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WEYL'S THEOREM FOR $f(T)$ WHEN T IS A DOMINANT OPERATOR

IN HO JEON, EUNGIL KO*

Department of Mathematics, Ewha Women's University, Seoul 120-750, Korea
e-mail: jih@math.ewha.ac.kr, eiko@mm.ewha.ac.kr

and HONG YOUL LEE**

Department of Mathematics, Jeonju Woosuk University, Korea
e-mail: hylee@core.woosuk.ac.kr

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Abstract. Let T be a dominant operator that is a quasi-affine transform of an M-hyponormal operator. In this paper we show that if f is a function analytic on a neighborhood of the spectrum of T, then Weyl's theorem holds for $f(T)$.

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1. Introduction. Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $B(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. Recall ([**10**], [**12**], [**13**]) that an operator $T \in B(\mathcal{H})$ is said to be *dominant* if for each $\lambda \in \mathbb{C}$ there exists a positive number M_λ such that

$$(T-\lambda)(T-\lambda)^* \leq M_\lambda (T-\lambda)^*(T-\lambda).$$

If the constants M_λ are bounded by a positive real number M, then T is said to be *M-hyponormal*. Also we note that if T is 1-hyponormal, then T is *hyponormal*. Evidently,

$$\text{hyponormal} \;\Rightarrow\; M\text{-hyponormal} \;\Rightarrow\; \text{dominant}. \tag{1.1}$$

If $T \in B(\mathcal{H})$, then we shall denote by $\sigma(T)$, $\sigma_p(T)$, $\pi_0(T)$, $\pi_{00}(T)$, and $\mathrm{iso}\,\sigma(T)$ the spectrum of T, the set of all eigenvalues of T, the set of all eigenvalues of finite multiplicity of T, the set of all isolated eigenvalues of finite multiplicity of T, and the set of all isolated points of $\sigma(T)$, respectively. An operator $T \in B(\mathcal{H})$ is called *Fredholm* if the range of T is closed and $\ker T$ and $\ker T^*$ are both finite dimensional. If T is Fredholm, then the *index* of T is defined by

$$\mathrm{ind}(T) = \dim \ker(T) - \dim \ker(T^*).$$

A Fredholm operator with index zero is called *Weyl* ([**5**], [**6**]). The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : \; T - \lambda \text{ is not Weyl}\}.$$

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Following L. A. Coburn [**2**] we say that *Weyl's theorem holds for T* if

$$\omega(T) = \sigma(T) - \pi_{00}(T).$$

In a vast literature, there exist several classes of operators for which Weyl's theorem holds. In particular, Coburn [**2**, Theorem (3.1)] showed that Weyl's theorem holds for hyponormal operators. This was extended to M-hyponormal operators by Arora and Kumer [**1**, Theorem 4]. On the other hand, using results of Oberai [**9**], Lee and Lee [**8**] showed that the spectral mapping theorem holds for $\omega(T)$ and Weyl's theorem holds for $f(T)$ when T is hyponormal and f is a function analytic on a neighborhood of $\sigma(T)$. Recently, this was also improved by Hou and Zhang [**7**] to show that the spectral mapping theorem holds for the Weyl spectrum of a dominant operator T and that Weyl's theorem holds for $f(T)$ when T is M-hyponormal and f is a function analytic on a neighborhood of $\sigma(T)$.

From the viewpoint of (1.1), it is natural to ask if Weyl's theorem holds for a dominant operator T. In general, however, the answer is negative. See Example 5. This is a motivation to write this paper, and so we shall consider a subclass of dominant operators for which the above question has an affirmative answer.

Recall ([**4**], [**14**]) that an operator $T \in B(\mathcal{H})$ is said to be a *quasi-affine transform* of $S \in B(\mathcal{H})$ if there exists an injection $X \in B(\mathcal{H})$ with dense range such that $SX = XT$, and this relation of T and S is denoted by $T \prec S$. If both $T \prec S$ and $S \prec T$, then we say that T and S are *quasi-similar*. In general, quasi-similarity preserves the point spectrum but not the spectrum, the set of all eigenvalues of finite multiplicity, the set of all isolated eigenvalues, nor the set of all isolated points of the spectrum. It is also well known that quasi-similarity (even similarity) does not preserve hyponormality, M-hyponormality, and dominantness of operators.

In this paper we shall prove the following result, which enables us to extend Theorem 3.5 in [**7**] or Theorem 2 in [**8**].

THEOREM. *Let T, S be a dominant operator and an M-hyponormal operator, respectively. If $T \prec S$ and f is a function analytic on a neighborhood of $\sigma(T)$, then Weyl's theorem holds for $f(T)$.*

2. Proof of Theorem.

We begin with an improvement of a result of Arora and Kumer [**1**, Theorem 4].

LEMMA 1. *Let T, S be a dominant operator and an M-hyponormal operator, respectively. If $T \prec S$, then Weyl's theorem holds for T.*

Proof. If T is dominant, then $T - \lambda$ is also dominant for $\lambda \in \mathbb{C}$, and so it suffices to prove that $0 \in \sigma(T) - \omega(T)$ if and only if $0 \in \pi_{00}(T)$. First, let $0 \in \sigma(T) - \omega(T)$. Then T is Weyl but not invertible, and so $0 \in \pi_0(T)$. Also, we have the following statement:

$$\ker(T) = \ker(T^*) = \operatorname{ran}(T)^\perp \;\Rightarrow\; \ker(T)^\perp = \operatorname{ran}(T).$$

Since $\ker(T)$ is a reducing subspace for T we have the following decompositions;

$$T = 0 \oplus T' \text{ with respect to } \mathcal{H} = \ker(T) \oplus \ker(T)^{\perp}. \tag{2.1}$$

Thus T' is onto, so that T' is invertible and $0 \in \text{iso}\sigma(T)$. Hence $0 \in \pi_{00}(T)$.

Conversely, let $0 \in \pi_{00}(T)$. We can consider a Riesz projection P_0^T corresponding to 0 such that $P_0^T T = TP_0^T$ [**11**]. Then the following is well known:

$$P_0^T(\mathcal{H}) = \mathcal{M}_T, \text{ where } \mathcal{M}_T = \{x \in \mathcal{H} : ||T^n x||^{\frac{1}{n}} \to 0\}.$$

Hence $T|_{P_0^T(\mathcal{H})}$ is also a dominant and quasi-nilpotent operator [**3**, Lemma p. 28]. Now, we claim that

$$P_0^T(\mathcal{H}) = \ker(T). \tag{2.2}$$

Since, by the assumption, there exists an injection X with dense range such that $SX = XT$, we can see that $\text{iso}\sigma(T) \subset \sigma(S)$ because if $\lambda \in \text{iso}\sigma(T)$, then by Theorem 2.5 in [**4**], we have

$$\{\lambda\} = \sigma(T|_{P_\lambda^T(\mathcal{H})}) \cap \sigma(S) \neq \emptyset.$$

Also since every M-hyponormal operator is similar to a subdecomposable operator [**14**, Theorem 1.1] we can see that $\sigma(S) \subset \sigma(T)$, by Theorem 2.1 in [**14**]. Thus $0 \in \text{iso}\sigma(T) \subset \text{iso}\sigma(S)$. We can also consider a Riesz projection P_0^S corresponding to 0 for S and then $P_0^S(\mathcal{H})$ is an invariant subspace of S. Also since

$$||S^n(Xx)||^{\frac{1}{n}} \leq ||X||^{\frac{1}{n}} ||T^n x||^{\frac{1}{n}} \text{ for } x \in \mathcal{H},$$

the quasi-nilpotency of $T|_{P_0^T(\mathcal{H})}$ implies the quasi-nilpotency of $S|_{P_0^S(\mathcal{H})}$. Thus $S|_{P_0^S(\mathcal{H})}$ is quasi-nilpotent and M-hyponormal; hence $S|_{P_0^S(\mathcal{H})}$ is a zero operator by [**10**, Corollary 5]. Since 0 is an isolated point in both $\sigma(T)$ and $\sigma(S)$, by Lemma 2.1 in [**4**] we have $P_0^S X = XP_0^T$. Thus we have

$$0 = SP_0^S X = SXP_0^T = XTP_0^T.$$

It follows that $T|_{P_0^T(\mathcal{H})}$ is also a zero operator. Hence $P_0^T(\mathcal{H}) \subset \ker(T)$ and, since the reverse inclusion is clear, we have equality (2.2). Then, from (2.1), T' is invertible, and so $\ker(T)^{\perp} = \overline{\text{ran}(T)}$. Hence, by the assumption $0 < \dim \ker(T) < \infty$, we get

$$\ker(T) = \overline{\text{ran}(T)}^{\perp} = \text{ran}(T)^{\perp} = \ker(T^*),$$

and thus T is Weyl but not invertible; that is, $0 \in \sigma(T) - \omega(T)$. $\square$

REMARK 2. In the middle of the proof of Lemma 1, we note that T is *isoloid*; i.e., $\text{iso}\sigma(T) \subset \sigma_p(T)$ because

$$0 \in \text{iso}\sigma(T) \Longrightarrow P_0^T(\mathcal{H}) = \ker(T) \neq \{0\}.$$

LEMMA 3 [**7**, Theorem 2.2]. *Let T be a dominant operator and f a function analytic on a neighborhood of $\sigma(T)$. Then $\omega(f(T)) = f(\omega(T))$.*

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LEMMA 4. *Let T, S be a dominant operator and an M-hyponormal operator, respectively. If $T \prec S$ and f is a function analytic on a neighborhood of $\sigma(T)$, then we have*

$$\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T)).$$

Proof. Since T is isoloid, as mentioned in Remark 2, applying it to Lemma in [**8**] the proof immediately follows. □

Proof of Theorem. Since T is dominant, by Lemma 3 we have $\omega(f(T)) = f(\omega(T))$. Thus from Lemma 1 it follows that

$$f(\sigma(T) - \pi_{00}(T)) = f(\omega(T)) = \omega(f(T)).$$

Hence, by Lemma 4, we have

$$\sigma(f(T)) - \pi_{00}(f(T)) = \omega(f(T)).$$ □

We conclude with the following example which shows that, in general, Weyl's theorem does not hold for a dominant operator (even isoloid) but not M-hyponormal operator.

EXAMPLE 5. We consider an operator $T = 0 \oplus W$ as a direct sum of a zero operator acting on a finite dimensional space and a unilateral weighted shift W acting on ℓ^2 defined by

$$W(x_1, x_2, x_3, \cdots) = (0, x_1/2, x_2/3, x_3/4, \cdots).$$

Since an operator $S \in B(\mathcal{H})$ is dominant if and only if $\text{ran}(S) \subset \text{ran}(S^*)$ [**12**], we can see that $T = 0 \oplus W$ is dominant. Furthermore, T is quasi-nilpotent but not zero. From a simple calculation, we deduce that

$$\sigma(T) = \{0\}, \;\; \omega(T) = \{0\}, \;\; \sigma_p(T) = \{0\}, \;\; \text{and } \pi_{00} = \{0\}.$$

We see that T is isoloid but Weyl's theorem does not hold for T. Hence T is not M-hyponormal.

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