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REFLECTION/REFRACTION OF SH-WAVES AT A CORRUGATED INTERFACE BETWEEN TWO DIFFERENT ANISOTROPIC AND VERTICALLY HETEROGENEOUS ELASTIC SOLID HALF-SPACES

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Abstract

We investigate the reflection and transmission of SH-waves at a corrugated interface between two different anisotropic, heterogeneous elastic solid half-spaces. Both the media are assumed to be transversely isotropic and vertically heterogeneous. Rayleigh's method is followed and expressions for the reflection and transmission coefficients are obtained in closed form for the first-order approximation of the corrugation. It is found that these coefficients depend on corrugation and are affected by the anisotropy and heterogeneity of the media. Numerical computations for a particular model have been performed.

1. Introduction

The study of the propagation of seismic waves and their reflection and refraction from discontinuities are of great practical importance in seismology. The amplitudes of seismic signals are of great help not only in investigating the internal structure of the earth, but also in exploration of valuable materials, oils, water, chemicals *etc.* Seismic waves can occur as a result of an earthquake. Mathematical study of seismic waves consists mainly of the study of propagation, reflection, refraction and diffraction of elastic waves from discontinuities present inside the earth.

Considerable work has been done by researchers concerning seismic wave propagation at a plane boundary. Bearing in mind the fact that earthquake-generated seismic waves encounter mountain basins, mountain roots and salt and ore bodies in their paths, such irregularities doubtlessly affect the reflection and refraction of elastic waves propagating through the earth. Thus the study of the reflection and refraction of elastic waves at various types of interfaces is of great practical importance.

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In reality, boundaries involving earth as a medium can never be flat but are always irregular to some extent. Moreover, the earth is not homogeneous and isotropic throughout. This has motivated us to consider the study of the reflection and refraction of SH-waves at a corrugated interface between two anisotropic and heterogeneous elastic half-spaces. The aim is to investigate how the reflection and refraction coefficients of SH-waves are affected by the corrugation of the interface, taking into account the anisotropy and heterogeneity of the two media.

Scattering of elastic waves from a corrugated surface has been studied by many researchers and there are several methods for dealing with such problems. Sato [14] analysed the problem of elastic wave scattering from a corrugated, traction-free boundary of an elastic half-space using the method introduced by Lord Rayleigh in his problem of sound and light. In Rayleigh's method expressions in the boundary conditions containing the function defining the corrugated boundary are expanded in Fourier series and the unknown coefficients in the solution are determined to any given order of approximation in terms of small parameter characteristics of the boundary.

Asano [3, 4, 5] also applied Rayleigh's method to problems involving the reflection and refraction of elastic waves at a corrugated interface between two half-spaces. Abubakar [2], Dunkin and Eringen [8] among others used a perturbation technique to study the problem of reflection of body waves from an arbitrary rough surface of a semi-infinite elastic solid. Abubakar [1] worked out the reflection and refraction of SH-waves at an irregular interface between two uniform elastic solid half-spaces using a perturbation technique.

A lot of literature is available on elastic wave propagation and reflection and refraction at a plane boundary between two anisotropic elastic half-spaces, including that by Musgrave [11], Thapliyal [17], Daley and Hron [7], Rokhlin *et al.* [13], Mandal [10], Crampin [6] and Pao [12]. Recently Tomar and Saini [18] studied the effect of transverse anisotropy on the reflection and transmission coefficients of SH-waves at a corrugated interface between two half-spaces.

Since anisotropy and heterogeneity are well-established phenomena within the earth, and keeping in view their importance, we have considered the problem of the reflection and refraction of SH-waves at a corrugated interface between two anisotropic heterogeneous elastic solids. Rayleigh's method is adopted and the expressions for the reflection and transmission coefficients are obtained in closed form for the first-order approximation of the corrugation. The results of Tomar and Saini [18] and Asano [3] reduce to particular cases of this problem.

2. Formulation of the problem and its solution

We consider a corrugated interface between two heterogeneous, transversely isotropic elastic solid half-spaces H_i $(i = 1, 2)$, with elastic constants, densities and

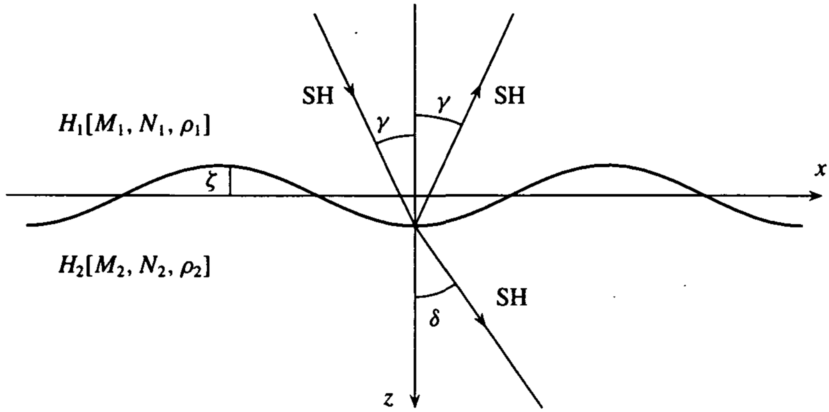

FIGURE 1. Geometry of the problem.

velocities (vertical and horizontal) given by M_i, N_i, ρ_i and β_{v_i}, β_{h_i} respectively. The factor N_i/M_i denotes the anisotropy factor and we let the variations of elastic parameters in H_i be

$$\begin{aligned}\{M_1, N_1, \rho_1\} &= \{M_0, N_0, \rho_0\}\cosh^2(z/b_1),\\ \{M_2, N_2, \rho_2\} &= \{M_0', N_0', \rho_0'\}\cosh^2(z/b_2),\end{aligned} \tag{2.1}$$

where the quantities within braces on the right-hand sides of (2.1) are constants and b_1^{-1}, b_2^{-1} are heterogeneity factors in H_1 and H_2 respectively. The x-and y-axes are horizontal and the z-axis is pointing vertically downward as shown in Figure 1.

Let the equation of a corrugated interface between two considered half-spaces be $z = \zeta$, where ζ is assumed to be a periodic function of x and independent of y, whose mean value is zero by assumption. The Fourier series representation of ζ is given by

$$\zeta = \sum_{n=1}^{\infty}[\zeta_n \exp(inpx) + \zeta_{-n}\exp(-inpx)]. \tag{2.2}$$

Taking $\zeta_1 = \zeta_{-1} = c/2$, $\zeta_{\pm n} = (c_n \mp is_n)/2$, we can write the expansion in (2.2) as

$$\zeta = c\cos px + c_2\cos 2px + s_2 \sin 2px + \cdots + c_n \cos npx + s_n \sin npx + \cdots .$$

In a special case, when the interface can be expressed by one cosine term, that is, $\zeta = c\cos px$, the wavelength of the corrugation is $2\pi/p$.

Neglecting body forces and assuming small deformations, the equation of motion for SH-waves in a transversely isotropic, heterogeneous medium can be written as

$$\frac{\partial}{\partial x}\left(N_i\frac{\partial V_i}{\partial x}\right) + \frac{\partial}{\partial z}\left(M_i\frac{\partial V_i}{\partial z}\right) = \rho_i\frac{\partial^2 V_i}{\partial t^2}, \tag{2.3}$$

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where V denotes the y-component of the displacement and the subscript i corresponds to the quantities in H_i. Consider the time harmonic waves and let

$$V_i = X_i(x)Z_i(z)\exp(i\omega t), \tag{2.4}$$

where ω is the circular frequency. Using (2.4) in (2.3) and assuming that elastic parameters are functions of z alone, we obtain

$$\frac{d^2X_i}{dx^2} + a_i^2X_i = 0, \quad M_i\frac{d^2Z_i}{dz^2} + \frac{dM_i}{dz}\frac{dZ_i}{dz} + \rho_i\omega^2\cos^2\theta Z_i = 0, \tag{2.5}$$

where a_i is the x-component of the wave number given by (Gupta [9])

$$a_i = \frac{\omega\sin\theta}{\beta_{h_i}}, \quad \beta_{h_i}^2 = \frac{N_i}{\rho_i},$$

where θ is the angle between the wave normal and the positive direction of the z-axis. Putting $Z_i = \overline{Z}_i/\sqrt{M_i}$ in $(2.5)_2$, we obtain

$$\frac{d^2\overline{Z}_i}{dz^2} - q_i^2\overline{Z}_i = 0,$$

where

$$q_i^2 = \frac{1}{b_i^2} - \frac{\omega^2}{\beta_{h_i}^2}\left(\frac{N_i}{M_i}\right)\cos^2\theta.$$

For a wave propagating along the positive direction of the x-axis, the solution of (2.5) is given by

$$X_i = A\exp(-ia_ix), \quad Z_i = \frac{1}{\sqrt{M_i}}\left[A_0e^{-q_iz} + B_0e^{q_iz}\right], \tag{2.6}$$

where A, A_0 and B_0 are constants.

Consider a plane SH-wave of unit amplitude and period $2\pi/\omega$, incident from the upper half-space H_1. Let γ be the angle between the z-axis and the incident wave normal and let the direction of propagation of the wave be the positive x-axis. With the help of (2.6), the solution (2.4) in the two elastic half-spaces is as follows. In medium H_1, the displacement is given by

$$V_1 = \left[\frac{e^{-qz} + B_0e^{qz}}{\sqrt{M_0}\cosh(z/b_1)}\right]\exp\left(i\omega\left(t - \frac{x\sin\gamma}{\beta_{h_1}}\right)\right).$$

In medium H_2, the displacement is given by

$$V_2 = \left[\frac{D_0e^{-rz}}{\sqrt{M_0'}\cosh(z/b_2)}\right]\exp\left(i\omega\left(t - \frac{x\sin\delta}{\beta_{h_2}}\right)\right), \tag{2.7}$$

where δ is the angle made by the refracted wave normal with the z-axis and is related to γ through Snell's law given by

$$\frac{\omega \sin\gamma}{\beta_{h_1}} = \frac{\omega \sin\delta}{\beta_{h_2}}$$

and

$$q^2 = \frac{1}{b_1^2} - \frac{\omega^2}{\beta_{h_1}^2}\frac{N_1}{M_1}\cos^2\gamma, \qquad r^2 = \frac{1}{b_2^2} - \frac{\omega^2}{\beta_{h_1}^2}\frac{N_2}{M_2}\left(\frac{\beta_{h_1}^2}{\beta_{h_2}^2} - \sin^2\gamma\right).$$

Since the interface is corrugated, it is necessary to take into account the effect of corrugation on the reflection and refraction of waves, in addition to the regularly reflected and refracted waves. Thus the total displacement V_1 in the upper medium is the sum of the incident, regularly reflected and irregularly reflected waves as

$$\begin{aligned} V_1 = {} & \left[\frac{e^{-qz} + B_0 e^{qz}}{\sqrt{M_0}\cosh(z/b_1)}\right]\exp\left(i\omega\left(t - \frac{x\sin\gamma}{\beta_{h_1}}\right)\right) \\ & + \sum_n B_n e^{q_n z}\exp\left(i\omega\left(t - \frac{x\sin\gamma_n}{\beta_{h_1}}\right)\right) \\ & + \sum_n B_n' e^{q_n' z}\exp\left(i\omega\left(t - \frac{x\sin\gamma_n'}{\beta_{h_1}}\right)\right). \end{aligned}$$

Similarly, the total displacement V_2 in the lower medium can be written as

$$\begin{aligned} V_2 = {} & \left[\frac{D_0 e^{-rz}}{\sqrt{M_0'}\cosh(z/b_2)}\right]\exp\left(i\omega\left(t - \frac{x\sin\delta}{\beta_{h_2}}\right)\right) \\ & + \sum_n D_n e^{-r_n z}\exp\left(i\omega\left(t - \frac{x\sin\delta_n}{\beta_{h_2}}\right)\right) \\ & + \sum_n D_n' e^{-r_n' z}\exp\left(i\omega\left(t - \frac{x\sin\delta_n'}{\beta_{h_2}}\right)\right), \end{aligned} \tag{2.8}$$

where γ_n, γ_n', δ_n and δ_n' are given by the Spectrum theorem (see Asano [3]),

$$\begin{aligned} \sin\gamma_n - \sin\gamma &= \frac{np\beta_{h_1}}{\omega}, & \sin\delta_n - \sin\delta &= \frac{np\beta_{h_2}}{\omega}, \\ \sin\gamma_n' - \sin\gamma &= -\frac{np\beta_{h_1}}{\omega}, & \sin\delta_n' - \sin\delta &= -\frac{np\beta_{h_2}}{\omega}, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} q_n^2 &= \frac{1}{b_1^2} - \frac{\omega^2}{\beta_{h_1}^2}\frac{N_1}{M_1}\cos^2\gamma_n, & r_n^2 &= \frac{1}{b_2^2} - \frac{\omega^2}{\beta_{h_1}^2}\frac{N_2}{M_2}\left(\frac{\beta_{h_1}^2}{\beta_{h_2}^2} - \sin^2\gamma_n\right), \\ q_n'^2 &= \frac{1}{b_1^2} - \frac{\omega^2}{\beta_{h_1}^2}\frac{N_1}{M_1}\cos^2\gamma_n', & r_n'^2 &= \frac{1}{b_2^2} - \frac{\omega^2}{\beta_{h_1}^2}\frac{N_2}{M_2}\left(\frac{\beta_{h_1}^2}{\beta_{h_2}^2} - \sin^2\gamma_n'\right). \end{aligned}$$

Using the relations given in (2.9), the displacements V_1 and V_2 become

$$
\begin{aligned}
V_1 = {} & \frac{1}{\sqrt{M_0}\cosh(z/b_1)}\left[e^{-qz} + B_0 e^{qz} + \sum_n B_n e^{q_n z} e^{-inpx} + \sum_n B_n' e^{q_n' z} e^{inpx}\right] \\
& \times \exp\left(i\omega\left(t - \frac{x\sin\gamma}{\beta_{h_1}}\right)\right), \qquad (2.10) \\
V_2 = {} & \frac{1}{\sqrt{M_0'}\cosh(z/b_2)}\left[D_0 e^{-rz} + \sum_n D_n e^{r_n z} e^{-inpx} + \sum_n D_n' e^{r_n' z} e^{inpx}\right] \\
& \times \exp\left(i\omega\left(t - \frac{x\sin\gamma}{\beta_{h_1}}\right)\right). \qquad (2.11)
\end{aligned}
$$

The unknowns D_0 in (2.7) and D_n, D_n' in (2.8) are related to the transmission coefficients T, T_n and T_n' through the relation (Singh *et al.* [16]),

$$
\{T, T_n, T_n'\} = \{D_0, D_n, D_n'\}\sqrt{\frac{M_0}{M_0'}\frac{\cosh(z/b_1)}{\cosh(z/b_2)}}. \qquad (2.12)
$$

3. Boundary conditions

The boundary conditions to be satisfied at the boundary surface $z = \zeta$ are the continuity of the displacement and stress, that is,

$$
V_1 = V_2, \qquad (3.1)
$$

$$
M_1\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_1}{\partial x}\zeta'\right) = M_2\left(\frac{\partial V_2}{\partial z} - \frac{\partial V_2}{\partial x}\zeta'\right), \qquad (3.2)
$$

where ζ' is the derivative of ζ with respect to x. Substituting (2.10) and (2.11) into (3.1) and (3.2), we can obtain the amplitudes of reflected and refracted waves at a plane interface and at a corrugated interface (see Appendix A).

We shall now consider a special case, when the boundary surface is given by $z = c\cos px$. In Appendix A, we have obtained the results for the corrugated interface of the form $z = \zeta$. In the special case, we have $\zeta_n = \zeta_{-n} = 0$, $n \neq 1$, and $\zeta_1 = \zeta_{-1} = c/2$. From (A.4) and (A.5) of Appendix A, we obtain the solution for B_0, D_0 given by

$$
B_0 = \left(Q\frac{M_0}{M_0'} - R\right)\left(Q\frac{M_0}{M_0'} + R\right)^{-1}, \quad D_0 = 2Q\sqrt{\frac{M_0}{M_0'}}\left(Q\frac{M_0}{M_0'} + R\right)^{-1}, \qquad (3.3)
$$

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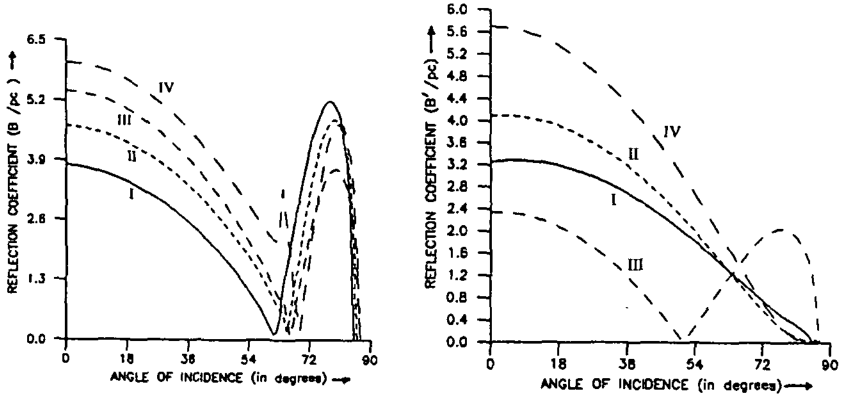


(a) (b)

FIGURE 2. Reflection variation with angle of incidence with anisotropy factors (a) $N_0/M_0 = 0.5$, $N_0'/M_0' = 0.7$ and (b) $N_0/M_0 = 0.7$, $N_0'/M_0' = 0.9$.

and hence using (2.12), the transmission coefficient in this case is

$$T = 2Q\frac{M_0}{M_0'}\left(Q\frac{M_0}{M_0'} + R\right)^{-1}. \tag{3.4}$$

Equations (3.3) and (3.4) are expressions for the reflection and transmission coefficients when SH-waves are incident at a plane interface between two transversely isotropic and vertically heterogeneous elastic solid half-spaces.

In order to obtain the solution of the first-order approximation of corrugation for B_1, D_1, B_1' and D_1', we solve (A.6)–(A.8) given in Appendix A and obtain the values

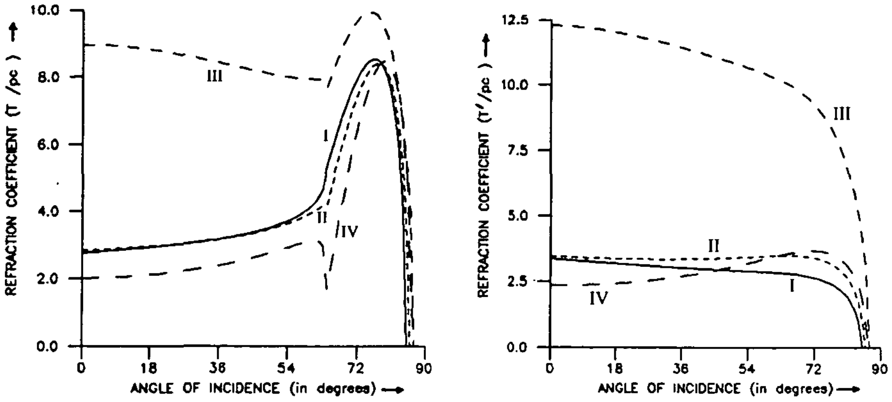
$$B_1 = d_1/d, \quad D_1 = d_2/d, \quad B_1' = d_1'/d', \quad D_1' = d_2'/d', \tag{3.5}$$

where the values of d_i, d_i', d and d' $(i = 1, 2)$ are given in Appendix B. The coefficients B_1, B_1' are reflection coefficients and using relation (2.12) the transmission coefficients are given by

$$\{T_1, T_1'\} = \{D_1, D_1'\}\sqrt{\frac{M_0}{M_0'}\frac{\cosh(c/2b_1)}{\cosh(c/2b_2)}}. \tag{3.6}$$

4. Particular cases

The heterogeneity of the half-spaces would be removed if we put $b_1^{-1} = b_2^{-1} = 0$ in all the expressions where they occur. Then the coefficients for the first approximation



(a) (b)

FIGURE 3. Reflection variation with angle of incidence with anisotropy factors (a) $N_0/M_0 = 1.5$, $N_0'/M_0' = 3.7$ and (b) $N_0/M_0 = 1.0$, $N_0'/M_0' = 1.0$.

of the corrugation given in (3.5) and (3.6) and the coefficients at the plane boundary given in (3.3) and (3.4) reduce to those at a corrugated interface between two homogeneous, transversely isotropic half-spaces. These reduced coefficients are found to be in full agreement with those obtained by Tomar and Saini [18].

When both anisotropy and heterogeneity vanish, then $N_i = M_i = \mu_i$, $\beta_{h_i} = \beta_{v_i} = \beta_i$ and the problem reduces to that of reflection and refraction of SH-waves at a corrugated boundary between two uniform elastic half-spaces with different properties. One can verify using simple algebra and appropriate notation that all the results in question are the same as those obtained by Asano [3].

In order to replace the corrugated interface by a plane interface, we put $\zeta = 0$, and removing heterogeneity and anisotropy as above, the problem reduces to SH-wave propagation at a plane interface between two uniform elastic half-spaces. In this case B_n, D_n, B_n' and D_n' vanish as they are proportional to ζ and we obtain

$$B_0 = \frac{m\cos\gamma - \sqrt{n^2 - \sin^2\gamma}}{m\cos\gamma + \sqrt{n^2 - \sin^2\gamma}}, \quad D_0 = \frac{2m\cos\gamma}{m\cos\gamma + \sqrt{n^2 - \sin^2\gamma}}, \tag{4.1}$$

where $n > \sin\gamma$, $m = \mu_1/\mu_2$ and $n = \beta_1/\beta_2$. These results are the same as those given in Savarensky [15, page 284].

5. Numerical results and discussion

In order to numerically study the effect of anisotropy and heterogeneity on the reflection and transmission coefficients we have computed these for a specific model

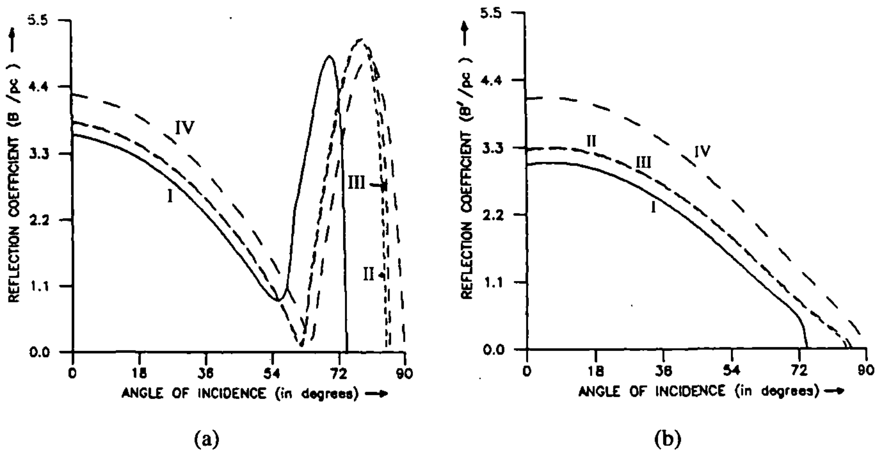
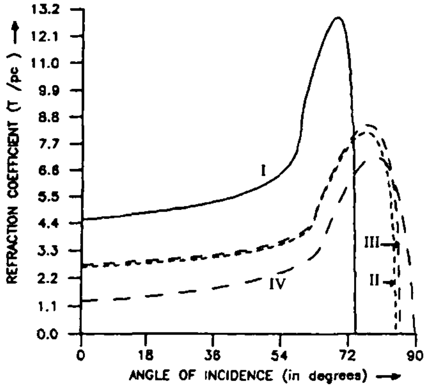


FIGURE 4. Reflection variation with angle of incidence with heterogeneity factors (a) $1/b_1 = 0.5$, $1/b_2 = 0.7$ and (b) $1/b_1 = 1.5$, $1/b_2 = 1.7$.

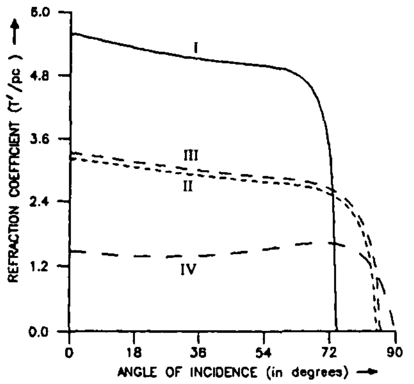
by varying the values of the anisotropy and inhomogeneity parameters. It is found that these coefficients are influenced by the anisotropy and inhomogeneity of the half-spaces. First, the anisotropy and inhomogeneities of the media are kept constant at values $1/b_1 = 1.5$, $1/b_2 = 2.0$, $\beta_{h_1}^2/\beta_{h_2}^2 = 1.5$, $\omega/p\beta_{h_1} = 10$ and the anisotropy factors of the media are varied. The variations of the reflection and transmission coefficients with angle of incidence γ are presented in Figures 2–3. Setting the values of the anisotropy factors equal to 1.0 corresponds to isotropy, that is, both media are then inhomogeneous and isotropic elastic half-spaces.

Next, the anisotropy of both media are kept fixed at values $N_0/M_0 = 0.5$, $N_0'/M_0' = 0.7$, $M_0/M_0' = 5.0$, $\beta_{h_1}^2/\beta_{h_2}^2 = 1.5$, $\omega/p\beta_{h_1} = 10$, and the heterogeneity factors are varied. The results obtained are presented in Figures 4–5. Setting the values of the inhomogeneity factors equal to zero corresponds to homogeneity, that is, both media are then homogeneous and transversely isotropic elastic half-spaces.

From Figures 2–3, the graphs of B_1/cp, T_1/cp, B_1'/cp and T_1'/cp show that the reflection and transmission coefficients at a corrugated interface are highly influenced by the anisotropies of the inhomogeneous media. These coefficients are most affected by the anisotropy at an angle of incidence of the SH-waves of $\gamma = 0$ degrees. As the angle of incidence approaches 90 degrees the effect decreases. This means that the reflection and transmission coefficients are almost unaffected near the grazing incidence. Since all the coefficients are zero at $\gamma = 90$ degrees, no reflection and transmission take place at the corrugated boundary. This shows that at this particular angle of incidence, the incident wave behaves in a similar fashion to the grazing incidence of elastic waves at the plane interface.



(a)



(b)

FIGURE 5. Reflection variation with angle of incidence with heterogeneity factors (a) $1/b_1 = 2.0$, $1/b_2 = 3.4$ and (b) $1/b_1 = 0.0$, $1/b_2 = 0.0$.

It can be noticed from Figures 4 (a)–(b) that an increase in the inhomogeneity factor results in an enhancement of the reflection coefficients, while Figures 5 (a)–(b) show that an increase in inhomogeneity results in a lowering of the transmission coefficients at the corrugated boundary of the anisotropic media. This is most dominant near normal incidence, that is, near $\gamma = 0$ degrees.

In the present paper, the effect of anisotropy and heterogeneity of the media on the reflection and refraction of SH-waves at a corrugated interface was considered using Rayleigh's method. The solutions were obtained for SH-waves with an arbitrary angle of incidence on a corrugated periodic boundary surface. The computations were carried out for a wave obliquely incident on the mean surface of a harmonic boundary $\zeta = c \cos px$. An important problem to be considered is that concerning the reflection and refraction of waves at an arbitrary form of boundary surface, particularly at troughs of large amplitude. If the results obtained above for various wavelengths of corrugation are superposed, one can obtain in principle the solutions to an arbitrary form of boundary surface, although the assumptions of small amplitude and periodicity concerning the corrugated boundary surface cannot be dispensed with. To do this may involve a cumbersome procedure and other methods may be applied more easily. It would be very interesting to solve the same problem by other methods and compare the results obtained.

Appendix A.

In this appendix, we present the boundary conditions (3.1) and (3.2) and the solutions for the first-order approximation of the corrugation.

Let us substitute

$$q = iQ, \quad r = iR, \quad q_n = iQ_n, \quad r_n = iR_n, \quad q'_n = iQ'_n, \quad r'_n = iR'_n,$$

where

$$Q^2 = \frac{\omega^2}{\beta_{h_1}^2}\frac{N_1}{M_1}\cos^2\gamma - \frac{1}{b_1^2}, \qquad R^2 = \frac{\omega^2}{\beta_{h_1}^2}\frac{N_2}{M_2}\left(\frac{\beta_{h_1}^2}{\beta_{h_2}^2} - \sin^2\gamma\right) - \frac{1}{b_2^2},$$

$$Q_n^2 = \frac{\omega^2}{\beta_{h_1}^2}\frac{N_1}{M_1}\cos^2\gamma_n - \frac{1}{b_1^2}, \qquad R_n^2 = \frac{\omega^2}{\beta_{h_1}^2}\frac{N_2}{M_2}\left(\frac{\beta_{h_1}^2}{\beta_{h_2}^2} - \sin^2\gamma_n\right) - \frac{1}{b_2^2},$$

$$Q_n'^2 = \frac{\omega^2}{\beta_{h_1}^2}\frac{N_1}{M_1}\cos^2\gamma'_n - \frac{1}{b_1^2}, \qquad R_n'^2 = \frac{\omega^2}{\beta_{h_1}^2}\frac{N_2}{M_2}\left(\frac{\beta_{h_1}^2}{\beta_{h_2}^2} - \sin^2\gamma'_n\right) - \frac{1}{b_2^2}$$

into (2.10) and (2.11). Inserting these values of V_1 and V_2 into (3.1) and (3.2), we obtain

$$\begin{aligned}
&\left[e^{-iQ\zeta} + B_0 e^{iQ\zeta} + \sum_n \left(B_n e^{iQ_n\zeta} e^{-inpx} + B'_n e^{iQ'_n\zeta} e^{inpx}\right)\right] \\
&\quad = \sqrt{\frac{M_0}{M'_0}}\,\frac{\cosh(\zeta/b_1)}{\cosh(\zeta/b_2)}\left[D_0 e^{-iR\zeta} + \sum_n \left(D_n e^{-iR_n\zeta} e^{-inpx} + D'_n e^{-iR'_n\zeta} e^{inpx}\right)\right]
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&\left[\left(\frac{\omega\zeta'\sin\gamma}{\beta_{h_1}} - Q\right)e^{-iQ\zeta} + \left(\frac{\omega\zeta'\sin\gamma}{\beta_{h_1}} + Q\right)B_0 e^{iQ\zeta}\right. \\
&\quad + \sum_n B_n\left\{Q_n + \left(\frac{\omega\sin\gamma}{\beta_{h_1}} + np\right)\zeta'\right\} e^{iQ_n\zeta} e^{-inpx} \\
&\quad \left. + \sum_n B'_n\left\{Q'_n + \left(\frac{\omega\sin\gamma}{\beta_{h_1}} - np\right)\zeta'\right\} e^{iQ'_n\zeta} e^{inpx}\right]\cosh(\zeta/b_1) \\
&\quad + \frac{1}{b_1}\left[e^{-iQ\zeta} + B_0 e^{iQ\zeta} + \sum_n \left(B_n e^{iQ_n\zeta} e^{-inpx} + B'_n e^{iQ'_n\zeta} e^{inpx}\right)\right][-\sinh(\zeta/b_1)] \\
&= \sqrt{\frac{M_0}{M'_0}}\left[\left(\frac{\omega\zeta'\sin\gamma}{\beta_{h_1}} - R\right)D_0 e^{-iR\zeta}\right. \\
&\quad + \sum_n D_n\left\{\left(\frac{\omega\sin\gamma}{\beta_{h_1}} + np\right)\zeta' - R_n\right\} e^{-iR_n\zeta} e^{-inpx} \\
&\quad \left. + \sum_n D'_n\left\{\left(\frac{\omega\sin\gamma}{\beta_{h_1}} - np\right)\zeta' - R'_n\right\} e^{-iR'_n\zeta} e^{inpx}\right]\cosh(\zeta/b_2) \\
&\quad + \frac{1}{b_2}\left[D_0 e^{-iR\zeta} + \sum_n \left(D_n e^{-iR_n\zeta} e^{-inpx} + D'_n e^{-iR'_n\zeta} e^{inpx}\right)\right][-\sinh(\zeta/b_2)].
\end{aligned} \tag{A.2}$$

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In order to obtain the solution of the first-order approximation we assume that the amplitude of corrugation, ζ, is very small so that the terms of an order higher than ζ are neglected. Therefore we have

$$e^{-iQ\zeta} = 1 - iQ\zeta. \tag{A.3}$$

The solution of B_0 and D_0 can be obtained by collecting the terms independent of x and ζ in (A.1) and (A.2),

$$1 + B_0 = D_0\sqrt{\frac{M_0}{M_0'}}, \tag{A.4}$$

$$Q[B_0 - 1] = -RD_0\sqrt{\frac{M_0'}{M_0}}. \tag{A.5}$$

These formulae give the amplitudes of reflected and refracted waves at a plane boundary between the two media in question.

If we collect together the coefficients of e^{-inpx} in (A.1) and (A.2), then we obtain the formulae determining the first approximation of B_n and D_n as

$$B_n - D_n\sqrt{\frac{M_0}{M_0'}} = i(1 - B_0)Q\zeta_{-n} - iD_0R\zeta_{-n}\sqrt{\frac{M_0}{M_0'}}, \tag{A.6}$$

$$\begin{aligned} B_nQ_n + \sqrt{\frac{M_0}{M_0'}}D_nR_n = {} & \left[i\left(\frac{np\omega\sin\gamma}{\beta_{h_1}} - Q^2\right) + \frac{1}{b_1^2}\right]\zeta_{-n}(1 + B_0) \\ & - \left[i\left(\frac{np\omega\sin\gamma}{\beta_{h_1}} - R^2\right) + \frac{1}{b_2^2}\right]\zeta_{-n}D_0\sqrt{\frac{M_0'}{M_0}}. \end{aligned} \tag{A.7}$$

Similarly the formulae concerning the first approximation of B_n' and D_n' can be obtained by collecting together the coefficients of e^{inpx} in (A.1) and (A.2):

$$B_n' - D_n'\sqrt{\frac{M_0}{M_0'}} = i(1 - B_0)Q\zeta_{-n} - iD_0R\zeta_n\sqrt{\frac{M_0}{M_0'}}, \tag{A.8}$$

$$\begin{aligned} B_n'Q_n' + D_n'R_n'\sqrt{\frac{M_0}{M_0'}} = {} & \left[-i\left(np\frac{\omega\sin\gamma}{\beta_{h_1}} + Q^2\right) + \frac{1}{b_1^2}\right]\zeta_n(1 + B_0) \\ & + \left[i\left(np\frac{\omega\sin\gamma}{\beta_{h_1}} + R^2\right) - \frac{1}{b_2^2}\right]\zeta_nD_0\sqrt{\frac{M_0'}{M_0}} \end{aligned} \tag{A.9}$$

where we have made use of the relations given in (2.9). The coefficients B_0 and D_0 involved in the above relations are given by (A.4) and (A.5).

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Appendix B.

We here give the formulae for the quantities d_1, d_2, d_1' and d_2' that appear in (3.5):

$$d_1 = \frac{ic}{2}\left[QR_1(1-B_0) + \frac{M_0}{M_0'}(1+B_0)\left(\frac{p\omega\sin\gamma}{\beta_{h_1}} - Q^2\right)\right.$$
$$\left. -D_0\sqrt{\frac{M_0}{M_0'}}\left(\frac{p\omega\sin\gamma}{\beta_{h_1}} - R^2 + RR_1\right)\right] + \frac{c(1+B_0)}{2b_1^2}\frac{M_0}{M_0'} - \frac{cD_0}{2b_2^2}\sqrt{\frac{M_0}{M_0'}},$$
$$d_2 = \frac{ic}{2}\left[\sqrt{\frac{M_0}{M_0'}}(1+B_0)\left(\frac{p\omega\sin\gamma}{\beta_{h_1}} - Q^2\right) - D_0\left(\frac{p\omega\sin\gamma}{\beta_{h_1}} - R^2\right)\right.$$
$$\left. +RQ_1D_0\frac{M_0}{M_0'} - QQ_1(1-B_0)\sqrt{\frac{M_0}{M_0'}}\right] - \frac{cD_0}{2b_2^2} + \frac{c(1+B_0)}{2b_1^2}\sqrt{\frac{M_0}{M_0'}},$$
$$d_1' = \frac{ic}{2}\left[QR_1'(1-B_0) - \frac{M_0}{M_0'}(1+B_0)\left(\frac{p\omega\sin\gamma}{\beta_{h_1}} + Q^2\right)\right.$$
$$\left. +D_0\sqrt{\frac{M_0}{M_0'}}\left(\frac{p\omega\sin\gamma}{\beta_{h_1}} + R^2 - RR_1\right) + \frac{c(1+B_0)}{2b_1^2}\frac{M_0}{M_0'} - \frac{cD_0}{2b_2^2}\sqrt{\frac{M_0}{M_0'}}\right],$$
$$d_2' = \frac{ic}{2}\left[-QQ_1'(1-B_0) - (1+B_0)\left(\frac{p\omega\sin\gamma}{\beta_{h_1}} + Q^2\right)\sqrt{\frac{M_0}{M_0'}}\right.$$
$$\left. +RQ_1D_0\frac{M_0}{M_0'} + D_0\left(\frac{p\omega\sin\gamma}{\beta_{h_1}} + R^2\right)\right] - \frac{cD_0}{2b_2^2} + \frac{c(1+B_0)}{2b_1^2}\sqrt{\frac{M_0}{M_0'}},$$

where $d = (M_0/M_0')Q_1 + R_1$ and $d' = (M_0/M_0')Q_1' + R_1'$.

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