APPLICATIONS OF EXTREMAL LENGTH TO CLASSIFICATION OF RIEMANN SURFACES

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Introduction

Let D be a subregion of a Riemann surface F, whose relative boundary consists of at most countable number of analytic curves which do not cluster in F. For a regular exhaustion $\{F_n\}$ of F, we put $D_n = D \cap (F - F_n)$, and define the extremal radius $R(P, \partial D_n)$ of the relative boundary ∂D_n of D_n, measured at a point $P(\in F_0)$ of F with respect to the connected component of $F - D_n$ which contains P. Let $K(|z| \leqq r)$ be a disk centered at P and contained in a parametric disk of P. And let $\lambda_{n,r}$ be the extremal length of the family of rectifiable curves which join ∂K and ∂D_n. Then, the extremal radius $R(P, \partial D_n)$ is defined as follows [2];

$$R(P, \partial D_n) = \lim_{r \to 0} re^{2\pi\lambda_{n,r}}.$$

And we put

$$R(P, B_D) = \lim_{n \to \infty} R(P, \partial D_n).$$

Taking F as D, we define the extremal radius $R(P, B) = \lim_{r \to 0} re^{2\pi\mu_r}$ of the ideal boundary B of F, where μ_r is the extremal length of the family of locally rectifiable curves which start from ∂K and tend to the ideal boundary B of F.

In § 1 we show that it is necessary and sufficient for F not to belong to the class O_{HD} that there exists a subregion D of F for which $\infty > R(P, B_D) > R(P, B)$.

In § 2 we consider a subregion W in place of the Riemann surface F. The corresponding extremal radii are denoted by $R'(P, B_D)$ and $R'(P, B)$.

Then, the existence of a subregion D of W such that $\infty > R'(P, B_D) > R'(P, B)$ is necessary and sufficient for W not to belong to the class NO_{HD}[1].

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[1] NO_{HD} (SO_{HD}) denotes the class of subregions on which there are no non-constant harmonic functions with finite Dirichlet integral whose normal derivatives are zero (which are zero, respectively) on the relative boundary.

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And we consider the extremal radius $R'(P, \partial W)$ of the relative boundary ∂W of W and the extremal radius $R'(P, \partial W \cup B)$ of the union of ∂W and the ideal boundary B of W. Then, W does not belong to the class SO_{HD} if and only if $R'(P, \partial W) > R'(P, \partial W \cup B)$.

Some applications of the theorems are also showed in this section.

§ 1. A criterion for the class O_{HD} of Riemann surfaces

In order to evaluate the extremal lengh $\lambda_{n,r}$, we consider the following harmonic function $U_{n,i}$ in $\{(F-D_n)\cap F_{n+i}\}-P$.[2)]

$$U_{n,i}: \begin{cases} U_{n,i} = -\log|z| + u_{n,i} & \text{in a neighborhood of } P, \text{ where } u_{n,i} \text{ is harmonic} \\ U_{n,i} = 0 & \text{on } \partial D_n \cap F_{n+i} \\ \dfrac{U_{n,i}}{\partial n} = 0 & \text{on } \partial F_{n+i} \cap (F-D_n). \end{cases}$$

Since the sequence $\{U_{n,i}\}_i$ converges in the sense of Dirichlet norm[3)], it converges uniformly to a limit function U_n on every compact set in $F-D_n$. The extremal length $\lambda_{n,i,r}$ of the family of curves which join ∂K and $\partial D_n \cap F_{n+i}$ decreases monotonely when i increases. So,

$$\lambda_{n,i,r} \geqq \lim_{i\to\infty} \lambda_{n,i,r} \geqq \lambda_{n,r}.$$

But, denoting by $U_{n,i,r}$ a harmonic function in $(F-D_n)\cap F_{n+i}-K$ which is zero on $\partial D_n \cap F_{n+i}$, equals $-\log r$ on ∂K, and whose normal derivative is zero on $\partial F_{n+i}\cap(F-D_n)$,

$$\lambda_{n,i,r} = \frac{(\log r)^2}{D(U_{n,i,r})}$$

and

$$\lambda_{n,r} \geqq \frac{(\log r)^2}{D(U_{n,r})} = \lim_{i\to\infty} \frac{(\log r)^2}{D(U_{n,i,r})},$$

where $U_{n,r} = \lim_{i\to\infty} U_{n,i,r}$.

Hence

$$\lambda_{n,r} = \lim_{i\to\infty} \lambda_{n,i,r} = \frac{(\log r)^2}{D(U_{n,r})}.$$

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2) When $\{(F-D_n)\cap F_{n+i}\}$ is not connected, we take a connected component which contains P.

3) $\lim_{i\to\infty} D(U_{n,i+p}-U_{n,i}) = 0$. (cf. Strebel [2] p. 8).

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While,

$$2\pi\frac{(\log r)^2}{D(U_{n,r})} = -\log r + u_n(0) + o(1),^{4)}$$

where $u_n = \lim_{i\to\infty} u_{n,i}$. We conclude that

$$R(P, \partial D_n) = e^{u_n(0)}.$$

And by our definition $R(P, B_D) = \lim_{n\to\infty} R(P, \partial D_n)$.

Using this extremal radius we get the following theorem.

THEOREM 1. *A Riemann surface F does not belong to the class O_{HD} if and only if there exists a subregion D of F such that*

$$\infty > R(P, B_D) > R(P, B).$$

For the proof of the theorem, we prove the following lemma.

LEMMA. *If the double $\hat{D}$ of D is not of the class O_G, the limit function $U_{B_D} = \lim_{n\to\infty} U_n$[5] is not constantly infinite.*

Proof of the lemma. By adding to D a suitable relatively compact subregion Δ which contains P, we build up a (connected) subregion $D' = D \cup \Delta$ whose double $\hat{D}'$ is not of the class O_G. The extremal length of the family of curves in $\hat{D}'$ which start from $\partial K \cup (\partial K)^{\sim}$ ($(\partial K)^{\sim}$ is a symmetric image of ∂K in $\hat{D}' - D'$) and tend to the ideal boundary of $\hat{D}'$ is finite because $\hat{D}' \notin O_G$. Then, by the method of symmetrization [3], the extremal length λ'_A, with respect to D', of the family A of curves in D' which start from ∂K and tend to the ideal boundary of D' is finite. Now, we consider a family B of curves in F, each curve of which contains a curve connecting ∂K and ∂D_n for all n. Then the family B contains the family A, so the extremal length λ_B of B with respect to F is smaller than the extremal length λ_A of A with respect to F. And,

$$\lambda_B \leqq \lambda_A = \lambda'_A < \infty.$$

But,

$$\frac{(\log r)^2}{D(U_{n,r})} = \lambda_{n,r} \leqq \lambda_B < \infty$$

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4) About these calculation, confer Strebel's paper ([2] p. 13).

5) According to Strebel, we call U_{BD} "Strömungspotential".

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and

$$\lim_{n\to\infty}\frac{(\log r)^2}{D(U_{n,r})}=\lim_{n\to\infty}\lambda_{n,r}\leqq\lambda_B<\infty.$$

So, $U_r=\lim_{n\to\infty}U_{n,r}$ is not a constant, and from

$$2\pi\frac{(\log r)^2}{D(U_{n,r})}=-\log r+u_n(0)+o(1),$$

$\lim_{n\to\infty}u_n(0)$ is finite.

Therefore, for a sufficiently large number L,

$$U_{BD}=\lim_{n\to\infty}U_n\leqq\lim_{n\to\infty}U_{n,r}+L$$

in $F-K$, and this shows that U_{BD} is not constantly infinite in F.

Proof of the Theorem. If F is not of the class O_{HD}, there are two disjoint subregions D and S neither of which is of the class SO_{HD}. And we suppose that the point P and its parametric disk K are contained in S.

For a regular exhaustion $\{F_n\}$ of F, we construct a harmonic function v_n in $F_n\cap S$[6] such that

$$v_n:\begin{cases}v_n & \text{has a positive logarithmic pole at } P\\ v_n=0 & \text{on } \partial S\cap F_n\\ \dfrac{\partial v_n}{\partial n}=0 & \text{on } \partial F_n\cap S.\end{cases}$$

v_n tends to a limit function $v=\lim_{n\to\infty}v_n$ as above, and v is not constant because v has a logarithmic pole at P and $v=0$ on ∂S. Let g be Green's function of S with a pole at P. Then, by Kuramochi's theorem (Kuramochi [1] p. 135),

$$v>g$$

because $S\notin SO_{HD}$.

On the other hand, since $D\notin SO_{HD}$, the double $\hat{D}$ does not belong to O_G. So, by the lemma, there exists a non-constant limit function U_{BD} of U_n. Now, we prove in the following that the inequality

$$U_{BD}-G\geqq v-g$$

holds in S, where G is Green's function of F. Let $G_{n,i}$ be Green's function of

[6] We take a connected component of $F_n\cap S$ which contains P.

$(F-D_n)\cap F_{n+i}$ with a pole at the point P, and g_{n+i} be Green's function of $F_{n+i}\cap S$ with a pole at P. We prove the above inequality in three steps.

1) Since $U_{n,i}-v_{n+i}$ is harmonic in $F_{n+i}\cap S$ and

$$\begin{cases} U_{n,i}-v_{n+i}\geqq 0 & \text{on } \partial S\cap F_{n+i} \\ \dfrac{\partial(U_{n,i}-v_{n+i})}{\partial n}=0 & \text{on } \partial F_{n+i}\cap S, \end{cases}$$

we have $U_{n,i}-v_{n+i}\geqq 0$ in $F_{n+i}\cap S$, especially on $\partial F_{n+i}\cap S$.

2) Since $v_{n+i}=g_{n+i}=0$ on $\partial S\cap F_{n+i}$,

$$U_{n,i}-G_{n,i}-(v_{n+i}-g_{n+i})=\begin{cases} U_{n,i}-G_{n,i}\geqq 0 & \text{on } \partial S\cap F_{n+i} \\ U_{n,i}-v_{n+i}\geqq 0 & \text{on } \partial F_{n+i}\cap S. \end{cases}$$

So, we have

$$U_{n,i}-G_{n,i}-(v_{n+i}-g_{n+i})\geqq 0 \qquad \text{on } \partial(S\cap F_{n+i}).$$

Hence,

$$U_{n,i}-G_{n,i}-(v_{n+i}-g_{n+i})\geqq 0 \qquad \text{in } S\cap F_{n+i}.$$

3) Here, let i tend to ∞, then

$$U_n-G_n\geqq v-g \qquad \text{in } S.$$

Since this inequality is valid for all n, we have

$$U_{B_D}-G\geqq v-g>0 \qquad \text{in } S.$$

And

$$U_{B_D}-G\geqq 0 \qquad \text{in } F$$

from the start. Consequently,

$$U_{B_D}-G>0 \qquad \text{in } F.$$

But, if we put $u=\lim\limits_{n\to\infty}\lim\limits_{i\to\infty}u_{n,i}$, then $U_{B_D}=-\log r+u$ in the neighborhood of P, and $R(P,B_D)=e^{u(0)}$. And

$$R(P,B)=e^{h(0)},$$

where $G=-\log r+h$ in the neighborhood of P. Therefore,

$$R(P,B_D)>R(P,B).$$

And $R(P,B_D)<\infty$ from the lemma.

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Conversely, we suppose that there exists a subregion D such that $\infty > R(P, B_D) > R(P, B)$. Then, $U_{B_D} - G$ is a non-constant harmonic function with finite Dirichlet integral. And F does not belong to O_{HD}.

Namely, if $U_{BD} - G$ is a constant, $D_{F-|z|<r}(U_{BD}) = D_{F-|z|<r}(G)$. And since

$$D(U_r - U_{B_D}) = o(1) \text{ and } D(G - G_r) = o(1)$$

we have $D(U_r) = D(G_r) + o(1)$. Here, G_r is a harmonic function in $F - (|z| \leqq r)$ with boundary values $\log 1/r$ on $|z| = r$ and zero on the ideal boundary of F. While, from $R(P, B_D) > R(P, B)$, we have

$$\lim_{n\to\infty} \lambda_{n,r} - \mu_r > \frac{1}{2\pi} \log\left(\frac{d}{re^{2\pi\mu_r}} + 1\right)$$

with a positive constant d, in $F - (|z| \leqq r)$ for sufficiently small r. This is a contradiction, because

$$\lim_{n\to\infty} \lambda_{n,r} - \mu_r = \frac{(\log r)^2}{D(U_r)} - \frac{(\log r)^2}{D(G_r)} = (\log r)^2 \frac{D(G_r) - D(U_r)}{D(U_r)D(G_r)}$$

and $D(U_r)D(G_r) \sim (\log r)^2$.

Remark. In the proof of Theorem 1, it is also proved that if there exist two such subregions D and S on a Riemann surface F that $\hat{D}$ is not of the class O_G and S is not of the class SO_{HD}, then, the Riemann surface F is not of the class O_{HD}.

§ 2. Subregion

In this section we consider a subregion W, and put $\overline{W} = W + \partial W$. We choose a sequence $\{W_n\}$ (exhaustion of W) of relatively compact subregions W_n such that the relative boundary ∂W_n of W_n consists of closed curves in W, cross-cuts ending at ∂W and parts of ∂W, and such that the intersection of the closures $\{\overline{W - W_n}\}$ of $\{W - W_n\}$ in W is empty. Then, the sequence $\{W - W_n\}$ defines the ideal boundary B of W. For a relatively non-compact subregion D of $\overline{W}$ we put $D_n = D \cap (W - W_n)$. Let $\lambda_{n,i,r}$ be the extremal length of the family of curves in $W_{n+i} - D_n$ which join ∂K and $\partial D_n \cap \overline{W}_{n+i}$. Then $\lambda_{n,r} = \lim_{i\to\infty} \lambda_{n,i,r}$ is the extremal length of the family of curves in $F - D_n$ which join ∂K and ∂D_n and we put $\lambda_r = \lim_{n\to\infty} \lambda_{n,r}$. And let μ_r be the extremal distance between ∂K and the ideal boundary B. By putting $R(P, B) = \lim_{r\to 0} re^{2\pi\mu_r}$ and

$R(P, B_D) = \lim_{r \to 0} re^{2\pi\lambda_r}$, we have the following theorem.

THEOREM 2. *W does not belong to NO_{HD} if and only if there exists a subregion D of W such that*

$$\infty > R(P, B_D) > R(P, B).$$

Proof. If W is not of the class NO_{HD}, the double $\hat{W}$ of W is not of the class O_{HD} and we can find two disjoint subregions D' and F' each of which is symmetric and not of the class SO_{HD}. We write $D' = D \cup \tilde{D}$ and $F' = F \cup \tilde{F}$.

As in the proof of Theorem 1, we construct the "Strömungspotential" U'_{B_D} with respect to D' and a point P in W and the "Strömungspotential" $U_{B_D}(\tilde{P})$ with respect to D' and the symmetric point $\tilde{P}$ of P. Let $G'(P)$ and $G'(\tilde{P})$ be Green's functions of W with the pole at P and the symmetric point $\tilde{P}$ of P, respectively. And we put

$$U_{B_D} = \frac{1}{2}(U'_{B_D}(P) + \tilde{U}'_{B_D}(\tilde{P})),$$

$$G = \frac{1}{2}(G'(P) + \tilde{G}'(\tilde{P})).$$

Then the normal derivatives of them along ∂W are zero. And since

$$U'_{B_D}(P) > G'(P) \text{ and } U'_{B_D}(\tilde{P}) > G'(\tilde{P}),$$

we have

$$U_{B_D} > G.$$

Hence, as in Theorem 1 we have

$$\infty > R(P, B_D) > R(P, B).$$

Converse is also true. If there exists a subregion D of W for which

$$\infty > R(P, B_D) > R(P, B),$$

we find as the proof of Theorem 1 that $U_{B_D} - G$ is a non-constant harmonic function with finite Dirichlet integral whose normal derivative on ∂W is zero. Hence, W is not of the class NO_{HD}.

Denoting by $R(P, \partial W)$ and $R(P, B(W))$ the extremal radii, measured at a point P, of the relative boundary ∂W of W and the whole boundary $B(W) = \partial W +$ (ideal boundary) of W, respectively, we have the following theorem as a direct consequence of Kuramochi's theorem.

THEOREM 3. *A subregion W is not of the class SO_{HD} if and only if*

$$R(P, \partial W) > R(P, B(W)).$$

As applications of Theorems 2 and 3 to the plane regions, we consider a closed set E on the unit circle $|z| = 1$. We set $W = |z| < 1$ and $\partial W = (|z| = 1) - E$. Then, W is of the class NO_{HD} if and only if capacity of E is zero. E is of the class N_D if and only if $R(P, |z| = 1) = R(P, \partial W)$ because if E is of the class N_D W is of the class SO_{HD} and vice versa.

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