get

$$(9) \qquad \left| \int_{T}^{t} \frac{1}{\rho(s)} w(s) ds \right| \leqq C \int_{t_0}^{t} \frac{ds}{\rho(s)} \quad \text{for every } t \geqq T^*,$$

where

$$T^* = \max\{T_0, T\} \quad \text{and} \quad C = K_1 + MK_2^2 + 2MK_2\sqrt{N} + MN.$$

Now, from (3) it follows that

$$\int_{T}^{t} \rho(s)a(s)ds \leqq -w(t) + w(T) + K, \quad t \geqq T$$

and thus, by taking into account (9), we obtain for $t \geqq T^*$

$$\int_{T}^{t} \frac{1}{\rho(s)} \int_{T}^{s} \rho(\tau)a(\tau)d\tau ds$$

$$\leqq -\int_{T}^{t} \frac{1}{\rho(s)} w(s)ds + [w(T) + K] \int_{T}^{t} \frac{ds}{\rho(s)}$$

$$\leqq \left| \int_{T}^{t} \frac{1}{\rho(s)} w(s)ds \right| + |w(T) + K| \int_{T}^{t} \frac{ds}{\rho(s)}$$

$$\leqq C^* \int_{t_0}^{t} \frac{ds}{\rho(s)},$$

where $C^* = C + |w(T) + K|$. Therefore, for every $t \geqq T^*$ we have

$$\int_{t_0}^{t} \frac{1}{\rho(s)} \int_{t_0}^{s} \rho(\tau)a(\tau)d\tau ds$$

$$= \int_{t_0}^{T} \frac{1}{\rho(s)} \int_{t_0}^{s} \rho(\tau)a(\tau)d\tau ds + \left[\int_{t_0}^{T} \rho(\tau)a(\tau)d\tau\right] \int_{T}^{t} \frac{ds}{\rho(s)}$$

$$+ \int_{T}^{t} \frac{1}{\rho(s)} \int_{T}^{s} \rho(\tau)a(\tau)d\tau ds$$

$$\leqq \left|\int_{t_0}^{T} \frac{1}{\rho(s)} \int_{t_0}^{s} \rho(\tau)a(\tau)d\tau ds\right| + \left|\int_{t_0}^{T} \rho(\tau)a(\tau)d\tau\right| \int_{T}^{t} \frac{ds}{\rho(s)}$$

$$+ C^* \int_{t_0}^{t} \frac{ds}{\rho(s)}$$

and consequently, in view of (8), one has

$$\int_{t_0}^{t} \frac{1}{\rho(s)} \int_{t_0}^{s} \rho(\tau)a(\tau)d\tau ds \leqq \hat{C} \int_{t_0}^{t} \frac{ds}{\rho(s)} \quad \text{for all } t \geqq T^*,$$

where

$$\hat{C} = \left|\int_{t_0}^{T} \frac{1}{\rho(s)} \int_{t_0}^{s} \rho(\tau)a(\tau)d\tau ds\right| + \left|\int_{t_0}^{t} \rho(\tau)a(\tau)d\tau\right| + C^*.$$

This contradicts condition $(\mathrm{A}_2[\rho])$.

Case 2. The integral

$$\int_{T}^{\infty} \frac{1}{\rho(s)} w^2(s) f'[x(s)]ds$$

is infinite. By condition $(\mathrm{A}_1[\rho])$, from (3) it follows that for some constant λ

$$-w(t) \geqq \lambda + \int_{T}^{t} \frac{1}{\rho(s)} w^2(s) f'[x(s)]ds \quad \text{for every } t \geqq T. \tag{10}$$

We can consider a $\hat{T} \geqq T$ such that

$$\Lambda \equiv \lambda + \int_{T}^{\hat{T}} \frac{1}{\rho(s)} w^2(s) f'[x(s)]ds > 0.$$

Then (10) ensures that w is negative on $[\hat{T}, \infty)$. Now, (10) gives

$$\frac{1}{\rho(t)} w^2(t) f'[x(t)] \Big/ \left\{\lambda + \int_{T}^{t} \frac{1}{\rho(s)} w^2(s) f'[x(s)]ds\right\}$$

$$\geqq -\frac{x'(t) f'[x(t)]}{f[x(t)]}, \quad t \geqq \hat{T}$$

and consequently for all $t \geqq \hat{T}$

$$\log \frac{\lambda + \int_T^t \frac{1}{\rho(s)} w^2(s) f'[x(s)]ds}{\Lambda} \geqq \log \frac{f[x(\hat{T})]}{f[x(t)]}.$$

Hence,

$$\lambda + \int_T^t \frac{1}{\rho(s)} w^2(s) f'[x(s)]ds \geqq \Lambda \frac{f[x(\hat{T})]}{f[x(t)]} \quad \text{for } t \geqq \hat{T}.$$

So, (10) yields

$$x'(t) \leqq -\Lambda^* \frac{1}{\rho(t)} \quad \text{for every } t \geqq \hat{T},$$

where

$$\Lambda^* = \Lambda f[x(\hat{T})] > 0.$$

Thus, we have

$$x(t) \leqq x(\hat{T}) - \Lambda^* \int_{\hat{T}}^t \frac{ds}{\rho(s)} \quad \text{for all } t \geqq \hat{T},$$

which, because of (7), leads to the contradiction

$$\lim_{t \to \infty} x(t) = -\infty.$$

THEOREM 2. *Suppose that* (F_2) *holds and let* ρ *be as in Theorem* 1. *In addition, assume that*

$$(R_1) \qquad \limsup_{t \to \infty} \frac{1}{t^2} \int_{t_0}^t \rho(s) \left[\int_{t_0}^s \frac{du}{\rho(u)} \right] ds < \infty.$$

Equation (E) *is oscillatory if* ($A_1[\rho]$) *holds and*

$$(A_3[\rho]) \quad \limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^t (t - s)^{n-1} \rho(s) a(s) ds = \infty$$

for some integer $n > 2$.

Proof. First of all, as in the proof of Theorem 1, we observe that (7) holds and consequently there exists a $T_0 > t_0$ such that (8) is fulfilled. Next, we consider a nonoscillatory solution x on an interval $[T, \infty)$, $T \geqq T_0$, of the differential equation (E). Without loss of generality, it can be assumed that $x(t) \neq 0$ for all $t \geqq T$. Furthermore, it is enough to restrict our attention to the case where x is positive on $[T, \infty)$. Now, we define

$$w(t) = \rho(t) x'(t) / f[x(t)], \quad t \geqq T.$$

As in the proof of Theorem 1, we obtain (1) and next we conclude that (2) and (3) are satisfied with

$$K = \rho'(T) \int_{x(T)}^{\infty} \frac{dy}{f(y)}.$$

We consider the following two cases:

Case 1.

$$\int_T^{\infty} \frac{1}{\rho(s)} w^2(s) f'[x(s)]ds < \infty.$$

Then (4) holds, where N is a positive constant. Furthermore, by the procedure of the proof of Theorem 1, inequality (5) can be obtained. Condition (F_2) ensures that

$$(11) \qquad \sqrt{f'[x(t)]} \int_{x(t)}^{\infty} \frac{\sqrt{f'(y)}}{f(y)} dy \geqq S \quad \text{for } t \geqq T,$$

where S is a positive constant. As in [**12**], we set

$$K^* = \int_{x(T)}^{\infty} \frac{\sqrt{f'(y)}}{f(y)} dy > 0$$

and, by (11), we obtain for $t \geqq T$

$$\begin{aligned} f'[x(t)] &\geqq S^2 \left[\int_{x(t)}^{\infty} \frac{\sqrt{f'(y)}}{f(y)} dy \right]^{-2} \\ &= S^2 \left[K^* - \int_{x(T)}^{x(t)} \frac{\sqrt{f'(y)}}{f(y)} dy \right]^{-2} \\ &= S^2 \left[K^* - \int_T^t \frac{x'(s)}{f[x(s)]} \sqrt{f'[x(s)]} ds \right]^{-2} \\ &\geqq S^2 \left[K^* + \left| \int_T^t \frac{x'(s)}{f[x(s)]} \sqrt{f'[x(s)]} ds \right| \right]^{-2}. \end{aligned}$$

So, by using (5), for every $t \geqq T$ we get

$$f'[x(t)] \geqq S^2 \left[K^* + \left\{ N \int_{t_0}^t \frac{ds}{\rho(s)} \right\}^{1/2} \right]^{-2}$$

and consequently, in view of (8), we have

$$(12) \qquad f'[x(t)] \geqq c \Big/ \int_{t_0}^t \frac{ds}{\rho(s)} \quad \text{for all } t \geqq T,$$

where

$$c = S^2(K^* + \sqrt{N})^{-2} > 0.$$

By (12), equation (1) gives

$$\rho(t)a(t) \leqq -w'(t) + \rho'(t)\frac{x'(t)}{f[x(t)]} - \frac{1}{\rho(t)}w^2(t)\frac{c}{R(t)}$$

for all $t \geqq T$,

where

$$R(t) = \int_{t_0}^{t} \frac{ds}{\rho(s)}, \quad t \geqq t_0.$$

Thus, for $t \geqq T$

$$\int_T^t (t - s)^{n-1}\rho(s)a(s)ds$$

$$\leqq -\int_T^t (t - s)^{n-1}w'(s)ds + \int_T^t (t - s)^{n-1}\rho'(s)\frac{x'(s)}{f[x(s)]}ds$$

$$- \int_T^t (t - s)^{n-1}\frac{c}{\rho(s)R(s)}w^2(s)ds.$$

But, because of (2), we have

$$\int_T^t (t - s)^{n-1}\rho'(s)\frac{x'(s)}{f[x(s)]}ds$$

$$= (n - 1)\int_T^t (t - s)^{n-2}\left[\int_T^s \rho'(u)\frac{x'(u)}{f[x(u)]}du\right]ds$$

$$\leqq K(n - 1)\int_T^t (t - s)^{n-2}ds = K(t - T)^{n-1}.$$

Hence, for every $t \geqq T$ we obtain

$$\int_T^t (t - s)^{n-1}\rho(s)a(s)ds$$

$$\leqq -\int_T^t (t - s)^{n-1}w'(s)ds + K(t - T)^{n-1}$$

$$- \int_T^t (t - s)^{n-1}\frac{c}{\rho(s)R(s)}w^2(s)ds$$

$$= (t - T)^{n-1}[w(T) + K] - (n - 1)\int_T^t (t - s)^{n-2}w(s)ds$$

$$- \int_T^t (t - s)^{n-1} \frac{c}{\rho(s)R(s)} w^2(s)ds$$

$$= (t - T)^{n-1}[w(T) + K] + \frac{(n - 1)^2}{4c}$$

$$\times \int_T^t (t - s)^{n-3}\rho(s)R(s)ds$$

$$- \int_T^t \left\{ (t - s)^{(n-1)/2}\left[\frac{c}{\rho(s)R(s)}\right]^{1/2} w(s) \right.$$

$$\left. + \frac{n - 1}{2}\left[\frac{\rho(s)R(s)}{c}\right]^{1/2}(t - s)^{(n-3)/2}\right\}^2 ds$$

$$\leqq (t - T)^{n-1}[w(T) + K] + \frac{(n - 1)^2}{4c}(t - T)^{n-3}$$

$$\times \int_T^t \rho(s)R(s)ds.$$

So, it holds

$$\frac{1}{t^{n-1}} \int_{t_0}^t (t - s)^{n-1}\rho(s)a(s)ds$$

$$\leqq \frac{1}{t^{n-1}} \int_{t_0}^T (t - s)^{n-1}\rho(s)|a(s)|ds$$

$$+ \frac{1}{t^{n-1}} \int_T^t (t - s)^{n-1}\rho(s)a(s)ds$$

$$\leqq \left(1 - \frac{t_0}{t}\right)^{n-1} \int_{t_0}^T \rho(s)|a(s)|ds + \left(1 - \frac{T}{t}\right)^{n-1}[w(T) + K]$$

$$+ \frac{(n - 1)^2}{4c}\left(1 - \frac{T}{t}\right)^{n-3}\left[\frac{1}{t^2} \int_{t_0}^t \rho(s)R(s)ds\right]$$

for all $t \geqq T$. Thus, by assumption (R_1), we have

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^t (t - s)^{n-1}\rho(s)a(s)ds$$

$$\leqq \int_{t_0}^T \rho(s)|a(s)|ds + w(T) + K$$

$$+ \frac{(n - 1)^2}{4c} \limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^t \rho(s)R(s)ds < \infty,$$

which contradicts condition $(A_3[\rho])$.

Case 2.

$$\int_T^{\infty} \frac{1}{\rho(s)} w^2(s) f'[x(s)]ds = \infty.$$

As exactly in the proof of Theorem 1, we can arrive at the contradiction

$$\lim_{t\to\infty} x(t) = -\infty.$$

Let us denote by $\mathscr{R}$ the function class defined as follows: ρ belongs to $\mathscr{R}$ if and only if ρ is a positive continuously differentiable function on the interval $[t_0, \infty)$ such that ρ' is nonnegative and decreasing on $[t_0, \infty)$. This class is sufficiently wide. For example, ρ belongs to $\mathscr{R}$ in each of the following cases (cf. **[11]**):

(i) $\rho(t) = t^{\beta}$, $t \geqq t_0$ for $\beta \in [0, 1]$.

(ii) $\rho(t) = \log^{\beta} t$, $t \geqq t_0$ for $\beta > 0$, where $t_0 > \max\{1, e^{\beta-1}\}$.

(iii) $\rho(t) = t^{\beta} \log t$, $t \geqq t_0$ for $\beta \in (0, 1)$, where

$$t_0 > \max\left\{1, \exp\left(\frac{1}{1-\beta} - \frac{1}{\beta}\right)\right\}.$$

(iv) $\rho(t) = t/\log t$, $t \geqq t_0$, where $t_0 \geqq e^2$.

(v) $\rho(t) = t^{1/2}[5 + \sin(\log t)]$, $t \geqq t_0$.

The more simple case of a function $\rho \in \mathscr{R}$ is that $\rho(t) = 1$, $t \geqq t_0$. In this case, condition (R_1) is satisfied. Hence, Theorems A and B can be obtained from Theorems 1 and 2 respectively for $\rho(t) = 1$, $t \geqq t_0$. Now, we shall concentrate our interest to the case (i), i.e., to the case where $e(t) = t^{\beta}$, $t \geqq e_0$ for $\beta \in [0, 1]$. It is easy to see that, when $0 \leqq \beta < 1$, condition (R_1) holds by itself. Hence, Corollaries 1 and 2 below are obtained from Theorems 1 and 2 respectively.

COROLLARY 1. *Suppose that* (F_1) *holds. Equation* (E) *is oscillatory if, for some* $\beta \in [0, 1]$, *we have*

$$(A_1(\beta)) \quad \liminf_{t\to\infty} \int_{t_0}^{t} s^{\beta} a(s)ds > -\infty$$

and

$$(A_2(\beta)) \quad \begin{cases} \limsup_{t\to\infty} \dfrac{1}{t^{1-\beta}} \displaystyle\int_{t_0}^{t} \frac{1}{s^{\beta}} \int_{t_0}^{s} \tau^{\beta} a(\tau)d\tau ds = \infty, & \textit{for } 0 \leqq \beta < 1 \\ \limsup_{t\to\infty} \dfrac{1}{\log t} \displaystyle\int_{t_0}^{t} \frac{1}{s} \int_{t_0}^{s} \tau a(\tau)d\tau ds = \infty, & \textit{for } \beta = 1. \end{cases}$$

COROLLARY 2. *Suppose that* (F_2) *holds. Equation* (E) *is oscillatory if, for some* $\beta \in [0, 1)$, $(A_1(\beta))$ *is satisfied and*

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$$(A_3(\beta)) \quad \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} s^{\beta} a(s)ds = \infty$$

for some integer $n > 2$.

Theorem A is a consequence of Theorem 1 and Theorem B follows from Theorem 2. On the other hand, Theorem 1 can be applied in some cases in which Theorem A is not applicable. Such a case is described in Example 1 below. Also, as it is demonstrated by Example 2 below, it is possible to have cases where Theorem 2 is applicable while Theorem B cannot be applied.

Example 1. Consider the differential equation (E) with (cf. [**11**])

$$a(t) = -t^{-1/2} \sin t + \frac{1}{2} t^{-3/2}(2 + \cos t), \quad t \geqq t_0 \equiv \pi/2.$$

For every $t \geqq t_0$, we get

$$\begin{aligned}\int_{t_0}^{t} sa(s)ds &= \int_{\pi/2}^{t} \left[-s^{1/2} \sin s + \frac{1}{2} s^{-1/2}(2 + \cos s)\right]ds \\ &= \int_{\pi/2}^{t} d[s^{1/2}(2 + \cos s)] \\ &= t^{1/2}(2 + \cos t) - 2(\pi/2)^{1/2} \geqq t^{1/2} - 2(\pi/2)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\log t} \int_{t_0}^{t} \frac{1}{s} \int_{t_0}^{s} \tau a(\tau)d\tau ds &\geqq \frac{1}{\log t} \int_{\pi/2}^{t} \left[s^{-1/2} - 2(\pi/2)^{1/2} \frac{1}{s}\right]ds \\ &= 2\frac{t^{1/2}}{\log t} - 2(\pi/2)^{1/2} \\ &\quad + 2(\pi/2)^{1/2}[\log(\pi/2) - 1]\frac{1}{\log t}\end{aligned}$$

and consequently

$$\liminf_{t\to\infty} \int_{t_0}^{t} sa(s)ds > -\infty \quad \text{and}$$

$$\limsup_{t\to\infty} \frac{1}{\log t} \int_{t_0}^{t} \frac{1}{s} \int_{t_0}^{s} \tau a(\tau)d\tau ds = \infty.$$

This means that $(A_1(1))$ and $(A_2(1))$ hold. Thus, from Corollary 1 (and so from Theorem 1) it follows that, when (F_1) is satisfied, our differential equation is oscillatory. On the other hand, Theorem A is not applicable for the equation under consideration. Indeed, for $t \geqq t_0$, we obtain

$$\begin{aligned}\int_{t_0}^{t} a(s)ds &= \int_{\pi/2}^{t}\left[-s^{-1/2}\sin s + \frac{1}{2}s^{-3/2}(2+\cos s)\right]ds \\ &\leqq \int_{\pi/2}^{t}\left(-s^{-1/2}\sin s + \frac{3}{2}s^{-3/2}\right)ds \\ &= t^{-1/2}\cos t + \int_{\pi/2}^{t}\left(\frac{1}{2}s^{-3/2}\cos s + \frac{3}{2}s^{-3/2}\right)ds \\ &\leqq t^{-1/2}\cos t + 2\int_{\pi/2}^{t} s^{-3/2}ds \\ &= t^{-1/2}\cos t - 4[t^{-1/2} - (\pi/2)^{-1/2}] \\ &\leqq -3t^{-1/2} + 4(\pi/2)^{-1/2}.\end{aligned}$$

Thus, for every $t \geqq t_0$

$$\begin{aligned}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} a(\tau)d\tau ds &\leqq \frac{1}{t}\int_{\pi/2}^{t}[-3s^{-1/2} + 4(\pi/2)^{-1/2}]ds \\ &= -\frac{6}{t^{1/2}} + 2(\pi/2)^{1/2}\frac{1}{t} + 4(\pi/2)^{-1/2}\end{aligned}$$

and hence

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} a(\tau)d\tau ds < \infty,$$

i.e., (A_2) fails.

Example 2. Consider the case of the differential equation (E), where

$$a(t) = -t^{-1/3}\sin t + \frac{1}{2}t^{-4/3}(2+\cos t), \quad t \geqq t_0 \equiv \pi/2.$$

For any $t \geqq t_0$, we obtain

$$\begin{aligned}\int_{t_0}^{t} s^{5/6}a(s)ds &= \int_{\pi/2}^{t}\left[-s^{1/2}\sin s + \frac{1}{2}s^{-1/2}(2+\cos s)\right]ds \\ &= \int_{\pi/2}^{t} d[s^{1/2}(2+\cos s)] \\ &= t^{1/2}(2+\cos t) - 2(\pi/2)^{1/2} \\ &\geqq t^{1/2} - 2(\pi/2)^{1/2}\end{aligned}$$

and hence

$$\liminf_{t\to\infty} \int_{t_0}^{t} s^{5/6}a(s)ds > -\infty,$$

i.e., $(A_1(5/6))$ is satisfied. Furthermore, we have for every $t \geqq t_0$

$$\begin{aligned}
&\frac{1}{t^2}\int_{t_0}^{t}(t-s)^2 s^{5/6}a(s)ds \\
&= \frac{2}{t^2}\int_{t_0}^{t}(t-s)\left[\int_{t_0}^{s}\tau^{5/6}a(\tau)d\tau\right]ds \\
&\geqq \frac{2}{t^2}\int_{\pi/2}^{t}(t-s)[s^{1/2}-2(\pi/2)^{1/2}]ds \\
&= \frac{8}{15}t^{1/2} - 2(\pi/2)^{1/2} + \frac{8}{3}(\pi/2)^{3/2}\frac{1}{t} - \frac{6}{5}(\pi/2)^{5/2}\frac{1}{t^2}
\end{aligned}$$

and consequently

$$\limsup_{t\to\infty}\frac{1}{t^2}\int_{t_0}^{t}(t-s)^2 s^{5/6}a(s)ds = \infty.$$

Thus, condition $(A_3(5/6))$ is also fulfilled (with $n = 3$). So, Corollary 2 (and hence Theorem 2) guarantees that the differential equation under consideration is oscillatory, provided that (F_2) holds. On the other hand, condition (A_3) fails and consequently Theorem B cannot be applied for the equation considered. In fact, for every $t \geqq t_0$ we derive

$$\begin{aligned}
\int_{t_0}^{t}a(s)ds &= \int_{\pi/2}^{t}\left[-s^{-1/3}\sin s + \frac{1}{2}s^{-4/3}(2+\cos s)\right]ds \\
&\leqq \int_{\pi/2}^{t}\left(-s^{-1/3}\sin s + \frac{3}{2}s^{-4/3}\right)ds \\
&= t^{-1/3}\cos t + \int_{\pi/2}^{t}\left(\frac{1}{3}s^{-4/3}\cos s + \frac{3}{2}s^{-4/3}\right)ds \\
&\leqq t^{-1/3}\cos t + \frac{11}{6}\int_{\pi/2}^{t}s^{-4/3}ds \\
&= t^{-1/3}\cos t - \frac{11}{2}[t^{-1/3} - (\pi/2)^{-1/3}] \\
&\leqq \frac{11}{2}(\pi/2)^{-1/3}.
\end{aligned}$$

Now, if n is an integer with $n > 2$, then we have for all $t \geqq t_0$

$$\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}a(s)ds$$

$$= \frac{n-1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-2} \left[\int_{t_0}^{s} a(\tau) d\tau \right] ds$$

$$\leqq \frac{11}{2} (\pi/2)^{-1/3} \frac{n-1}{t^{n-1}} \int_{\pi/2}^{t} (t-s)^{n-2} ds$$

$$= \frac{11}{2} (\pi/2)^{-1/3} \left(1 - \frac{\pi/2}{t} \right).$$

This gives

$$\limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} a(s) ds < \infty,$$

i.e., condition (A_3) is not satisfied.

3. An extension: Oscillation in the damped case. Theorems 1 and 2 can be extended to differential equations with damped term of the form (E′). More precisely, we have the following more general theorems.

THEOREM 1′. *Suppose that* (F_1) *holds and let* ρ *be a positive continuously differentiable function on the interval* $[t_0, \infty)$ *such that*

$$(R_2) \qquad \int^{\infty} \frac{dt}{\rho(t)} = \infty.$$

Moreover, let $\rho' - \rho q$ *be nonnegative and decreasing on* $[t_0, \infty)$. *Equation* (E′) *is oscillatory if* ($A_1[\rho]$) *and* ($A_2[\rho]$) *hold.*

Proof. Let x be a nonoscillatory solution on an interval $[T, \infty)$, $T \geqq t_0$, of the differential equation (E′). We suppose, without loss of generality, that $x(t) \neq 0$ for all $t \geqq T$. Furthermore, we observe that the substitution $z = -x$ transforms (E′) into the equation

$$z''(t) + q(t)z(t) + a(t)\hat{f}[z(t)] = 0,$$

where $\hat{f}(y) = -f(-y)$, $y \in \mathbf{R}$. The function $\hat{f}$ is subject to the assumptions posed on f. Thus, we can restrict ourselves only to the case where x is positive on $[T, \infty)$. If w is defined as in the proof of Theorem 1, then for every $t \geqq T$ we have

$$w'(t) = \rho'(t)\frac{x'(t)}{f[x(t)]} + \rho(t)\left[\frac{x''(t)}{f[x(t)]} - \left\{\frac{x'(t)}{f[x(t)]}\right\}^2 f'[x(t)]\right]$$

$$= \rho'(t)\frac{x'(t)}{f[x(t)]}$$

$$+ \rho(t)\left[-a(t) - q(t)\frac{x'(t)}{f[x(t)]} - \left\{\frac{x'(t)}{f[x(t)]}\right\}^2 f'[x(t)]\right]$$

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and hence

$$(1)' \qquad \rho(t)a(t) = -w'(t) + [\rho'(t) - \rho(t)q(t)]\frac{x'(t)}{f[x(t)]} - \frac{1}{\rho(t)}w^2(t)f'[x(t)] \quad \text{for every } t \geqq T.$$

Thus, for $t \geqq T$

$$\int_T^t \rho(s)a(s)ds = -w(t) + w(T) + \int_T^t [\rho'(s) - \rho(s)q(s)]\frac{x'(s)}{f[x(s)]}ds - \int_T^t \frac{1}{\rho(s)}w^2(s)f'[x(s)]ds.$$

But, by using the Bonnet theorem, for a fixed $t \geqq T$ there exists a $\xi \in [T, t]$ so that

$$\int_T^t [\rho'(s) - \rho(s)q(s)]\frac{x'(s)}{f[x(s)]}ds = [\rho'(T) - \rho(T)q(T)]\int_T^\xi \frac{x'(s)}{f[x(s)]}ds = [\rho'(T) - \rho(T)q(T)]\int_{x(T)}^{x(t)} \frac{dy}{f(y)}$$

and consequently

$$(2)' \qquad \int_T^t [\rho'(s) - \rho(s)q(s)]\frac{x'(s)}{f[x(s)]}ds \leqq K \quad \text{for all } t \geqq T,$$

where

$$K = [\rho'(T) - \rho(T)q(T)]\int_{x(T)}^{\infty} \frac{dy}{f(y)}.$$

Therefore, (3) can be obtained and the remainder of the proof proceeds as in the proof of Theorem 1.

We observe that, if ρ is as in Theorem 1, then (R_2) holds by itself and hence Theorem 1 can be obtained from Theorem 1′ for $q = 0$. The following result follows also from Theorem 1: Suppose that (F_1) holds and let ρ be as in Theorem 1. Moreover, let q be nonpositive and ρq be increasing on the interval $[t_0, \infty)$. Equation (E′) is oscillatory if $(A_1[\rho])$ and $(A_2[\rho])$ hold. In particular, we have the next result: Suppose that (F_1) holds and q is nonpositive on $[t_0, \infty)$. Equation (E′) is oscillatory if, for some $\beta \in [0, 1]$, $(A_1(\beta))$ and $(A_2(\beta))$ hold and $t^\beta q(t)$ is increasing for $t \geqq t_0$.

THEOREM 2′. *Suppose that* (F_2) *holds and let* ρ *be a positive continuously differentiable function on the interval* $[t_0, \infty)$ *such that* (R_1) *and* (R_2) *are satisfied. Moreover, let* $\rho' - \rho q$ *be nonnegative and decreasing on* $[t_0, \infty)$. *Equation* (E′) *is oscillatory if* $(A_1[\rho])$ and $(A_3[\rho])$ *hold.*

Proof. By (R_2), we can choose a $T_0 > t_0$ such that (8) is satisfied. Let x be a nonoscillatory solution on an interval $[T, \infty)$, $T \geqq T_0$, of the differential equation (E′). Without loss of generality, we suppose that $x(t) \neq 0$ for all $t \geqq T$ and next we restrict our discussion to the case where x is positive on $[T, \infty)$. Now, we define the function w as in the proof of Theorem 2. Then, by the procedure of the proof of Theorem 1′, we obtain (1)′ and next we verify that (2)′ and (3) are fulfilled with

$$K = [\rho'(T) - \rho(T)q(T)] \int_{x(T)}^{\infty} \frac{dy}{f(y)}.$$

Following the arguments of the proof of Theorem 2 with $\rho' - \rho q$ in place of ρ', we can complete the proof.

Since (R_2) is satisfied when ρ is as in Theorem 1, for $q = 0$ Theorem 2′ leads to Theorem 2. Also, the following result is a consequence of Theorem 2′: Suppose that (F_2) holds and let ρ be as in Theorem 1. Moreover, assume that (R_1) is satisfied and let q be nonpositive and ρq be increasing on the interval $[t_0, \infty)$. Equation (E′) is oscillatory if $(A_1[\rho])$ and $(A_3[\rho])$ hold. This result gives the following particular one: Suppose that (F_2) holds and q is nonpositive on $[t_0, \infty)$. Equation (E′) is oscillatory if, for some $\beta \in [0, 1)$, $(A_1(\beta))$ and $(A_3(\beta))$ hold and $t^\beta q(t)$ is increasing for $t \geqq t_0$.

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