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Reducibility of the Principal Series for $\widetilde{\mathrm{Sp}_2}(F)$ over a p-adic Field

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Abstract. Let $G_n = \mathrm{Sp}_n(F)$ be the rank n symplectic group with entries in a nondyadic p-adic field F. We further let $\widetilde{G}_n$ be the metaplectic extension of G_n by $\mathbb{C}^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$ defined using the Leray cocycle. In this paper, we aim to demonstrate the complete list of reducibility points of the genuine principal series of $\widetilde{G}_2$. In most cases, we will use some techniques developed by Tadić that analyze the Jacquet modules with respect to all of the parabolics containing a fixed Borel. The exceptional cases involve representations induced from unitary characters χ with $\chi^2 = 1$. Because such representations π are unitary, to show the irreducibility of π, it suffices to show that $\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}}(\pi, \pi) = 1$. We will accomplish this by examining the poles of certain intertwining operators associated to simple roots. Then some results of Shahidi and Ban decompose arbitrary intertwining operators into a composition of operators corresponding to the simple roots of $\widetilde{G}_2$. We will then be able to show that all such operators have poles at principal series representations induced from quadratic characters and therefore such operators do not extend to operators in $\mathrm{Hom}_{\widetilde{G}_2}(\pi, \pi)$ for the π in question.

1 Introduction

Let $G_n = \mathrm{Sp}_n(F)$ be the split rank n symplectic group over a p-adic field F with $p \neq 2$. Further, let $\widetilde{G}_n$ be the metaplectic extension of G_n by the group $\mathbb{C}^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$ defined using the Leray cocycle (see [10, 11]). Moreover, we give $\mathbb{C}^1$ the metric topology. We choose this particular cover $\widetilde{G}_n$ because it offers us some splittings that allow us to define the genuine principal series of $\widetilde{G}_n$ in a way that is completely analogous to the principal series of the linear group G_n. For $\widetilde{G}_2$, we will denote such representations as

$$\mathrm{Ind}_{\widetilde{P}_\varnothing}^{\widetilde{G}_2}(\chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0) = \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0,$$

where χ_i are unitary characters of $F^\times$, τ_0 is the identity character of $\mathbb{C}^1$, $\nu(x) = |x|$ is the absolute value character, and $s, t \in \mathbb{R}$. We will use the notation $\xi_a(x) = (x, a)_F$ for some $a \in F^\times$ where $(\cdot\,,\,\cdot)_F$ is the Hilbert symbol for F. We use $\widetilde{P}_\varnothing$ to denote the full inverse image of a fixed Borel subgroup $P_\varnothing$. As we will show below, for a proper choice of Borel,

$$\widetilde{P}_\varnothing = \widetilde{M}_\varnothing \mathbf{N}_\varnothing,$$

where $\widetilde{M}_\varnothing \simeq (F^\times)^2 \times \mathbb{C}^1$ as groups. Ultimately, we aim to prove the following theorem.

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Theorem 1.1 *Let $\pi = \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ be a genuine principal series representation of $\widetilde{G}_2$. Then π is irreducible unless one of the following hold.*

(i) *π is a Weyl conjugate of $\chi\nu^{s+\frac{1}{2}} \times \chi\nu^{s-\frac{1}{2}} \rtimes \tau_0$ where one of the following holds.*

(a) *$\chi^2 \neq 1$ and s arbitrary.*

(b) *$\chi^2 = 1$ and $s \notin \{0, \pm 1\}$.*

This representation has two irreducible constituents,

$$\chi\nu^s St_{\mathrm{GL}_2} \rtimes \tau_0 \quad \textit{and} \quad \chi\nu^s \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0.$$

(ii) *π is a Weyl conjugate of $\chi\nu^s \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0$ where one of the following holds.*

(a) *$\chi^2 \neq 1$ and s arbitrary.*

(b) *$\chi = \xi_b$ with $ab^{-1} \notin (F^\times)^2$ and $s \notin \{\pm\frac{1}{2}\}$.*

(c) *$\chi = \xi_a$ and $s \notin \{\pm\frac{1}{2}, \pm\frac{3}{2}\}$.*

This representation has two irreducible constituents,

$$\chi\nu^s \rtimes \omega_a^+ \quad \textit{and} \quad \chi\nu^s \rtimes sp_a,$$

where ω_a^+ and sp_a are the irreducible constituents of the genuine principal series of $\widetilde{G}_1$ induced from the character $\xi_a\nu^{\frac{1}{2}} \otimes \tau_0$.

(iii) *π is a Weyl conjugate of $\xi_a\nu^{\frac{3}{2}} \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0$. This representation has four irreducible constituents,*

$$sp_{a,2}, \quad Q(\xi_a\nu St_{\mathrm{GL}_2}, \tau_0), \quad Q(\xi_a\nu^{\frac{3}{2}}, sp_a), \quad \textit{and} \quad \omega_{a,2}^+.$$

(iv) *π is a Weyl conjugate of $\xi_a\nu^{\frac{1}{2}} \times \xi_a\nu^{-\frac{1}{2}} \rtimes \tau_0$. This representation has four irreducible constituents,*

$$T_1(\xi_a St_{\mathrm{GL}_2}, \tau_0), \quad T_2(\xi_a St_{\mathrm{GL}_2}, \tau_0), \quad Q(\xi_a\nu^{\frac{1}{2}}, sp_a), \quad \textit{and} \quad Q(\xi_a\nu^{\frac{1}{2}}, \xi_a\nu^{-\frac{1}{2}}).$$

(v) *π is a Weyl conjugate of $\xi_a\nu^{\frac{1}{2}} \times \xi_b\nu^{\frac{1}{2}} \rtimes \tau_0$ with $ab^{-1} \in F^\times \setminus (F^\times)^2$. This representation has four irreducible constituents,*

$$T(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0), \quad Q(\xi_a\nu^{\frac{1}{2}}, sp_b), \quad Q(\xi_b\nu^{\frac{1}{2}}, sp_a), \quad \textit{and} \quad Q(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0).$$

Notice that $\chi_1 \times \chi_2 \rtimes \tau_0$ is irreducible for all χ_i unitary.

This paper is divided into several sections but contains three main arguments. In the first three sections, we will explain our notation and explore the properties of our realization of $\widetilde{G}_2$. In particular, our construction offers us a splitting of a Siegel parabolic subgroup as well as the maximal compact subgroup $\mathrm{Sp}_2(\mathcal{O})$. We will show that we can define parabolic subgroups of $\widetilde{G}_n$ in a completely analogous way to the linear group G_n. Moreover, all of our theory pertaining to parabolic induction and Jacquet modules for the linear group translate to the covering group in a natural way.

The next two sections are devoted to proving reducibility and irreducibility using a technique of Tadić [16, 17]. This technique computes Jacquet modules with respect to all of the standard parabolics for the various constituents of the principal series. In particular, we prove the irreducibility of certain constituents by demonstrating that their Jacquet modules exhaust all available constituents. This technique will cover all the cases except for

$$\chi_1, \chi_2 \in \{\xi_a \mid a \in F^\times\} \quad \text{and} \quad s = t = 0.$$

These cases comprise the last section of the paper and require a different approach. In that section, we aim to show that for π of this form,

$$\mathrm{Hom}_{\widetilde{G}_2}(\pi, \pi) = \mathbb{C} \cdot \mathrm{id}_\pi .$$

Because the representations are induced from unitary characters (and are thus completely reducible), this suffices to show that π is irreducible. Our technique involves computing the poles and zeros for the standard intertwining operators associated with Weyl group elements corresponding to simple roots. We then use some results of Shahidi to factor arbitrary intertwining operators into the ones we computed. Then we establish that these operators have poles at all of the appropriate values to establish that none of these nontrivial intertwining operators can extend to $\mathrm{Hom}_{\widetilde{G}_2}(\pi, \pi)$ for $\pi = \xi_a \times \xi_b \rtimes \tau_0$.

2 The Symplectic Group

Let F be an arbitrary finite extension of $\mathbb{Q}_p$ for $p \neq 2$. This restriction is due to some technical reasons that we will address as they become relevant. We also let $\mathcal{O}$ be the ring of integers of F, $\mathcal{P}$ be its maximal ideal and ϖ be a uniformizing element. Moreover, let $\mathbb{F}_q$ be the residue field with $|\mathbb{F}_q| = q = p^k$ for some $k \in \mathbb{Z}$. We also fix an additive character ψ that is trivial on $\mathcal{O}$ but nontrivial on $\mathcal{P}^{-1}$. Finally, we define the group G_n to be the rank n symplectic group $\mathrm{Sp}_n(F)$ defined as

$$\mathrm{Sp}_n(F) = \left\{ g \in \mathrm{GL}_{2n}(F) \mid {}^t g \left(\begin{smallmatrix} & I_n \\ -I_n & \end{smallmatrix}\right) g = \left(\begin{smallmatrix} & I_n \\ -I_n & \end{smallmatrix}\right) \right\},$$

where I_n is the $n \times n$ identity matrix. It is worth noting that in this notation, $G_1 \simeq \mathrm{SL}_2(F)$. The paper is primarily concerned with the case where $n = 2$, but will occasionally need results pertaining to the case where $n = 1$.

Let T_n be the diagonal torus and let B_n be the Borel subgroup of G_n with Levi factor T_n and having the unipotent radical of the form

$$\begin{pmatrix} A & \\ & {}^t A^{-1} \end{pmatrix} \begin{pmatrix} I & X \\ & I \end{pmatrix},$$

where A is an upper triangular unipotent matrix in $\mathrm{GL}_n(F)$ and $X \in \mathrm{Sym}_n(F)$. Let Δ_n be the set of simple roots corresponding to our choice of Borel and torus. If $n = 2$, $\Delta_2 = \{\alpha, \beta\}$ where β denotes the longer root. Finally, we set $W_{G_n} = N_{G_n}(T)/T$ to be the Weyl group of G_n. For $n = 2$, it is generated by two elements w_α and w_β corresponding to the simple roots. We fix a section $W_{G_2} \to G_2$ by

$$w_\alpha \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad w_\beta \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

on the generators. For an arbitrary $w \in W_{G_2}$, the section is defined by taking a reduced expression for w in W_{G_2} in terms of the simple roots and taking the corresponding product of the images of the simple roots. Notice that since we have the braid relation $(w_\alpha w_\beta)^2 = (w_\beta w_\alpha)^2$, a simple counting argument shows that we have at most 9 minimal length words in W_{G_2}. However, $|W_{G_2}| = 8$, so that all but one of our elements in W_{G_2} has a unique minimal length expression. In particular, the long Weyl group element w_2 is expressible as

$$w_2 = w_\alpha w_\beta w_\alpha w_\beta = w_\beta w_\alpha w_\beta w_\alpha,$$

and a routine calculation shows that the image of w_2 via our given section is independent of this factorization.

Finally, we will define the various standard parabolics of G_n. In particular, for each $\Omega \subset \Delta_n$, there is a standard method for constructing a parabolic P_Ω containing our fixed Borel B_n (see Springer [14]). Notice that for any n, $B_n = P_\varnothing$ for $\varnothing \subset \Delta_n$. For $n = 2$, we get the following parabolic subgroups:

- Borel $P_\varnothing = B = M_\varnothing N_\varnothing$ where $M_\varnothing = T \simeq (F^\times)^2$;
- Siegel parabolic $P_\alpha = M_\alpha N_\alpha$ where $M_\alpha \simeq \mathrm{GL}_2(F)$ and $N_\alpha \simeq \mathrm{Sym}_2(F)$;
- Klingen parabolic $P_\beta = M_\beta N_\beta$ where $M_\beta \simeq F^\times \times \mathrm{Sp}_1(F)$ and N_β is a Heisenberg group on a two-dimensional symplectic space (see [9, Chapter I]).

Because it relates to a construction we will perform on the metaplectic group, it is worth noting that an arbitrary parabolic $P \subset G_n$ has a Levi decomposition $P = MN$ such that

$$M \simeq \mathrm{GL}_{r_1}(F) \times \mathrm{GL}_{r_2}(F) \times \cdots \times \mathrm{GL}_{r_m}(F) \times \mathrm{Sp}_{n'}(F),$$

where $n' = n - \sum_{i=1}^m r_i$. If $r_1 = n$, we call this parabolic the Siegel parabolic. Finally, because it will be useful for the next few sections, we choose to fix the following notation. Let

$$m\colon\ \mathrm{GL}_{r_1}(F) \times \mathrm{GL}_{r_2}(F) \times \cdots \times \mathrm{GL}_{r_m}(F) \times \mathrm{Sp}_{n'}(F) \to M$$

be the map that identifies the product of reductive groups with the Levi factor of $P = MN$. Notice that we do not differentiate the m attached to different parabolics, but rather allow the reader to determine the proper m according to context.

3 The Metaplectic Group

While the metaplectic cover of $\mathrm{Sp}_n(F)$ is well studied, we employ a less traditional realization of this group. We will denote this group by $\widetilde{G}_n$. For our purposes, $\widetilde{G}_n$ satisfies an exact sequence

$$1 \to \mathbb{C}^1 \to \widetilde{G}_n \to G_n \to 1.$$

Furthermore, there exists a section $G_n \to \widetilde{G}_n$ so that $\widetilde{G}_n = G_n \times \mathbb{C}^1$ as a set with multiplication

$$[g_1, z_1] \cdot [g_2, z_2] = [g_1 g_2, z_1 z_2 c_L(g_1, g_2)]$$

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where $c_L(\cdot\,,\,\cdot)$ denotes the Leray cocycle. This cocycle is defined as follows (see [11]).

Let W be the symplectic vector space on which G_n acts by right multiplication. There exists a complete polarization $W = X+Y$ where X and Y are maximal isotropic subspaces and Y is such that $\mathrm{Stab}_{G_n}(Y)$ is our block upper triangular, Siegel, parabolic subgroup. Then we define the Leray cocycle as

$$c_L(g_1, g_2) := \gamma(\psi \circ \frac{1}{2} q_L(Yg_1, Y, Yg_2^{-1})),$$

where γ denotes the Weil index of a character of second degree and $q_L(Yg_1, Y, Yg_2^{-1})$ is a quadratic form called the Leray invariant associated with the triple of maximal isotropic subspaces (Yg_1, Y, Yg_2^{-1}). For more details, one can consult [11, Chapters 2 and 4] for definitions and explicit formulas.

This cocycle differs from the more utilized Rao cocycle (see [11, Chapter 5]) that is valued in $\{\pm 1\}$ and is used to define the double cover $\widetilde{G}_n^{(2)}$. In particular, the Leray cocycle is valued in the eighth roots of unity μ_8. It is shown in [10, 11] that the two cocycles are equivalent and thus give rise to isomorphic covers when extending G_n by $\mathbb{C}^1$. In order to avoid confusion regarding cocycles, we will use the notation $[g, z]_L$ to denote elements of $\widetilde{G}_n$ when using the Leray cocycle and $[g, z]_R$ to denote elements of $\widetilde{G}_n$ when using the Rao cocycle. For the purposes of this paper, we will almost exclusively use the Leray cocycle. It is worth noting that the two cocycles are related as follows (see [9, Section I.4]):

$$[g, z]_R = [g, z\varsigma(g)]_L,$$

where $\varsigma(g)$ is a coboundary defined using Weil indices and some other quantities defined in [11]. Explicit formulas for $\varsigma(g)$ can also be found in [9, Chapter 1, Theorem 4.5].

The utility of using the larger cover is the existence of various splittings; the utility of using the different cocycle is that these splittings will be much less complicated to express.

3.1 Splittings of Subgroups of G_n

Let $P_n \subset G_n$ be the Siegel parabolic subgroup $P_n = \mathrm{Stab}_{G_n}(Y)$. Moreover, let $P_\varnothing$ be the Borel subgroup defined in Section 2. Notice that $P_\varnothing \subset P_n$. A simple calculation (see [11, Theorem 4.1]) shows that for any $p \in P_n$ and $g \in G_n$

$$c_L(p, g) = c_L(g, p) = 1. \tag{3.1}$$

So we can define a splitting $P_n \to \widetilde{G}_n$ by $p \mapsto [p, 1]_L$. Moreover, if we let $\widetilde{P}_n$ be the full inverse image of P_n in $\widetilde{G}_n$, then $\widetilde{P}_n \simeq P_n \times \mathbb{C}^1$ as groups. In fact, (3.1) allows us to define the standard parabolic subgroups of $\widetilde{G}_n$ in a very natural way. Now let P be any parabolic subgroup of G_n with $P_\varnothing \subset P$. As in the last section, we see that $P = MN$ with

$$M \simeq \mathrm{GL}_{r_1}(F) \times \mathrm{GL}_{r_2}(F) \times \cdots \times \mathrm{GL}_{r_m}(F) \times \mathrm{Sp}_{n'}(F),$$

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where $n' = n - \sum_{i=1}^{m} r_i$. Let $A_1, A_2 \subset M$ be such that

$$A_1 \simeq \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \times \cdots \times \mathrm{GL}_{r_m} \times \{I_{n'}\} \text{ and}$$
$$A_2 \simeq \{I_{n-n'}\} \times \mathrm{Sp}_{n'}(F)$$

so that $M = A_1 \times A_2$. Notice that $A_1 \subset P_n$, so for all $a \in A_1$ and $g \in G_n$,

$$c_L(a, g) = c_L(g, a) = 1.$$

Moreover, we can restrict the splitting on P_n to A_1 to yield the subgroup

$$\mathbf{A}_1 = \{[a, 1]_L \mid a \in A_1\} \subset \widetilde{G}_n.$$

Now consider the full inverse image of A_2 in $\widetilde{G}_n$, which we denote $\widetilde{A}_2$. We would like to establish the following lemma.

Lemma 3.1 *Let $\widetilde{A}_2$ be the full inverse image of A_2 in $\widetilde{G}_n$. Then $\widetilde{A}_2 \simeq \widetilde{G}_{n'}$, where $\widetilde{G}_{n'}$ is defined using the Leray cocycle for on* $\mathrm{Sp}_{n'}(F)$.

Proof The candidate for this homomorphism is

$$\widetilde{A}_2 \to \widetilde{G}_{n'} \quad [(I_{n-n'}, g'), z]_L \mapsto [g', z]_L.$$

We need only verify that the cocycle on $\widetilde{A}_2$ is the same as the Leray cocycle on $\widetilde{G}_{n'}$. To do this, let $W' \subset W$ be the symplectic vector space of dimension $2n'$ such that $\mathrm{Sp}(W')$ is identified with A_2 via the isomorphism that we used to identify $\mathrm{Sp}(W)$ with $\mathrm{Sp}_n(F)$. Further, we define $Y' = Y \cap W$. We note that

$$\mathrm{Stab}_{A_2}(Y') = A_2 \cap P_n = \{I_{n-n'}\} \times P_{n'}.$$

So the maximal isotropic subspace $Y' \subset W'$ is stabilized by the upper triangular Siegel parabolic in A_2.

Next, we endeavor to compute the cocycle on $\widetilde{A}_2$. For $i \in \{1, 2\}$, let $g_i \in A_2 \subset \mathrm{Sp}_n(F)$. Then there exists $g_i' \in \mathrm{Sp}_{n'}(F)$ so that $g_i = (1_{n-n'}, g_i')$. Now if we follow Rao's formula in [11, Section 2.4] that computes the Leray invariant, we see that

$$q_L(Yg_1, Y, Yg_2^{-1}) = q_L(Y'g_1', Y', Y'(g_2')^{-1}).$$

Since the Leray cocycle is simply the Weil index of the Leray invariant, we see that the restriction of the Leray cocycle on $\widetilde{G}_n$ to A_2 (given by the Weil index of $q_L(Yg_1, Y, Yg_2^{-1})$) is precisely equal to the Leray cocycle on $\widetilde{G}_{n'}$ (given by the Weil index of $q_L(Y'g_1', Y', Y'(g_2')^{-1})$). ∎

So finally, let us consider any element $[p, z]_L \in \widetilde{P}$. First notice that

$$[p, z]_L = [mn, z]_L = [m, z]_L \cdot [n, 1]_L$$

since $n \in P_n$. If we define $\mathbf{N} = \{[n, 1]_L \mid n \in N\} \subset \widetilde{G}_n$, then $\widetilde{P} = \widetilde{M}\mathbf{N}$. Moreover, we see that

$$[m, z]_L = [a_1 a_2, z]_L = [a_1, 1]_L \cdot [a_2, z]_L$$

so that $\widetilde{M} = \mathbf{A}_1 \widetilde{A}_2$. Furthermore, a routine verification shows that

$$\mathbf{A}_1 \cap \widetilde{A}_2 = \{1_{\widetilde{G}_n}\} \text{ and } [a_1, 1]_L \cdot [a_2, z]_L = [a_2, z]_L \cdot [a_1, 1]_L$$

for all $a_1 \in A_1$, $a_2 \in A_2$ and $z \in \mathbb{C}^1$. Therefore,

$$\widetilde{M} = \mathbf{A}_1 \times \widetilde{A}_2 \simeq \mathrm{GL}_{r_1}(F) \times \mathrm{GL}_{r_2}(F) \times \cdots \times \mathrm{GL}_{r_m}(F) \times \widetilde{\mathrm{Sp}_{n'}}(F)$$

So we will define a standard parabolic subgroup $\widetilde{P}$ of $\widetilde{G}_n$ to be the full inverse image of the parabolic subgroup $P \subset G_n$ in $\widetilde{G}_n$ where $P_\varnothing \subset P$.

There are also splittings of certain compact open subgroups of G_n to $\widetilde{G}_n$. If F is an extension of $\mathbb{Q}_p$ with $p \neq 2$, let $K = \mathrm{Sp}_n(\mathcal{O})$. Then there exists a splitting

$$K \to \widetilde{G}_n \qquad k \mapsto [k, \lambda(k)]_L.$$

The map $\lambda\colon K \to \mathbb{C}^1$ is defined in [10, Section 8.4.1].

Remark 3.2 If $F = \mathbb{Q}_2$, there is no splitting of the maximal compact subgroup. Instead there is a splitting of the subgroup $I_\alpha(4)$, the set of matrices in K that reduce to the Siegel parabolic mod 4. In fact, when $n = 2$, $I_\alpha(4)$ is of index two in a larger compact open subgroup for which a splitting exists. So we get an analogous map

$$I_\alpha(4) \to \widetilde{G}_2 \quad k \mapsto [k, \lambda(k)]_L.$$

This λ is also defined in [10, Section 8.4.5] and is essentially the generalization of the λ found in the nondyadic case to the case where $F = \mathbb{Q}_2$.

Finally, it is worth noting that we have a couple of necessary decompositions of the group $\widetilde{G}_n$. For general $\widetilde{G}_n$ we have a *Bruhat decomposition* of $\widetilde{G}_n$ that we define using the Bruhat decomposition for $\mathrm{Sp}_n(F)$. Furthermore, in the case where $p \neq 2$, we have an *Iwasawa decomposition of* $\widetilde{G}_n$ that is defined using the Iwasawa decomposition on $\mathrm{Sp}_n(F)$. In particular, we have the the following lemma.

Lemma 3.3 (Bruhat and Iwasawa Decomposition for $\widetilde{G}_n$) *Let F be a finite extension of $\mathbb{Q}_p$ and let $\widetilde{P}$, $\widetilde{P}'$ be standard parabolics of $\widetilde{G}_n$. Then*

$$\widetilde{G}_n = \bigcup_{[W_M \backslash W_{G_n} / W_{M'}]} \widetilde{P}\mathbf{w}\widetilde{P}'$$

where $W_M \subset W_{G_n}$ is the Weyl group of the Levi factor M of $P = MN$. Furthermore, if F is nondyadic, let $\widetilde{P}$ be a standard parabolic subgroup of $\widetilde{G}_n$ and let $\mathbf{K}$ *denote image of K under the splitting $k \mapsto [k, \lambda(k)]_L$. Then*

$$\widetilde{G}_n = \widetilde{P}\mathbf{K}. \tag{3.2}$$

Proof Both of these statements follow from factoring the $g \in \mathrm{Sp}_n$ component of $[g, z]_L$ in $\mathrm{Sp}_n(F)$ and then adapting that to the cover using our splitting map and equation (3.1). For $g \in \mathrm{Sp}_n(F)$, the Iwasawa decomposition of $\mathrm{Sp}_n(F)$ tells us that $g = pwp'$ for some $p \in P$ and $p' \in P'$. Thus for $[g, z]_L \in \widetilde{G}_n$, we have

$$[g, z]_L = [pwp', z]_L = [p, zc_L(p, wp')]_L[w, 1]_L[p', c_L(w, p')^{-1}]_L$$
$$= [p, z]_L[w, 1]_L[p', 1]_L = [p, 1]_L[w, 1]_L[p', z]_L.$$

This demonstrates the Bruhat decomposition. Next, let $[g, z]_L \in \widetilde{G}_n$ and F is non-dyadic. Then we have a the splitting of K to $\widetilde{G}_n$ defined before Remark 3.2. Since $g \in \mathrm{Sp}_n(F)$, for a parabolic $P \subset \mathrm{Sp}_n(F)$ there exists $p \in P$ and $k \in K$ such that $g = pk$. So we see that

$$[g, z]_L = [pk, z]_L = [p, zc_L(p, k)^{-1}\lambda(k)^{-1}]_L[k, \lambda(k)]_L = [p, z\lambda(k)^{-1}]_L[k, \lambda(k)]. \quad \blacksquare$$

Now that we have discussed the various splittings, let us codify the following notation that we have already had occasion to employ. When discussing a subgroup or group element that is subject to a splitting, we will use boldface notation to denote the image of the splitting. For instance $\mathbf{K}$ denotes the image of K under the splitting map, and the image of $k \in K$ is the element $\mathbf{k} = [k, \lambda(k)]_L$. When considering the full inverse image of a subgroup, we will use the tilde notation. So, as we have seen previously, $\widetilde{P}$ is the full inverse image of a parabolic subgroup $P \subset G_n$ in $\widetilde{G}_n$.

4 The Genuine Principal Series of $\widetilde{G}_n$

It is worth mentioning that $\widetilde{G}_n$ is not an l-group in the sense of Bernstein and Zelevinsky (see [3, 4]). In particular, not every neighborhood of the identity contains a compact-open subgroup since $\mathbb{C}^1$ does not have any nontrivial compact open subgroups when given the usual metric topology. However, $\widetilde{G}_n$ is closely related to a group for which the Bernstein–Zelevinsky results apply. In particular, since our Leray cocycle is valued in $\mu_8 \in \mathbb{C}^1$ (the eighth roots of unity), we can form the eight-fold cover $\widetilde{G}_n^{(8)}$. So the image of all of our splittings in the previous section are contained in $\widetilde{G}_n^{(8)}$. Moreover, we give $\widetilde{G}_n^{(8)}$ and $\widetilde{G}_n$ the following topologies.

For the group G_n, we have a neighborhood basis of the identity $\{K_m\}_{m\in\mathbb{N}}$ where $K_m \subset K$ is the kernel of the map that reduces the entries of K mod $\mathcal{P}^m$ (by convention $K_0 = K$). We then denote $\mathbf{K}_m$ to be the image of K_m under the previously described splitting map for K. We then give $\widetilde{G}_n^{(8)}$ a topology given by $\{\mathbf{K}_m\}_{m\in\mathbb{N}}$ as a neighborhood basis for $[1_{G_n}, 1]_L$. With this topology, our splitting map for K is continuous so that the set $\{\mathbf{K}_m\}_{m\in\mathbb{N}}$ is actually a neighborhood basis of compact-open subgroups. Thus, $\widetilde{G}_n^{(8)}$ is an l-group in the sense of Bernstein and Zelevinsky.

Now one can see, given the natural inclusions $\mu_8 \hookrightarrow \widetilde{G}_n^{(8)}$ and $\mu_8 \hookrightarrow \mathbb{C}^1$, that $\widetilde{G}_n = \widetilde{G}_n^{(8)} \times_{\mu_8} \mathbb{C}^1$ is the pushout of these two inclusions. If we give μ_8 the discrete topology, then the two inclusions are also continuous maps. Thus the natural topology for $\widetilde{G}_n$ is the topology inherited as the topological pushout of $\widetilde{G}_n^{(8)}$ and $\mathbb{C}^1$. This discussion of topologies mirrors [9, Chapter 1] where Kudla constructs the $\mathbb{C}^\times$ extension of G_n and then realizes it as the pushout of $\mathbb{C}^\times$ and the metaplectic double cover $\widetilde{G}_n^{(2)}$.

More generally, if $H \subset G_n$ is any subgroup and $\widetilde{H}$ (resp. $\widetilde{H}^{(8)}$) is its full inverse image in $\widetilde{G}$ (resp. $\widetilde{H}^{(8)}$), then $\widetilde{H} = \widetilde{H}^{(8)} \times_{\mu_8} \mathbb{C}^1$ is the pushout of the obvious inclusions.

As with $\widetilde{G}_n^{(8)}$, if $H \subset G_n$ is an l-group, then $\widetilde{H}^{(8)}$ is an l-group as well. Thus we have a definition for admissible representation of $\widetilde{H}^{(8)}$ given by [3, 4]. Given these two facts, we have the following definition.

Definition 4.1 (i) Let $H \subset G_n$ be any subgroup and let (π, V_π) be a complex representation of $\widetilde{H}$ (resp. $\widetilde{H}^{(8)}$). We call such a representation *genuine* if

$$\pi([h, z]_L)v = z\pi([h, 1]_L)v$$

for any $v \in V_\pi$, $h \in H$ and $z \in \mathbb{C}^1$ (resp. $z \in \mu_8$).

(ii) Let $H \subset G_n$ be an l-group and let (π, V_π) be a representation of $\widetilde{H}$. We call such a representation *admissible* if its restriction to $\widetilde{H}^{(8)}$ is an admissible representation of $\widetilde{H}^{(8)}$.

Let us use $\mathcal{A}^{gen}(\widetilde{H})$ (resp. $\mathcal{A}^{gen}(\widetilde{H}^{(8)})$) to denote the category of the genuine admissible representations of $\widetilde{H}$ (resp. $\widetilde{H}^{(8)}$) with the usual intertwining maps as morphisms. The main idea that we use in this section is the following.

Proposition 4.2 *There is a natural isomorphism of categories $\mathcal{A}^{gen}(\widetilde{H}) \simeq \mathcal{A}^{gen}(\widetilde{H}^{(8)})$.*

Proof Because we are trying to prove an isomorphism (rather than an equivalence) of categories, we would like to establish some natural transformations between these categories that are inverse to each other. One of these natural transformations is obvious. We let $\mathcal{A}^{gen}(\widetilde{H}) \to \mathcal{A}^{gen}(\widetilde{H}^{(8)})$ be the map that restricts the representation (π, V_π) of $\widetilde{H}$ to the subgroup $\widetilde{H}^{(8)}$. Furthermore, for any representations (π, V_π) and (σ, V_σ), this natural transformation induces the following canonical inclusion

$$\mathrm{Hom}_{\widetilde{H}}(V_\pi, V_\sigma) \subset \mathrm{Hom}_{\widetilde{H}^{(8)}}(V_\pi, V_\sigma).$$

Now let us define the natural transformation $\mathcal{A}^{gen}(\widetilde{H}^{(8)}) \to \mathcal{A}^{gen}(\widetilde{H})$. This map is almost as straightforward as the previous one. Let $(\pi, V_\pi) \in \mathcal{A}^{gen}(\widetilde{H}^{(8)})$, then π is a homomorphism $\pi\colon \widetilde{H}^{(8)} \to \mathrm{GL}(V_\pi)$ by which $\mu_8 \hookrightarrow \widetilde{H}^{(8)}$ operates as a scalar matrix. So if we define a map $\tau\colon \mathbb{C}^1 \to \mathrm{GL}(V_\pi)$ by $\tau(z) = z \cdot \mathrm{id}_{V_\pi}$, then the universal property of pushouts induces a unique map $\pi'\colon \widetilde{H} \to \mathrm{GL}(V_\pi)$. It is routine to verify that

(i) $\pi'|_{\widetilde{H}^{(8)}} = \pi$, so that (π', V_π) is an admissible representation of $\widetilde{H}$, and
(ii) $\pi'([h, z]_L)v = z\pi'([h, 1]_L)v$ for all $h \in H$, $z \in \mathbb{C}^1$, and $v \in V_\pi$. So (π', V_π) is genuine.

So we have shown that $(\pi', V_\pi) \in \mathcal{A}^{gen}(\widetilde{H})$. Moreover, since $\mathbb{C}^1 \hookrightarrow \widetilde{H}$ acts as scalars via π', we get a canonical map on morphisms

$$\mathrm{Hom}_{\widetilde{H}^{(8)}}(V_\pi, V_\sigma) \subset \mathrm{Hom}_{\widetilde{H}}(V_\pi, V_\sigma).$$

Finally, it is routine to verify that these natural transformations are mutual inverses of each other. ■

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Thus, we have demonstrated that while $\widetilde{G}_n$ is not an l-group, in the typical sense, the category $\mathcal{A}^{gen}(\widetilde{G}_n)$ is isomorphic to a category of genuine admissible representations on the l-group $\widetilde{G}_n^{(8)}$. Ultimately, we want to use the previous isomorphism in conjunction with various constructions in [3, 4] to define both the parabolic induction functor and Jacquet functor for $\widetilde{G}_n$.

4.1 Parabolic Induction for $\widetilde{G}_n$

Since we have defined parabolic subgroups of $\widetilde{G}_n$ and $\widetilde{G}_n^{(8)}$, we would like to define genuine parabolically induced representations for these groups. As pointed on in [8, Section 1], because $\widetilde{G}_n^{(8)}$ is an l-group we have both parabolic induction and Jacquet functors in the sense of Bernstein and Zelevinsky (see [3] for un-normalized and [4] for normalized constructions). In particular, let $\widetilde{P}^{(8)} = \widetilde{M}^{(8)}\mathbf{N}$ be a parabolic subgroup with

$$\widetilde{M}^{(8)} \simeq \mathrm{GL}_{r_1}(F) \times \mathrm{GL}_{r_2}(F) \times \cdots \times \mathrm{GL}_{r_m}(F) \times \widetilde{G}_{n'}^{(8)}.$$

Further, let δ_P be the modulus character of $P \subset G_n$. Finally, let (π_i, V_i) be admissible representations of $\mathrm{GL}_{r_i}(F)$ and let (σ_0, V_0) be a genuine admissible representation of $\widetilde{G}_{n'}^{(8)}$. Then we define

$$\mathrm{Ind}_{\widetilde{P}^{(8)}}^{\widetilde{G}_n^{(8)}}(\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_m \otimes \sigma_0)$$

as functions $f \colon \widetilde{G}_n^{(8)} \to (\bigotimes_{i=1}^m V_i) \otimes V_0$ such that

$$f(\mathbf{n}[m(g_1, g_2, \ldots, g_m, g_0), z]_L g') = \delta(m(g_1, g_2, \ldots, g_m, g_0))^{\frac{1}{2}} \left(\bigotimes_{i=1}^m \pi_i(g_i) \right) \otimes \sigma_0([g_0, z]_L) f(g')$$

for all $g_i \in \mathrm{GL}_{r_i}(F)$, $g_0 \in G_{n'}$, $z \in \mu_8$ and $g' \in \widetilde{G}_n^{(8)}$. Furthermore, f must also be right $\mathbf{K}'$ invariant for some $K' \subset K$. We also let $\widetilde{G}_n$ act on this space of functions by right translation. It is easily verified that since σ_0 is genuine, the resulting induced representation is also genuine. Finally, using the isomorphism of categories from Proposition 4.2, we see that the representation we get by removing the superscript (8) from all of our subgroups is an element of $\mathcal{A}^{gen}(\widetilde{G}_n)$.

To simplify our notation, we will borrow a convention of Tadić in [16, 17] where he denotes various induced representations in the following way. The representation that we just discussed will be denoted by $\pi_1 \times \pi_2 \times \cdots \times \pi_m \rtimes \sigma_0$. It is possible that $n' = 0$ (for instance if $P = P_n$, the Siegel parabolic). In this case, we will set $\widetilde{G}_0 = \mathbb{C}^1$. Since our representations (π, V_π) of $\widetilde{G}_n$ are genuine, we require that every $z \in \mathbb{C}^1$ act as $z \cdot \mathrm{id}_{V_\pi}$, so we define the map

$$\tau_0 \colon \mathbb{C}^1 \to \mathbb{C}^1 \qquad \tau_0(z) = z.$$

The representations that are most significant for this paper are those induced from characters on the diagonal torus in $\widetilde{M}_\varnothing$. These are the principal series representations of $\widetilde{G}_n$ and have the form $\chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \times \cdots \times \chi_n\nu^{s_n} \rtimes \tau_0$, where χ_i are unitary characters of $F^\times$ and $s_i \in \mathbb{R}$ for all i. We call such data $\chi_1\nu^{s_1} \otimes \chi_2\nu^{s_2} \otimes \cdots \otimes \chi_n\nu^{s_n}$ *regular* if for any $w \in W_{G_n}$,

$$w \cdot (\chi_1\nu^{s_1} \otimes \chi_2\nu^{s_2} \otimes \cdots \otimes \chi_n\nu^{s_n}) \neq \chi_1\nu^{s_1} \otimes \chi_2\nu^{s_2} \otimes \cdots \otimes \chi_n\nu^{s_n},$$

where $(w \cdot \chi)(m) = \chi(w^{-1}mw)$. Otherwise, the data will be called *irregular*. Also, the set of representations of the form $\chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \times \cdots \times \chi_n\nu^{s_n} \rtimes \sigma_0$, with the χ_i unramified for all i will be referred to as the *unramified principal series* of $\widetilde{G}_n$.

4.2 Jacquet Functors and Spherical Vectors

Now that we have a definition of principal series representations for $\widetilde{G}_n$, we would like to show that most of the theory for linear groups generalizes to our metaplectic cover in the most natural way. One such piece of machinery is the Jacquet functor. First, let (π, V) be a genuine admissible representation of $\widetilde{G}_n$. For a parabolic subgroup $\widetilde{P} = \widetilde{M}\mathbf{N}$, define $V(\mathbf{N}) = \operatorname{span}_{\mathbb{C}}(v - \pi(\mathbf{n})v)$. We then have the following proposition.

Proposition 4.3 (Jacquet modules for $\widetilde{G}_n$) *Let $V_{\mathbf{N}} = V/V(\mathbf{N})$ and define $\pi_{\mathbf{N}} \colon \widetilde{M} \to \mathrm{GL}(V_{\mathbf{N}})$ via the action*

$$\pi_{\mathbf{N}}([m,z]_L)(v + V(\mathbf{N})) = \pi([m,z]_L)v + V(\mathbf{N}).$$

Then $(\pi_{\mathbf{N}}, V_{\mathbf{N}})$ is a genuine admissible representation of $\widetilde{M}$.

Proof As before, we use the isomorphism of categories $\mathcal{A}^{gen}(\widetilde{M}^{(8)}) \simeq \mathcal{A}^{gen}(\widetilde{M})$ and the basic theory from [3]. If we restrict π to $\widetilde{G}_n^{(8)}$, then one can verify that the Jacquet module of the restriction is a genuine admissible representation of $\widetilde{M}^{(8)}$ as follows from [3, Section 2]. Thus, $(\pi_{\mathbf{N}}, V_{\mathbf{N}}) \in \mathcal{A}^{gen}(\widetilde{M})$ via Proposition 4.2. ∎

For $(\pi, V) \in \mathcal{A}^{gen}(\widetilde{G}_n)$ and parabolic subgroup $\widetilde{P} = \widetilde{M}\mathbf{N}$, we call the representation $(\pi_{\mathbf{N}}, V_{\mathbf{N}})$ the *Jacquet module* of π with respect to $\widetilde{P}$. The functor from $\mathcal{A}^{gen}(\widetilde{G}_n) \to \mathcal{A}^{gen}(\widetilde{M})$ that makes this assignment is called the *Jacquet functor* with respect to $\widetilde{P}$.

Furthermore, we let $r_{\widetilde{P}}^{\widetilde{G}}$ denote the functor taking an admissible representation of $\widetilde{G}_n$ to its Jacquet module with respect to $\widetilde{M}$ and twisted by $\delta_P^{-\frac{1}{2}}$ (*i.e.,* the normalized Jacquet functor, see [4]). We then get the usual Frobenius Reciprocity.

Proposition 4.4 (Frobenius reciprocity for $\widetilde{G}_n$) *Let (π, V) be a genuine admissible representation of $\widetilde{G}_n$ and (σ, W) be a genuine admissible representation of $\widetilde{M}$. Then we have a natural isomorphism*

$$\operatorname{Hom}_{\widetilde{G}_n}(\pi, \operatorname{Ind}_{\widetilde{P}}^{\widetilde{G}_n}(\sigma)) \simeq \operatorname{Hom}_{\widetilde{M}}(r_{\widetilde{P}}^{\widetilde{G}_n}(\pi), \sigma).$$

Proof As before, this result holds when we restrict π and $\operatorname{Ind}_{\widetilde{P}}^{\widetilde{G}_n}(\sigma)$ to $\widetilde{G}^{(8)}$ as well as $r_{\widetilde{P}}^{\widetilde{G}_n}(\pi)$ and σ to $\widetilde{M}^{(8)}$ according to [4, Section 1]. The isomorphism of categories in Proposition 4.2 then transfers the result back to the larger covers. ∎

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Finally, when F is nondyadic, we can also define spherical and Iwahori-spherical representations of $\widetilde{G}_n$. Let $P_\varnothing$ be our standard Borel subgroup of G_n. We want to show that for unramified characters $\{\chi_i\}_{i=1}^n$, the representation

$$\pi = \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \times \cdots \times \chi_n\nu^{s_n} \rtimes \tau_0$$

will have a nonzero vector fixed by $\mathbf{K}$. Notice that since the representation is genuine, we cannot expect to have vectors fixed by the larger compact open subgroup $\widetilde{K}$ since $\mathbb{C}^1 \hookrightarrow \widetilde{K}$ acts as scalars.

To extend this idea, let $\Omega \subset \Delta_n$. As in Section 2, we can associate with Ω a standard parabolic subgroup of $\mathrm{Sp}_n(\mathbb{F}_q)$, which we will denote as $P_\Omega(\mathbb{F}_q)$. Now we let $I_\Omega = \mathrm{proj}^{-1}(P_\Omega(\mathbb{F}_q))$, where $\mathrm{proj}\colon K \to \mathrm{Sp}_n(\mathbb{F}_q)$ is the map that reduces the entries of K mod $\mathcal{P}$. We call I_Ω the *standard parahoric subgroup* associated with Ω. Further, we call the subgroup $I_\varnothing$ the *standard Iwahori subgroup* of G_n. Lastly, let $\mathbf{I}_\Omega \subset \widetilde{G}_n$ be the image of the splitting λ restricted to I_Ω. With this we have the following definition.

Definition 4.5 Let (π, V_π) be a genuine admissible representation of $\widetilde{G}_n$. We will call the representation spherical (resp. $\mathbf{I}_\Omega$-spherical) if

$$\dim_{\mathbb{C}} V_\pi^{\mathbf{K}} \neq 0 \qquad (\text{resp. } \dim_{\mathbb{C}} V_\pi^{\mathbf{I}_\Omega} \neq 0)$$

where $\pi^{\mathbf{K}}$ (resp. $\pi^{\mathbf{I}_\varnothing}$) represented the space of vectors fixed by $\mathbf{K}$ (resp. $\mathbf{I}_\varnothing$).

With this definition and the argument preceding it, we have the following proposition.

Proposition 4.6 *Let F be a nondyadic field and let*

$$\pi = \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \times \cdots \times \chi_n\nu^{s_n} \rtimes \tau_0$$

be an unramified principal series representation of $\widetilde{G}_n$. Then we have

$$\dim_{\mathbb{C}}(\pi^{\mathbf{K}}) = 1 \quad \textit{and} \quad \dim_{\mathbb{C}}(\pi^{\mathbf{I}_\varnothing}) = |W_{G_n}|,$$

where W_{G_n} is the Weyl group of G_n.

Proof Recall that our splitting $K \to \widetilde{G}_n$ only exists for F nondyadic (see Remark 3.2), which explains our condition on the field F.

Next, we endeavor to construct the unique space of $\mathbf{K}$-invariant vectors. By the Iwasawa decomposition for $\widetilde{G}_n$, $\widetilde{G}_n = \widetilde{P}_\varnothing\mathbf{K}$. So π will have a vector fixed by $\mathbf{K}$ as long as $f(\mathbf{p}) = f(1_{\widetilde{G}_n})$ for $\mathbf{p} \in \widetilde{P}_\varnothing \cap \mathbf{K}$. However, by a simple computation in [10, Remark 8.5.2], we see that for $\mathbf{p} \in \widetilde{P}_\varnothing \cap \mathbf{K}$, $\mathbf{p} = [p, \lambda(p)]_L$ with $\lambda(p) = 1$. Let us now slightly alter some previous notation. In particular, let $m\colon \mathrm{GL}_n(F) \to P_n$ be the map that identifies $\mathrm{GL}_n(F)$ with the Levi factor of P_n for the remainder of the paper. Then we can write $p = nm(\mathrm{diag}(a_1, a_2, \dots, a_n))$ with $a_i \in \mathcal{O}^\times$. Therefore, we compute

$$\begin{aligned} f(\mathbf{p}) &= f([p,1]_L) = f([nm(\mathrm{diag}(a_1,a_2,\dots,a_n)),1]_L) \\ &= \prod_{i=1}^n \chi_i(a_i)|a_i|^{s_i+n-\frac{2i+1}{2}} f(1_{\widetilde{G}_n}) = f(1_{\widetilde{G}_n}). \end{aligned}$$

Next we would like to show that $\dim_{\mathbb{C}}(\pi^{\mathbf{I}_\varnothing}) = |W_{G_n}|$. However, this follows from the decomposition

$$K = \bigcup_{w \in W_{G_2}} I_\varnothing w I_\varnothing = \bigcup_{w \in W_{G_2}} N_\varnothing(\mathcal{O}) w I_\varnothing$$

This derives from the inverse image of the Bruhat decomposition of $\mathrm{Sp}_n(\mathbb{F}_q)$ (see [6, Proposition 1.3]) under the map proj: $K \longrightarrow \mathrm{Sp}_n(\mathbb{F}_q)$. As such, we see that the space $\pi^{\mathbf{I}_\varnothing}$ is spanned by the vectors indexed by $w \in W_{G_n}$. In particular, we have the functions $f_{\mathbf{w}}$ with $\operatorname{supp}(f_{\mathbf{w}}) \subset \widetilde{P}_\varnothing \mathbf{w} \mathbf{I}_\varnothing$ and $f_{\mathbf{w}}(\mathbf{wk}) = 1$ for all $\mathbf{k} \in \mathbf{I}_\varnothing$. ■

Notice that this last proposition does not apply when $F = \mathbb{Q}_2$ due to the lack of splitting. As one can see in [10] for $\widetilde{G}_1$, the dyadic case is much more sensitive.

4.3 Compatibility of the Bernstein–Zelevinsky Constructions with the Metaplectic Group

We have spent much of this section defining the analogs to the various standard tools of the representation theory of p-adic groups (like parabolic induction and the Jacquet functor). Now, we will further justify that these constructions inherit all of the relevant properties that make them useful in the p-adic group setting. In particular, we would like to justify that the parabolic induction and Jacquet functors are exact and are subject to the various results of [4, Sections 2 and 3]. The critical result in these sections is [4, Proposition 2.3], whose metaplectic analog would be the following.

Proposition 4.7 *Let $\widetilde{P} = \widetilde{M}\mathbf{N}$ and $\widetilde{P}' = \widetilde{M}'\mathbf{N}'$ be standard parabolics of $\widetilde{G}_n$ with $\widetilde{P}' \subset \widetilde{P}$. Then the following statements hold:*

(i) *The functors $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}_n}$ and $r_{\widetilde{P}}^{\widetilde{G}_n}$ are exact.*

(ii) *The functor $r_{\widetilde{P}}^{\widetilde{G}_n}$ is left adjoint to $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}_n}$.*

(iii) *We have $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}_n} \circ \mathrm{Ind}_{\widetilde{M} \cap \widetilde{P}'}^{\widetilde{M}} = \mathrm{Ind}_{\widetilde{P}'}^{\widetilde{G}_n}$ $\quad r_{\widetilde{M} \cap \widetilde{P}'}^{\widetilde{M}} \circ r_{\widetilde{P}}^{\widetilde{G}_n} = r_{\widetilde{P}'}^{\widetilde{G}_n}$.*

(iv) $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}_n}(\rho)^\vee = \mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}_n}(\rho^\vee)$ *for $\rho \in \mathcal{A}^{gen}(\widetilde{M})$ where $\rho^\vee$ denotes the contragradient of ρ.*

(v) $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}_n}$ *and $r_{\widetilde{P}}^{\widetilde{G}_n}$ carry admissible representations into admissible ones.*

Proof In the proof of the corresponding proposition for p-adic groups [4, Proposition 2.3], the authors only rely on a few properties of a p-adic group and its parabolic subgroups then apply a previous proposition [4, Proposition 1.9] that pertains to general l-groups. For us to prove this result, we first invoke our isomorphism of categories from $\mathcal{A}^{gen}(\widetilde{G}_n)$ (resp. $\mathcal{A}^{gen}(\widetilde{M})$) to $\mathcal{A}^{gen}(\widetilde{G}_n^{(8)})$ (resp. $\mathcal{A}^{gen}(\widetilde{M}^{(8)})$) and then verify that the standard parabolic subgroups of $\widetilde{G}_n^{(8)}$ satisfy the properties cited in [4, Proposition 2.3]. Since $\widetilde{G}_n^{(8)}$ is an l-group, [4, Proposition 1.9] would then imply our desired result just as it did in the p-adic group case.

The three properties that need to be satisfied are as follows. First, we need that for any compact set $C \subset \mathbf{N}$, there exists a compact subgroup $U \subset \mathbf{N}$ with $C \subset U$. However, $\mathbf{N}$ is isomorphic to N via $[n,1]_L \leftrightarrow n$, and we know that this property

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holds for N since it is the unipotent radical of a standard parabolic of a algebraic group. Next, we need that $\widetilde{G}_n^{(8)}$ is compact modulo $\widetilde{P}^{(8)}$. This is clear from equation (3.2) that established the Iwasawa decomposition of $\widetilde{G}_n$ (which adapts to $\widetilde{G}_n^{(8)}$ exactly as written). Finally, we need to establish that $\widetilde{G}_n^{(8)}$ is unimodular. But such a measure can be constructed as the product of a unimodular measure on G_n with the counting measure on μ_8. ■

Although the various results regarding composition of parabolic induction and Jacquet module functors in [4, Sections 2–5] are stated in terms of p-adic groups, the results can be extended to our covering groups. Our eightfold metaplectic cover is an l-group having the correct properties to invoke the results of Bernstein and Zelevinsky.

Finally, our methods for ascertaining whether a principal series representation of $\widetilde{G}_2$ is reducible will depend greatly on our ability to compute Jacquet modules. To do this, we would like to use the machinery outlined by Casselman in [6, Sections 6 and 7]. This is also explained in [15, Sections 5 and 8]. However, these calculations ultimately utilize the previously mentioned results of Bernstein and Zelevinsky in [4] along with the Bruhat decomposition of a p-adic group. Since we have established that the Bernstein–Zelevinsky theory applies to $\widetilde{G}_n$ and we have a Bruhat decomposition of $\widetilde{G}_n$, we are able to use the Casselman machinery to compute the Jordan–Hölder constituents of the various Jacquet modules for our principal series representations.

Casselman's method for computing Jacquet modules is very explicit. Because it is complicated, we refer the reader to [6, Sections 6 and 7] and [15, Sections 5 and 8] for the full details, but we will give a basic sketch of the main ideas. For a parabolic $\widetilde{P} = \widetilde{M}\mathbf{N}$ and an irreducible representation σ of $\widetilde{M}$, we can form the representation $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\sigma)$. To compute the constituents of the Jacquet module with respect to a parabolic $\widetilde{P}' = \widetilde{M}'\mathbf{N}'$, we construct a filtration of $\widetilde{M}'$-representations

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = \mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\sigma)(\sigma)_{\mathbf{N}'},$$

where the spaces in the filtration are indexed by the double cosets $\widetilde{P}\backslash\widetilde{G}_n/\widetilde{P}'$. Recall that $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}_n}(\sigma)_{\mathbf{N}'}$ consists of the $\mathbf{N}'$-coinvariants of the induced representation defined in Proposition 4.3. In particular, these $\widetilde{M}'$-representations come from the $\mathbf{N}'$-coinvariants of the various subspaces $I(\widetilde{P}\mathbf{w}\widetilde{P}') \subset \mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}_n}(\sigma)$ that contain functions $f \in \mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}_n}(\sigma)$ with $\mathrm{supp}(f) \subset \widetilde{P}\mathbf{w}\widetilde{P}'$. Finally, if we set $\widetilde{P} = \widetilde{P}' = \widetilde{P}_\varnothing$ and σ to be some character of the torus $(F^\times)^n$ and follow the procedures of Casselman and Tadić, we get the following lemma (which is essentially [15, Theorem 8.1(i)]).

Lemma 4.8 *The Jordan–Hölder series of* $r_{\widetilde{P}_\varnothing}^{\widetilde{G}_n}(\mathrm{Ind}_{\widetilde{P}_\varnothing}^{\widetilde{G}_n}(\sigma))$ *consists of all* $(w \cdot \sigma) \rtimes \tau_0$, *where* $w \in W_{G_n}$ *and* $(w \cdot \sigma)(m) = \sigma(w^{-1}mw)$.

It is precisely this lemma along with the various properties of parabolic induction and Jacquet functors outlined in Proposition 4.7 that we will leverage in our analysis of the principal series for $\widetilde{G}_2$.

5 Reducibility Results for $\widetilde{G}_2$-Regular Case

From this point forward, our $n \in \{1, 2\}$. We would like to create a complete list of reducibility for the genuine principal series of $\widetilde{G}_2$. However, to do this, we will need to leverage some known results for $\widetilde{G}_1$. We will also break up the arguments into two cases depending on the size of $\mathrm{Stab}_{W_{G_2}}(\chi_1\nu^s \otimes \chi_2\nu^t)$, where the Weyl group acts on the inducing data by

$$w \cdot (\chi_1 \otimes \chi_2)(m) := \chi_1 \otimes \chi_2(w^{-1}mw).$$

When this stabilizer is small, computing and comparing the Jacquet modules for the various standard parabolics will suffice to determine reducibility points. Much of this section will follow a technique of Tadić [16, 17]. When the stabilizer is large, which occurs for representations induced from quadratic characters, we study the set $\mathrm{Hom}_{\widetilde{G}_2}(\pi, \pi)$. We will then use some results of Shahidi [12] in order to decompose some intertwining operators as a composition of relative, rank-one, intertwining operators. We will then show that such operators have poles at the relevant values.

Before we begin, let us recall some results regarding the principal series of $\widetilde{G}_1$ (see [10, Chapter 8]).

Lemma 5.1 *Let $I(\chi, s)$ denote the representation $\mathrm{Ind}_{\widetilde{P}_\varnothing}^{\widetilde{G}_1}(\chi\nu^s \otimes \tau_0)$, then $I(\chi, s)$ is irreducible unless $\chi = \xi_a$ and $s \in \{\pm\frac{1}{2}\}$. $I(\xi_a, \frac{1}{2})$ has two constituents. It has a spherical quotient $\omega_{a,\psi}^+$ that corresponds to the even part of the Weil representation $(\omega_{a,\psi}, S)$ (see [9, Section I.1]). $I(\xi_a, \frac{1}{2})$ also has a (nonspherical) submodule sp_a. $I(\xi_a, -\frac{1}{2})$ has the same constituents, except their roles as submodules and quotients reverse. Moreover, $sp_a \simeq sp_b$ and $\omega_{a,\psi}^+ \simeq \omega_{b,\psi}^+$ if and only if $\xi_a = \xi_b$ if and only if $ab^{-1} \in (F^\times)^2$.*

Also note that since we have fixed an additive character ψ, we can drop it from the notation $\omega_{a,\psi}^+$.

More generally, we will denote $\omega_{a,n}^+$ to be the even part of the Weil representation of $\widetilde{G}_n$ on $S(V^n)$ where V is the quadratic space (F, q) with $q(x) = ax^2$. Kudla shows in [9] that $\omega_{a,n}^+$ is a constituent of (indeed a quotient of)

$$\pi = \xi_a\nu^{n-\frac{1}{2}} \times \xi_a\nu^{n-\frac{3}{2}} \times \cdot \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0.$$

In order to keep our notation as consistent as possible, we will regard the irreducible submodule of this π as $sp_{a,n}$.

5.1 Tadić's Technique

In order to discuss Tadić's technique, we should really work with the Grothendieck group of $\widetilde{G}_2$ denoted $\mathfrak{R}(\widetilde{G}_2)$. For our purposes, $\mathfrak{R}(\widetilde{G}_2)$ is the free abelian group on the irreducible genuine admissible representations of $\widetilde{G}_2$ modulo the equivalence that $\pi_1 = \pi_2 \in \mathfrak{R}(\widetilde{G}_2)$ if $m(\rho, \pi_1) = m(\rho, \pi_2)$ for all irreducible $\rho \in \mathfrak{R}(\widetilde{G}_2)$, where $m(\rho, \pi)$ is the multiplicity of π in the Jordan–Hölder series for π. This group also has a partial order as follows. For $\pi, \pi' \in \mathfrak{R}(\widetilde{G}_2)$, we have $\pi \leq \pi'$ if $m(\rho, \pi) \leq m(\rho, \pi)$ for all irreducible $\rho \in \mathfrak{R}(\widetilde{G}_2)$.

For a parabolic $\widetilde{P} \subset \widetilde{G}_2$ with $\widetilde{P} = \widetilde{M}\mathbf{N}$, the corresponding (normalized) induction and Jacquet functors provide maps

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}_2} : \mathfrak{R}(\widetilde{M}) \to \mathfrak{R}(\widetilde{G}_2) \quad \text{and} \quad r_{\widetilde{P}}^{\widetilde{G}_2} : \mathfrak{R}(\widetilde{G}_2) \to \mathfrak{R}(\widetilde{M}).$$

As n increases, G_n (resp. $\widetilde{G}_n$) has an increasing number of parabolic subgroups containing a fixed Borel. While the Jacquet modules for various parabolics encode certain information about the representation, Tadić developed irreducibility criteria that take into account relationships between Jacquet modules across several parabolics (see [16, 17]). In particular, Tadić notes the following facts about Jacquet modules. Let G be a p-adic group and $P_0 = M_0N_0$ a parabolic subgroup. Then for an irreducible admissible representation (σ, V) of M_0 and π a constituent of $\mathrm{Ind}_{P_0}^{G}(\sigma)$, we have that

(i) $r_P^G(\pi) \neq 0$ for any parabolic $P \supset P_0$;
(ii) If $\pi = \pi_1 + \pi_2 \in \mathfrak{R}(G)$ and $P_0 \subset P$, then $r_P^G(\pi) = r_P^G(\pi_1) + r_P^G(\pi_2) \in \mathfrak{R}(M)$;
(iii) $r_{P_0}^G(\pi) = r_{P_0\cap M}^M r_P^G(\pi)$ for any parabolic P with $P_0 \subset P = MN$.

These three properties give some very restrictive structure on the various Jacquet modules of π. Moreover, they give us a technique to prove irreducibility that basically amounts to accounting for all of the Jacquet modules of a given representation that we outline in the following proposition.

Proposition 5.2 *Let π be an arbitrary constituent of $\mathrm{Ind}_{P_0}^{G}(\sigma)$ and let π_0 be an irreducible constituent of π. If there exists a $P \supset P_0$ with $r_P^G(\pi_0) = r_P^G(\pi)$, then $\pi_0 = \pi$ and thus π is irreducible.*

Proof Suppose that $\pi = \pi_0 + \pi' \in \mathfrak{R}(G)$ with $\pi' \geq 0$, then $r_P^G(\pi) = r_P^G(\pi_0) + r_P^G(\pi')$. However, $r_P^G(\pi) = r_P^G(\pi_0)$ in $\mathfrak{R}(M)$. Thus, $m(\rho, r_P^G(\pi)) = m(\rho, r_P^G(\pi_0))$ for all irreducible $\rho \in \mathfrak{R}(M)$. Thus, $m(\rho, r_P^G(\pi')) = 0$ for all irreducible $\rho \in \mathfrak{R}(M)$ and therefore $r_P^G(\pi') = 0$. However, this contradicts property (i) above if π' is a nonzero constituent of $\mathrm{Ind}_{P_0}^{G}(\sigma)$. ■

Remark 5.3 It is worth remarking that Tadić's criteria and our Proposition 5.2 are stated for p-adic groups. However, their formulation requires only the basic properties of the parabolic induction and Jacquet functors that have metaplectic analogs in Proposition 4.7. Thus, we are justified in extending Tadić's criteria and Proposition 5.2 to our metaplectic groups.

Let us apply Proposition 5.2 to a specific case to help illustrate its utility. First, let B be the upper triangular Borel subgroup of $\mathrm{GL}_2(F)$.

Proposition 5.4 *Let χ_1 and χ_2 be unitary characters of $F^\times$ such that*

$$\chi_1\nu^s \times \chi_2\nu^t := \mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi_1\nu^s \otimes \chi_2\nu^t)$$

is an irreducible representation of $\mathrm{GL}_2(F)$ and both $\chi_1\nu^s \rtimes \tau_0$ and $\chi_2\nu^t \rtimes \tau_0$ are irreducible representations of $\widetilde{G}_1$. Further, suppose that

$$\chi_1\nu^s \otimes \chi_2\nu^t \neq w \cdot (\chi_1\nu^s \otimes \chi_2\nu^t)$$

for any $w \in W_{G_2}$. Then $\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ is an irreducible representation of $\widetilde{G}_2$.

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Proof Let π_0 be a nonzero irreducible constituent of $\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$. Thus, $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0) \geq 0$ and $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0)$ must contain one of the irreducible constituents of

$$r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0).$$

Without loss of generality, suppose that $\chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0 \leq r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0)$. Our goal is to show that $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0)$ contains all the remaining irreducible constituents of

$$r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0).$$

First, we see that if $\chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0 \leq r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0)$, then

$$\mathrm{Hom}_{\widetilde{M}_\varnothing}(r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0), \chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0) \neq 0.$$

But applying Frobenius reciprocity with respect to both $\widetilde{M}_\varnothing$ and $\widetilde{M}_\alpha$ yields that

$$\begin{aligned} 0 \neq \mathrm{Hom}_{\widetilde{M}_\varnothing}(r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0), \chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0) &\simeq \mathrm{Hom}_{\widetilde{G}_2}(\pi_0, \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0) \\ &\simeq \mathrm{Hom}_{\widetilde{M}_\alpha}(r^{\widetilde{G}_2}_{\widetilde{P}_\alpha}(\pi_0), \chi_1\nu^s \times \chi_2\nu^t \otimes \tau_0). \end{aligned}$$

But $\chi_1\nu^s \times \chi_2\nu^t \otimes \tau_0$ is an irreducible representation of $\widetilde{M}_\alpha$, so $\chi_1\nu^s \times \chi_2\nu^t \otimes \tau_0$ is a constituent of $r^{\widetilde{G}_2}_{\widetilde{P}_\alpha}(\pi_0)$ and

$$\chi_1\nu^s \times \chi_2\nu^t \otimes \tau_0 \leq r^{\widetilde{G}_2}_{\widetilde{P}_\alpha}(\pi_0).$$

Finally, Tadić's property (iii) and the transitivity of Jacquet functors tell us that

$$\begin{aligned} \chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0 + \chi_2\nu^t \otimes \chi_1\nu^s \otimes \tau_0 &= r^{\widetilde{M}_\alpha}_{\widetilde{M}_\alpha \cap \widetilde{P}_\varnothing}(\chi_1\nu^s \times \chi_2\nu^t \otimes \tau_0) \\ &\leq r^{\widetilde{M}_\alpha}_{\widetilde{M}_\alpha \cap \widetilde{P}_\varnothing}(r^{\widetilde{G}_2}_{\widetilde{P}_\alpha}(\pi_0)) = r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0). \end{aligned}$$

A nearly identical argument using the parabolic $\widetilde{P}_\beta$ shows that

$$\begin{aligned} \chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0 + \chi_1\nu^s \otimes \chi_2^{-1}\nu^{-t} \otimes \tau_0 &= r^{\widetilde{M}_\beta}_{\widetilde{M}_\beta \cap \widetilde{P}_\varnothing}(\chi_1\nu^s \otimes \chi_2\nu^t \rtimes \tau_0) \\ &\leq r^{\widetilde{M}_\beta}_{\widetilde{M}_\beta \cap \widetilde{P}_\varnothing}(r^{\widetilde{G}_2}_{\widetilde{P}_\beta}(\pi_0)) = r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0). \end{aligned}$$

Now we repeat the argument above beginning with

$$\chi_1\nu^s \otimes \chi_2^{-1}\nu^{-t} \otimes \tau_0 \leq r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0) \qquad (\text{resp. } \chi_2\nu^t \otimes \chi_1\nu^s \otimes \tau_0 \leq r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0)).$$

This shows that

$$(\chi_1\nu^s \times \chi_2^{-1}\nu^{-t}) \otimes \tau_0 \leq r^{\widetilde{G}_2}_{\widetilde{P}_\alpha}(\pi_0) \qquad (\text{resp. } \chi_2\nu^t \otimes (\chi_1\nu^s \rtimes \tau_0) \leq r^{\widetilde{G}_2}_{\widetilde{P}_\beta}(\pi_0)).$$

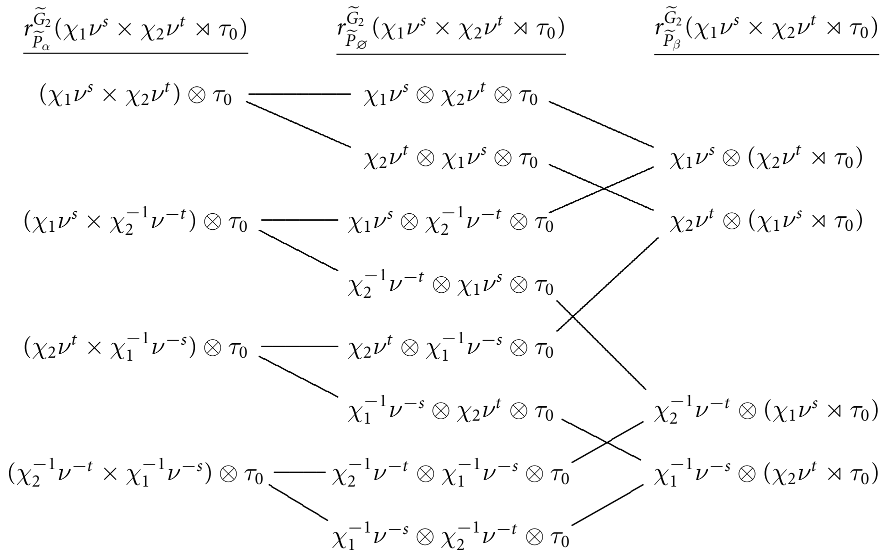

Figure 1

This then implies that

$$\chi_2^{-1}\nu^{-t} \otimes \chi_1\nu^s \otimes \tau_0 \le r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0) \qquad (\text{resp. } \chi_2\nu^t \otimes \chi_1^{-1}\nu^{-s} \otimes \tau_0 \le r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0).$$

By repeating the argument for each new constituent of $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0)$ that we find, we eventually show that

$$r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0) = r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi_0).$$

Thus, by Proposition 5.2, we have that $\pi_0 = \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$, and irreducibility is proved. ■

For a more visually satisfying interpretation of the proof, consider the diagram in Figure 1, where each column contains the irreducible constituents of the Jacquet modules corresponding to a different standard parabolic subgroup; the irreducible constituents of the Jacquet modules for the maximal parabolics are the outer columns and the irreducible constituents of the Jacquet modules for the Borel subgroup form the center column. The lines connect representations on the Levi factors of the maximal parabolics with the irreducible constituents of their Jacquet modules on the torus. This figure is very similar to one in Tadić's paper [16].

The argument in the proof of Proposition 5.4 essentially tells us that any Jacquet modules that are connected via a series of line segments must belong to the same irreducible representation π. So for any irreducible constituent π of $\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ with $\chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0 \le r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi)$, then $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi)$ must be as great as any submodule of $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0)$ that we can connect to $\chi_1 \otimes \chi_2 \otimes \tau_0$ in our diagram via a series of line segments. However, in this case, one can join any two irreducible submodules of $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0)$, so all must be submodules of $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi)$. This picture and argument suggests a graph theoretic criterion for proving the irreducibility of principal series representations.

Definition 5.5 Let (π, V) be a constituent of a principal series representation of $\widetilde{G}_2$. We define a labeled graph as follows.

- The number of vertices should be the sum of the lengths of the Jordan–Hölder series for $r^{\widetilde{G}_2}_{\widetilde{P}}(\pi)$ for each standard parabolic $\widetilde{P}$.
- The vertices should be labeled by the irreducible constituents of $r^{\widetilde{G}_2}_{\widetilde{P}}(\pi)$ for all the standard parabolics taking into account multiplicities.
- A vertex labeled by σ, an irreducible constituent of $r^{\widetilde{G}_2}_{\widetilde{P}}(\pi)$ for a maximal parabolic, connects (via an edge) to a vertex labeled by σ', an irreducible constituent of $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi)$, whenever $r^{\widetilde{M}}_{\widetilde{M}\cap\widetilde{P}_\varnothing}(\sigma) \geq \sigma'$. Moreover, these are the only edges in the labeled graph.

We will refer to such a graph as the *Jacquet module graph* of (π, V), and it will be denoted by $JMG(\pi)$.

As we will eventually see, if π is a constituent of a principal series representation with a large amount of symmetry under the Weyl group, then the $JMG(\pi)$ will not accurately reflect the relationship between the Jacquet modules of the Borel subgroup with the Jacquet modules of larger parabolics. Such Jacquet module graphs will have spurious edges. Consider the following example.

Example Let $\pi = \xi_a \times \xi_b \rtimes \tau_0$, where ξ_a and ξ_b are inequivalent quadratic characters. Then the $JMG(\pi)$ is given in Figure 2.

Note that each vertex corresponding to an irreducible constituent σ of the Jacquet module with respect to the maximal parabolic has more edges emanating from it than the length of $r^{\widetilde{M}}_{\widetilde{M}\cap\widetilde{P}_\varnothing}(\sigma)$. So one can see that this Jacquet module graph cannot be used to illustrate an argument as in Proposition 5.2.

This phenomenon stems from the fact that Tadić's technique involves working in the Grothendieck group, where we lose some of the information about Jacquet modules. In particular, when working in the Grothendieck group, we are actually computing the semi-simplification of $r^{\widetilde{G}_2}_{\widetilde{P}}(\pi)$ where we see only the Jordan–Hölder constituents but not the entire Jordan-Hölder series. Consequently, for σ an irreducible constituent of $r^{\widetilde{G}_2}_{\widetilde{P}}(\pi)$ for P a maximal parabolic, the number of edges emanating from σ may be greater than the length of $r^{\widetilde{M}}_{\widetilde{M}\cap\widetilde{P}_\varnothing}(\sigma)$. This happens if the inducing data for which π is a constituent has a great deal of symmetry under the Weyl group.

So we offer the following definition.

Definition 5.6 For π a constituent of a principal series representation of $\widetilde{G}_2$, we will call the Jacquet module graph $JMG(\pi)$ *proper* if for any irreducible constituent σ of $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi)$ we have a unique irreducible constituent σ_α of $r^{\widetilde{G}_2}_{\widetilde{P}_\alpha}(\pi)$ and a unique irreducible

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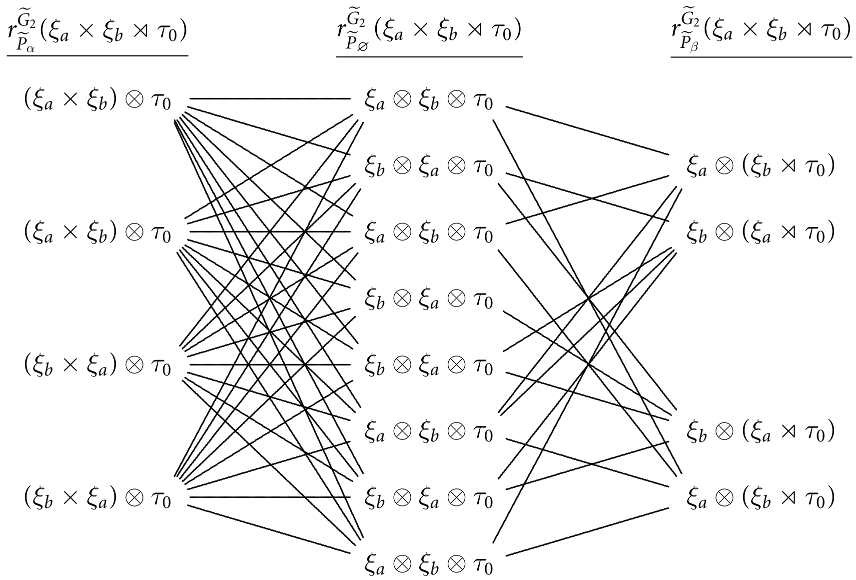

Figure 2: $JMG(\pi)$

constituent σ_β of $r^{\widetilde{G}_2}_{\widetilde{P}_\beta}(\pi)$ that join σ via edges.

With this definition, it should be relatively clear that Proposition 5.2 and the proof of Proposition 5.2 yield the following corollary.

Corollary 5.7 *Let (π, V) be a constituent of the principal series of $\widetilde{G}_2$. If $JMG(\pi)$ is both a proper Jacquet module graph and connected graph, then (π, V) is irreducible.*

This technique is quite powerful, as demonstrated by Proposition 5.2, which has the following corollary.

Corollary 5.8 *Let χ_1 and χ_2 be unitary quasi-characters of $F^\times$. Then $\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ is an irreducible representation of $\widetilde{G}_2$, if none of the following conditions holds.*

(i) $\chi_1^2 = 1$ *and* $2s \in \{0, \pm 1\}$.
(ii) $\chi_2^2 = 1$ *and* $2t \in \{0, \pm 1\}$.
(iii) $\chi_1 = \chi_2$ *and* $s - t \in \{0, \pm 1\}$.
(iv) $\chi_1\chi_2 = 1$ *and* $s + t \in \{0, \pm 1\}$.

Proof Notice that the conditions involving $2s, 2t, s \pm t \notin \{\pm 1\}$ ensure that $\chi_1\nu^s \times \chi_2\nu^t$ is an irreducible representation of $\mathrm{GL}_2(F)$ (see [5, Lemma 4.5.1]) and that $\chi_1\nu^s \rtimes \tau_0$ and $\chi_2\nu^t \rtimes \tau_0$ are irreducible representations of $\widetilde{G}_1$ (see [9, Example VII.1.5]). Moreover, the conditions involving $s, t, s \pm t \neq 0$ ensure that the inducing data is regular. Thus, the inducing data satisfies all the hypotheses of Proposition 5.2. ■

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As we can see, we have determined nearly all of the irreducible principal series. We will spend the remainder of the paper resolving the cases not covered in Corollary 5.7. In fact, most of the remaining cases are reducible.

Proposition 5.9 *Let χ be a unitary character of $F^\times$. Then the representation $\chi\nu^{s+\frac{1}{2}} \times \chi\nu^{s-\frac{1}{2}} \rtimes \tau_0$ has exactly two irreducible constituents if either of the following holds.*

(i) $\chi^2 \neq 1$ *and* $s \in \mathbb{R}$ *is arbitrary.*
(ii) $\chi^2 = 1$ *and* $s \notin \{0, \pm 1\}$.

In this case, it has an irreducible representation $\chi\nu^s St_{\mathrm{GL}_2} \rtimes \tau_0$ and an irreducible quotient $\chi\nu^s \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0$, where St_{GL_2} is the Steinberg representation of $\mathrm{GL}_2(F)$ and $\mathbb{1}_{\mathrm{GL}_2}$ is the trivial representation of $\mathrm{GL}_2(F)$.

Proof The proof of this can be reduced to using the exactness of parabolic induction to show the existence of the two constituents above then computing the Jacquet module graph of these constituents to establish the irreducibility of each constituent.

Consider the representation

$$\chi\nu^{s+\frac{1}{2}} \times \chi\nu^{s-\frac{1}{2}} := \mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi\nu^{s+\frac{1}{2}} \otimes \chi\nu^{s-\frac{1}{2}}).$$

It is known (see [5, Lemma 4.5.1]) that this representation has length two with

$$1 \to \chi\nu^s St_{\mathrm{GL}_2} \to \chi\nu^{s+\frac{1}{2}} \times \chi\nu^{s-\frac{1}{2}} \to \chi\nu^s \mathbb{1}_{\mathrm{GL}_2} \to 1$$

an exact sequence of $\mathrm{GL}_2(F)$ representations. Thus,

$$1 \to \chi\nu^s St_{\mathrm{GL}_2} \otimes \tau_0 \to \chi\nu^{s+\frac{1}{2}} \times \chi\nu^{s-\frac{1}{2}} \otimes \tau_0 \to \chi\nu^s \mathbb{1}_{\mathrm{GL}_2} \otimes \tau_0 \to 1$$

is an exact sequence of representations of $\widetilde{M}_\alpha \simeq \mathrm{GL}_2(F) \times \mathbb{C}^1$. Finally by Proposition 4.7, we have that parabolic induction is exact. Thus, we find that

$$1 \to \chi\nu^s St_{\mathrm{GL}_2} \rtimes \tau_0 \to \chi\nu^{s+\frac{1}{2}} \times \chi\nu^{s-\frac{1}{2}} \rtimes \tau_0 \to \chi\nu^s \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0 \to 1$$

is an exact sequence of representations of $\widetilde{G}_2$.

Next, to show the irreducibility of the two constituents, one just applies Corollary 5.7. We leave it to the reader to show that both $\chi\nu^s St_{\mathrm{GL}_2} \rtimes \tau_0$ and $\chi\nu^s \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0$ have connected proper Jacquet module graphs; however, we do supply the Jacquet modules for the maximal parabolic subgroups in Tables 2 and 3 in the appendix. ∎

This proof contains the general technique that we will use for all of the cases that our $\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ is reducible. As we will see, reducibility of our induced representation will come from reducibility of inducing data for various representations on the Levi factors of our maximal parabolic subgroups.

Proposition 5.10 *Let $a \in F^\times$ and let χ be a unitary character of $F^\times$. Then the representation $\chi\nu^s \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0$ has length two as long as any of the following holds.*

(i) $\chi^2 \neq 1$ *and* $s \in \mathbb{R}$ *is arbitrary.*

(ii) $\chi = \xi_b$ *with* ab^{-1} *not a square in* $F^\times$ *and* $s \notin \{\pm\frac{1}{2}\}$.
(iii) $\chi = \xi_a$ *and* $s \notin \{\pm\frac{1}{2}, \pm\frac{3}{2}\}$.

In this case, it has an irreducible submodule $\chi\nu^s \rtimes sp_a$ *and an irreducible quotient* $\chi\nu^s \rtimes \omega_a^+$.

Proof The proof of this result is nearly identical to the proof of the previous proposition, except we now examine the reducibility of a representation induced from the Klingen parabolic $\widetilde{P}_\beta$. In particular, [9, Example VII.1.5] shows us that

$$1 \to sp_a \to \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0 \to \omega_a^+ \to 1$$

is an exact sequence of $\widetilde{G}_1$ representations. Thus,

$$1 \to \chi\nu^s \otimes sp_a \to \chi\nu^s \otimes \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0 \to \chi\nu^s \otimes \omega_a^+ \to 1$$

is an exact sequence of $\widetilde{M}_\beta$ representations. Finally, we invoke the exactness of parabolic induction outlined in Proposition 4.7. This tells us that

$$1 \to \chi\nu^s \rtimes sp_a \to \chi\nu^s \rtimes \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0 \to \chi\nu^s \times \omega_a^+ \to 1$$

is an exact sequence of $\widetilde{G}_2$ representations. Moreover, both $\chi\nu^s \rtimes sp_a$ and $\chi\nu^2 \rtimes \omega_a^+$ have connected, proper, Jacquet module graphs. ∎

The previous two propositions account for all the representations $\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ of length exactly two. It is worth noting that both propositions contain infinite families of representations indexed by the unitary character χ and $s \in \mathbb{R}$. We now turn our attention to some finite families of representations where the representations induced from characters on the Borel have length four.

Proposition 5.11 *Let* $a \in F^\times$. *Then the representation* $\xi_a\nu^{\frac{3}{2}} \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0$ *has length four. It has an irreducible submodule* $sp_{a,2}$, *an irreducible quotient* $\omega_{a,2}^+$, *and two more irreducible constituents* $Q(\xi_a\nu St_{\mathrm{GL}_2}, \tau_0)$ *and* $Q(\xi_a\nu^{\frac{3}{2}}, sp_a)$.

Proof The proof is essentially due to the fact that $\xi_a\nu^{\frac{3}{2}} \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0$ can be realized as induced from reducible representations on the Levi components of both maximal parabolics. In particular, note that

$$1 \to \xi_a\nu St_{\mathrm{GL}_2} \to \xi_a\nu^{\frac{3}{2}} \times \xi_a\nu^{\frac{1}{2}} \to \xi_a\nu \mathbb{1}_{\mathrm{GL}_2} \to 1$$

is an exact sequence of $\mathrm{GL}_2(F)$ representations (see [5, Lemma 4.5.1]), and

$$1 \to sp_a \to \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0 \to \omega_a^+ \to 1$$

is an exact sequence of $\widetilde{G}_1$ representations (see [9, Example VII.1.5]). So, as before,

$$1 \to \xi_a\nu St_{\mathrm{GL}_2} \rtimes \tau_0 \to \xi_a\nu^{\frac{3}{2}} \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0 \to \xi_a\nu \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0 \to 1$$

and

$$1 \to \xi_a\nu^{\frac{3}{2}} \times sp_a \to \xi_a\nu^{\frac{3}{2}} \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0 \to \xi_a\nu^{\frac{3}{2}} \times \omega_a^+ \to 1$$

are both exact sequences of $\widetilde{G}_2$ representations. Moreover, a simple examination of Jacquet modules shows that no two of the following representations are equivalent:

$$\xi_a\nu St_{\mathrm{GL}_2} \rtimes \tau_0,\ \xi_a\nu \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0,\ \xi_a\nu^{\frac{3}{2}} \times sp_a,\ \xi_a\nu^{\frac{3}{2}} \times \omega_a^+.$$

Now if we choose two representations from this list, each induced from a different maximal parabolic, we find a common constituent between the two. The various common constituents are summarized in the following table:

		Submod.	Quotient
	$\xi_a\nu^{\frac{3}{2}} \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0$	$\xi_a\nu St_{\mathrm{GL}_2} \rtimes \tau_0$	$\xi_a\nu \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0$
Submod.	$\xi_a\nu^{\frac{3}{2}} \rtimes sp_a$	$sp_{a,2}$	$Q(\xi_a\nu^{\frac{3}{2}}, sp_a)$
Quotient	$\xi_a\nu^{\frac{3}{2}} \rtimes \omega_a^+$	$Q(\xi_a\nu St_{\mathrm{GL}_2}, \tau_0)$	$\omega_{a,2}^+$

Notice that as one reads from left to right in a given row (resp. top to bottom in a given column), one finds a representation followed by a submodule followed by a quotient. As mentioned in the paragraph after Lemma 5.1, a result of Kudla ([9, Example III.5.4]) identifies the bottom right representation in our table as $\omega_{a,2}^+$. Finally, one can show that any constituent claimed to be irreducible in this proposition has a connected, proper, Jacquet module graph and is thus irreducible by Corollary 5.7. ∎

Remark 5.12 In principle, to compute the various Jacquet modules for representations such as $sp_{a,2}$ or $Q(\xi_a\nu^{\frac{3}{2}}, sp_a)$, we actually compute the Jacquet modules for the representations induced from irreducible representations on the Levi factors of the maximal parabolics using some techniques of [6] and [15]. Given one such representation induced from the Siegel parabolic and another induced from the Klingen parabolic, their Jacquet modules with respect to the Borel will have some overlap. That overlap represents the Jacquet modules for their shared constituent.

We also have the following representations induced from characters of the form $\xi_a\nu^{\frac{1}{2}}$ and $\xi_b\nu^{\frac{1}{2}}$, where $ab^{-1} \in F^\times \setminus (F^\times)^2$.

Proposition 5.13 *Let $a, b \in F^\times$ with $ab^{-1} \in F^\times \setminus (F^\times)^2$. Then the representation $\xi_a\nu^{\frac{1}{2}} \times \xi_b\nu^{\frac{1}{2}} \rtimes \tau_0$ has length four. It has an irreducible submodule $T(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0)$, an irreducible quotient $Q(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0)$, and two further subquotients $Q(\xi_a, sp_b)$ and $Q(\xi_b, sp_a)$.*

Proof We proceed as in Proposition 5.11 and start with the following sequences of $\widetilde{G}_1$ representations

$$1 \to sp_a \to \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0 \to \omega_a^+ \to 1$$

and

$$1 \to sp_b \to \xi_b\nu^{\frac{1}{2}} \rtimes \tau_0 \to \omega_b^+ \to 1.$$

As before, [9, Example VII.1.5] establishes their exactness. Then we use the exactness of parabolic induction from Proposition 4.7 to yield the following exact sequences of $\widetilde{G}_2$ representations:

$$1 \longrightarrow \xi_a\nu^{\frac{1}{2}} \rtimes sp_b \longrightarrow \xi_a\nu^{\frac{1}{2}} \times \xi_b\nu^{\frac{1}{2}} \rtimes \tau_0 \longrightarrow \xi_a \rtimes \omega_b^+ \longrightarrow 1,$$
$$1 \longrightarrow \xi_b\nu^{\frac{1}{2}} \rtimes sp_a \longrightarrow \xi_b\nu^{\frac{1}{2}} \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0 \longrightarrow \xi_b \rtimes \omega_a^+ \longrightarrow 1.$$

However, a simple examination of Jacquet modules shows that

$$\xi_a\nu^{\frac{1}{2}} \rtimes sp_b \not\cong \xi_b\nu^{\frac{1}{2}} \rtimes sp_a \quad \text{and} \quad \xi_a\nu^{\frac{1}{2}} \rtimes \omega_b^+ \not\cong \xi_b\nu^{\frac{1}{2}} \rtimes \omega_a^+.$$

So we have the following subquotients summarized in the following table.

		Submod.	Quotient
	-	$\xi_a\nu^{\frac{1}{2}} \rtimes sp_b$	$\xi_a\nu^{\frac{1}{2}} \rtimes \omega_b^+$
Submod.	$\xi_b\nu^{\frac{1}{2}} \rtimes sp_a$	$T(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0)$	$Q(\xi_b\nu^{\frac{1}{2}}, sp_a)$
Quotient	$\xi_b\nu^{\frac{1}{2}} \rtimes \omega_a^+$	$Q(\xi_a\nu^{\frac{1}{2}}, sp_b^+)$	$Q(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0)$

This table uses the same convention for submodules and quotients as in Proposition 5.4. Finally, we check irreducibility by computing Jacquet module graphs. ■

This exhausts all of the representations induced from regular data. We now turn our attention to representations induced from irregular data. It turns out that we will still be able to get some mileage out of Jacquet module graphs even in the irregular case. However, this method will not be sufficient for us to determine the irreducibility of all the constituents of the principal series.

6 Reducibility Results for $\widetilde{G}_2$-Irregular Case Using Tadić's Technique

Let us consider the simplest case of principal series representations induced from irregular characters.

Proposition 6.1 *Let $a \in F^\times$ and let χ be a unitary character of $F^\times$. If $\chi^2 \neq 1$ or $2s \notin \{0, \pm 1\}$, then the representation $\chi\nu^s \times \chi\nu^s \rtimes \tau_0$ is irreducible. On the other hand, suppose one of the following conditions holds.*

- *$\chi^2 \neq 1$ and $s \in \mathbb{R}$ arbitrary.*
- *$\chi = \xi_b$ with ab^{-1} not a square in $F^\times$ and $2s \notin \{0, \pm 1\}$.*
- *$\chi = \xi_a$ and $2s \notin \{0, \pm 1, \pm 2\}$.*

Then the representation $\chi\nu^s \times \xi_a \rtimes \tau_0$ is irreducible.

Proof In principle, this proposition can be proved using Tadić's technique (and thus proper Jacquet module graphs). However, because the characters are irregular, we have to keep track of the multiplicities of the various Jacquet modules. We will demonstrate with the case of $\chi\nu^s \times \chi\nu^s \rtimes \tau_0$, and the other representations will follow in a similar way.

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Let π be a nonzero irreducible subquotient of $\chi\nu^s \times \chi\nu^s \rtimes \tau_0$. As in Proposition 5.2, we have that $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi) \neq 0$. So without loss of generality, suppose that

$$\chi\nu^s \otimes \chi\nu^s \otimes \tau_0 \leq r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi).$$

As in Proposition 5.2, we apply Frobenius reciprocity twice to yield

$$\begin{aligned} 0 \neq \mathrm{Hom}_{\widetilde{M}_\varnothing}(r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi), \chi\nu^s \otimes \chi\nu^s \otimes \tau_0) &\simeq \mathrm{Hom}_{\widetilde{G}_2}(\pi, \chi\nu^s \times \chi\nu^s \rtimes \tau_0) \\ &\simeq \mathrm{Hom}_{\widetilde{M}_\alpha}(r^{\widetilde{G}_2}_{\widetilde{P}_\alpha}(\pi), \chi\nu^s \times \chi\nu^s \otimes \tau_0) \\ 0 \neq \mathrm{Hom}_{\widetilde{M}_\varnothing}(r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi), \chi\nu^s \otimes \chi\nu^s \otimes \tau_0) &\simeq \mathrm{Hom}_{\widetilde{G}_2}(\pi, \chi\nu^s \times \chi\nu^s \rtimes \tau_0) \\ &\simeq \mathrm{Hom}_{\widetilde{M}_\beta}(r^{\widetilde{G}_2}_{\widetilde{P}_\beta}(\pi), \chi\nu^s \otimes \chi\nu^s \rtimes \tau_0). \end{aligned}$$

However, our restrictions on χ and s assure that

$$\chi\nu^s \times \chi\nu^s \otimes \tau_0 \qquad \text{and} \qquad \chi\nu^s \otimes \chi\nu^s \rtimes \tau_0$$

are irreducible representations of their respective Levi factors (see [5, Lemma 4.5.1] for the $\mathrm{GL}_2(F)$ case and [5, Example VII.1.5] for the $\widetilde{G}_1$ case). So we see that

$$\chi\nu^s \times \chi\nu^s \otimes \tau_0 \leq r^{\widetilde{G}_2}_{\widetilde{P}_\alpha}(\pi) \quad \text{and} \quad \chi\nu^s \otimes (\chi\nu^s \rtimes \tau_0) \leq r^{\widetilde{G}_2}_{\widetilde{P}_\beta}(\pi).$$

By a simple computation of Jacquet modules for $\chi\nu^s \times \chi\nu^s$, an admissible representation of $\mathrm{GL}_2(F)$, the first relation tells us that

$$2 \cdot \chi\nu^s \otimes \chi\nu^s \otimes \tau_0 = r^{\widetilde{M}_\alpha}_{\widetilde{M}_\alpha \cap \widetilde{P}_\varnothing}(\chi\nu^s \times \chi\nu^s \otimes \tau_0) \leq r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi).$$

By a similar computation for $\chi\nu^s \rtimes \tau_0$, a genuine admissible representation of $\widetilde{G}_1$, the second relation tell us that

$$\chi\nu^s \otimes \chi^{-1}\nu^{-s} \otimes \tau_0 \leq r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi).$$

Again, we can follow the argument in Proposition 5.2 to show irreducibility. As one progresses through the computation, one realizes that $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi)$ must contain each irreducible submodule with multiplicity two. Thus,

$$r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi) = r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\chi\nu^s \times \chi\nu^s \rtimes \tau_0),$$

and Proposition 5.2 gives us the irreducibility of π. The proof for $\chi\nu^s \times \xi_a \rtimes \tau_0$ is completely analogous. ■

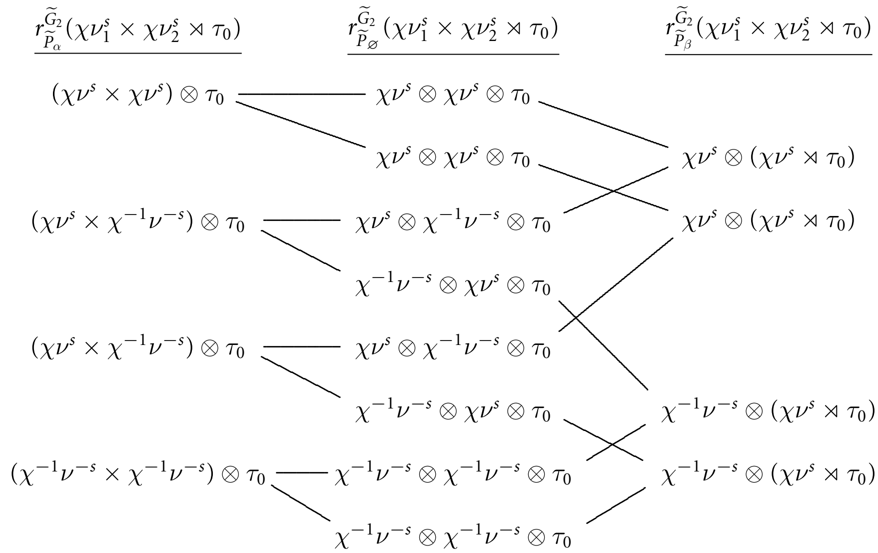

Figure 3: $JMG(\chi\nu^s \times \chi\nu^s \rtimes \tau_0)$

Remark 6.2 There are two key ideas coming up in this proof. First, $\chi\nu^s \times \chi\nu^s$ and $\chi\nu^s \rtimes \tau_0$ are both irreducible representations of their respective groups. Thus, we have the minimal possible number of irreducible constituents belonging to the Jacquet modules with respect to maximal parabolics. Second, the inducing data are only stabilized by one nontrivial element of $W_{\widetilde{G}_2}$. Thus, each irreducible submodule of $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\chi\nu^s \times \chi\nu^s \rtimes \tau_0)$ must have multiplicity two; however, each vertex coming from a constituent of a Jacquet module with respect to a maximal parabolic subgroup is allowed two edges emanating from it. Thus, this additional symmetry of the inducing data need not prevent us from arguing as in Proposition 5.2.

As before, we include $JMG(\chi\nu^s \times \chi\nu^s \rtimes \tau_0)$ in order to help the reader visualize the proof (Figure 3).

Notice that this is a connected, proper, Jacquet module graph.

Now let us consider a reducible representation coming from irregular inducing data.

Proposition 6.3 *Let $a \in F^\times$ be as above. Then the representation $\xi_a\nu^{\frac{1}{2}} \times \xi_a\nu^{-\frac{1}{2}} \rtimes \tau_0$ has length four. In particular, it has the following submodules*

$$\xi_a St_{\mathrm{GL}_2} \rtimes \tau_0 = T_1(\xi_a St_{\mathrm{GL}_2}, \tau_0) \oplus T_2(\xi_a St_{\mathrm{GL}_2}, \tau_0)$$

and the following quotients

$$\xi_a \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0 = Q(\xi_a\nu^{\frac{1}{2}}, sp_a) \oplus Q(\xi_a\nu^{\frac{1}{2}}, \xi_a\nu^{-\frac{1}{2}}, \tau_0).$$

Moreover, all representations appearing as a direct summand are irreducible.

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Proof The representation $\xi_a\nu^{\frac{1}{2}} \times \xi_a\nu^{-\frac{1}{2}} \rtimes \tau_0$ is another example of a representation induced from reducible representations of Levi factors of both maximal parabolics. In particular, we have that

$$1 \longrightarrow \xi_a St_{\mathrm{GL}_2} \longrightarrow \xi_a\nu^{\frac{1}{2}} \times \xi_a\nu^{-\frac{1}{2}} \longrightarrow \xi_a \mathbb{1}_{\mathrm{GL}_2} \longrightarrow 1$$

is an exact sequence of representations of $\mathrm{GL}_2(F)$ and

$$1 \longrightarrow \omega_a^+ \longrightarrow \xi_a\nu^{-\frac{1}{2}} \rtimes \tau_0 \longrightarrow sp_a \longrightarrow 1$$

is an exact sequence of representations of $\widetilde{G}_1$. As before, our irreducible subquotients are realized as common constituents of pairs of representations where each representation in the pair is induced from a different maximal parabolic. Again, we capture this information in the following table

	Submod.	Quotient
$\xi_a\nu^{\frac{1}{2}} \times \xi_a\nu^{-\frac{1}{2}} \rtimes \tau_0$	$\xi_a St_{\mathrm{GL}_2} \rtimes \tau_0$	$\xi_a \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0$
$\xi_a\nu^{\frac{1}{2}} \rtimes \omega_a^+$ (submod.)	$T_1(\xi_a St_{\mathrm{GL}_2}, \tau_0)$	$Q(\xi_a\nu^{\frac{1}{2}}, \xi_a\nu^{-\frac{1}{2}}, \tau_0)$
$\xi_a\nu^{\frac{1}{2}} \rtimes sp_a$ (quotient)	$T_2(\xi_a St_{\mathrm{GL}_2}, \tau_0)$	$Q(\xi_a\nu^{\frac{1}{2}}, sp_a)$

As before, if one reads from left to right in a given row (resp. top to bottom in the left column), one finds a representation followed by a submodule followed by a quotient. Also notice that representations in the top of the remaining columns are unitary, so they break up as a direct sum of the representations below them in the column.

Finally, to show the irreducibility of the representations appearing as direct summands, we simply verify that they have connected, proper, Jacquet module graphs. ∎

Remark 6.4 For inducing data $\eta = \chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0$, let $\mathrm{Stab}(\eta)$ be the stabilizer of η in W_{G_2}, the Weyl group of $\widetilde{G}_2$. One can easily show that if $[W_{G_2} : \mathrm{Stab}(\eta)] = 8$, then the data is regular and if $[W_{G_2} : \mathrm{Stab}(\eta)] = 4$, then the representation induced from η is Weyl conjugate to one of the induced representations found in Propositions 6.1 and 6.3.

So we have finally come to the point where we have exhausted the representations that we could study using Jacquet module graphs. However, there are only a finite number of representations that we have not accounted for. In particular, we still need to consider representations of the form $\pi = \xi_a \times \xi_b \rtimes \tau_0$. Notice that $\mathrm{Stab}(\xi_a \otimes \xi_b \otimes \tau_0)$ has either order 4 or order 8. This tells us that $r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi)$ has at most two distinct constituents with multiplicity at least four. The example in Section 5 shows that such a representation will not have a proper Jacquet module graph. However, we do gain the advantage that π is now unitary. So we can show irreducibility by demonstrating that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}_2}(\pi, \pi) = \mathbb{C} \cdot \mathbb{1}_\pi.$$

In the next section, we will do this via some decomposition results of Shahidi [12] as well as some more direct computations.

7 Poles of Certain Intertwining Operators

The goal of our last section is to show the irreducibility of $\pi = \xi_a \times \xi_b \rtimes \tau_0$ by establishing that $\mathrm{Hom}_{\widetilde{G_2}}(\pi, \pi) = \mathbb{C} \cdot \mathbb{1}_\pi$. Because π is a unitary representation, irreducibility of π is equivalent to this condition on $\mathrm{Hom}_{\widetilde{G_2}}(\pi, \pi)$. We will accomplish this goal with the following steps.

(i) For $\pi = \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ with $s, t \in \mathbb{C}$, we will define the standard intertwining operators corresponding to w_α and w_β for $\mathrm{Re}(s) > RE(t) > 0$. We then meromorphically continue these operators to arbitrary s and t.

(ii) We analyze the poles of the extended intertwining operators. In particular, we show that such a pole exists when $\pi = \xi_a \times \xi_b \rtimes \tau_0$. Consequently, these operators do not analytically continue to the representations in question.

(iii) Lastly, we use a factorization result of Shahidi [12] to reduce the case of an arbitrary intertwining operator to the previously analyzed case. We shall see that all such intertwining operators have a pole at $\pi = \xi_a \times \xi_b \rtimes \tau_0$ and consequently do not extend to an element of $\mathrm{Hom}_{\widetilde{G_2}}(\pi, \pi)$.

Let χ_1 and χ_2 be unitary characters of $F^\times$ and $\pi = \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ be a principal series representation of $\widetilde{G}_2$ with $s, t \in \mathbb{C}$. For any $w \in W_{G_2}$ and any $\mathrm{Re}(s) > \mathrm{Re}(t) > 0$, we can follow the construction of [12, Section 2] to define the standard intertwining operator

$$A(\mathbf{w}, \chi_1, s, \chi_2, t, \cdot)\colon \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0 \to w \cdot (\chi_1\nu^s \times \chi_2\nu^t) \rtimes \tau_0$$

given by the integral

$$A(\mathbf{w}, \chi_1, s, \chi_2, t, f)(g') := \int_{\mathbf{N}^{\mathbf{w}}} f(\mathbf{w}^{-1}\mathbf{n}g')d\mathbf{n},$$

where $\mathbf{w} = [w, \lambda(w)]_L$ and $\mathbf{N}^{\mathbf{w}} = \mathbf{N}_\varnothing \cap \mathbf{w}^{-1}\overline{\mathbf{N}}_\varnothing\mathbf{w}$. This operator converges absolutely in the region $\mathrm{Re}(s) > \mathrm{Re}(t) > 0$ (see [6, Section 6.4]). Hence, by Frobenius reciprocity (Proposition 4.4), such an operator corresponds to a linear functional

$$\Lambda(\mathbf{w}, \chi_1, s, \chi_2, t, \cdot)\colon \pi \to w \cdot (\chi_1\nu^s \otimes \chi_2\nu^t) \otimes \tau_0$$

that is given by the integral

$$\Lambda(\mathbf{w}, \chi_1, s, \chi_2, t, f) = \int_{\mathbf{N}^{\mathbf{w}}} f(\mathbf{w}^{-1}\mathbf{n})d\mathbf{n}.$$

Let us begin by analyzing the case where $w = w_\alpha$.

7.1 The Intertwining Operator $A_0(\mathbf{w}_\alpha, \chi, s, \chi, t, \cdot)$

Our first task is to meromorphically continue our functional $\Lambda(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \cdot)$ to arbitrary s and t. In order to do this, we will rely on some results of Casselman [6, Sections 6 and 7] involving filtration of our induced representations via Bruhat cells. Consider the representation $\pi = \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$, where $\mathrm{Re}(s-t) > 0$. Then we know that the functional

$$f \overset{\Lambda(\mathbf{w}_\alpha,\chi_1,s,\chi_2,t,\,\cdot\,)}{\longrightarrow} \int_{\mathbf{N}^{w_\alpha}} f(\mathbf{w}_\alpha^{-1}\mathbf{n})\,d\mathbf{n} = \int_{\mathbf{N}^{w_\alpha}} f(\mathbf{w}_\alpha \mathbf{n})\,d\mathbf{n}$$

converges absolutely. Notice that $w_\alpha = w_\alpha^{-1}$. For any $f \in V_\pi$, we have that f is compactly supported mod $\widetilde{P}_\varnothing$. Let us we denote $I(\widetilde{P}_\varnothing \mathbf{w}_\alpha \widetilde{P}_\varnothing) \subset V_\pi$ to be the set of functions supported on the double coset $\widetilde{P}_\varnothing \mathbf{w}_\alpha \widetilde{P}_\varnothing$. Because $w_\alpha^{-1} m w_\alpha \in M_\varnothing$ for all $m \in M_\varnothing$ and $w_\alpha^{-1} n w_\alpha \in N_\varnothing$ for all $n \in N_\varnothing \setminus N^{w_\alpha}$, we have that for any $p_1, p_2 \in P_\varnothing$

$$[p_1 w_\alpha p_2, z]_L = [p' w_\alpha n', z]_L = [p', z]_L [w_\alpha, 1]_L [n', 1]_L$$

for some $p' \in P_\varnothing$ and $n' \in N^{w_\alpha}$. Thus, $\widetilde{P}_\varnothing \mathbf{w}_\alpha \widetilde{P}_\varnothing = \widetilde{P}_\varnothing \mathbf{w}_\alpha \mathbf{N}^{\mathbf{w}_\alpha}$ and f is compactly supported mod $\widetilde{P}_\varnothing$. We see that there is a compact subgroup $N_f \subset N^{w_\alpha}$ such that f is supported on the double coset $\widetilde{P}_\varnothing \mathbf{w}_\alpha \mathbf{N}_f$. Because of this compact support, we have that, as a distribution on $I(\widetilde{P}_\varnothing \mathbf{w}_\alpha \widetilde{P}_\varnothing)$,

$$f \longrightarrow \int_{\mathbf{N}^{w_\alpha}} f(\mathbf{w}_\alpha \mathbf{n})\,d\mathbf{n}$$

converges for arbitrary values of s and t.

Next, notice that $\widetilde{P}_\alpha$ is the full inverse image of the closed set $P_\alpha \subset G_2$ under the projection $\widetilde{G}_2 \to G_2$. So $\widetilde{P}_\alpha$ is a closed set containing $\widetilde{P}_\varnothing \mathbf{w}_\alpha \widetilde{P}_\varnothing$. Furthermore, for $p \in P_\varnothing$, let us define a sequence of elements

$$t_p(n) := p \begin{pmatrix} -\varpi^n & 1 \\ & \varpi^{-n} \end{pmatrix} w_\alpha m \begin{pmatrix} 1 & \varpi^n \\ & 1 \end{pmatrix} = p \begin{pmatrix} 1 & \\ \varpi^{-n} & 1 \end{pmatrix}.$$

So the sequence $\{t_p(n)\}_{n\in\mathbb{N}}$ is a sequence of elements in $P_\varnothing w_\alpha P_\varnothing$ with

$$\lim_{n\to\infty} t_p(n) = p \in P_\varnothing.$$

As such, we see that the closure of $P_\varnothing w_\alpha P_\varnothing$ also contains $P_\varnothing$. However, $P_\alpha = P_\varnothing \cup P_\varnothing w_\alpha P_\varnothing$, so the closure of $P_\varnothing w_\alpha P_\varnothing$ is P_α. Taking the inverse image of these sets in the covering group $\widetilde{G}_2$ tells us that the closure of $\widetilde{P}_\varnothing \mathbf{w}_\alpha \widetilde{P}_\varnothing$ is the set $\widetilde{P}_\alpha$.

Now, our goal will be to define a new functional that is defined beyond the region $\mathrm{Re}(s-t) > 0$ and agrees with $\Lambda(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \cdot)$ when $\mathrm{Re}(s-t) > 0$. In fact, we will define our functional on the set of functions supported on $\widetilde{P}_\alpha$ (we will denote this set as $I(\widetilde{P}_\alpha)$), and we can further extend this functional to all of V_π by defining it to be zero for $f \in V_\pi$ with $\mathrm{supp}(f) \cap \widetilde{P}_\alpha = \varnothing$. To this end, for any $f \in I(\widetilde{P}_\alpha) \subset V_\pi$, define

$$f' := f - \chi_1(\varpi)^{-1} |\varpi|^{-s-2} \pi\Big(\mathbf{m}\big(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\big)\Big) f.$$

Notice that $f'(1_{\widetilde{G}_2}) = 0$ for any $f \in I(\widetilde{P}_\alpha)$. Further since f' is an element of $\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$. Thus, we see that for all $p \in \widetilde{P}_\varnothing$

$$f'(p) = \chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0(p) f'(1_{\widetilde{G}_2}) = 0.$$

So we see that $f' \in I(\widetilde{P}_{\varnothing}\mathbf{w}_\alpha\widetilde{P}_{\varnothing})$. So we define

$$\Lambda_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f) := \Lambda(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f').$$

Furthermore, when $\mathrm{Re}(s-t) > 0$, the functional $\Lambda(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f)$ converges absolutely for all $f \in V_\pi$. So we see that for $\mathrm{Re}(s-t) > 0$ and $f \in V_\pi$,

$$\begin{aligned}
\Lambda_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f) &= \Lambda(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f') \\
&= \Lambda(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f) - \chi_1^{-1}(\varpi)|\varpi|^{-s-2}\Lambda\Big(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \pi\Big(\mathbf{m}\big(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\big)\Big) f\Big) \\
&= \Lambda(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f) - \chi_1^{-1}\chi_2(\varpi)|\varpi|^{-s+t-1}\delta_{P_\varnothing}^{\frac{1}{2}}\Big(\mathbf{m}\big(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\big)\Big)\Lambda(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f) \\
&= (1 - \chi_1^{-1}\chi_2(\varpi)q^{s-t})\Lambda(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f).
\end{aligned}$$

Thus, the functional

$$\frac{1}{1 - \chi_1^{-1}\chi_2(\varpi)q^{s-t}}\Lambda_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \cdot)$$

continues the functional $\Lambda(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \cdot)$ to $s, t \in \mathbb{C}$ with possible poles if

$$\chi_1^{-1}\chi_2(\varpi)q^{s-t} = 1.$$

Using Frobenius Reciprocity as we did previously, this functional gives us a normalized intertwining operator

$$A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \cdot)\colon \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0 \to \chi_2\nu^t \times \chi_1\nu^s \rtimes \tau_0$$

as

$$A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f)(g) := \frac{1}{1 - \chi_1^{-1}\chi_2(\varpi)q^{s-t}}\Lambda_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \pi(g)f).$$

Now that we have meromorphically continued the standard intertwining operator $A(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \cdot)$ to arbitrary $s, t \in \mathbb{C}$, we would like to ascertain the poles of this intertwining operator by evaluating it on specific test functions that we describe next. It is worth noting that if $f \in V_\pi$ is an eigenfunction for right translation under some subgroup, then $A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f)$ will be a similar eigenfunction as well. In particular, we would like to consider eigenfunctions for the action of the Iwahori subgroup.

Let $\varsigma\colon I_\varnothing \to \mathbb{C}^1$ be a character of the Iwahori subgroup of G_2. Because $I_\varnothing \simeq \mathbf{I}_\varnothing \subset \widetilde{G}_2$, we can abuse notation and use ς to denote the character of $\mathbf{I}_\varnothing$ given by $\varsigma([k, \lambda(k)]_L) := \varsigma(k)$ that composes ς with the isomorphism between $I_\varnothing$ and $\mathbf{I}_\varnothing$ coming from restricting the splitting $K \to \widetilde{G}_2$ to $I_\varnothing \subset K$. Finally, we define the set $V_\pi^{(\mathbf{I}_\varnothing, \varsigma)}$ to be the set of functions in V_π such that $f(g'[k, \lambda(k)]_L) = \varsigma(k)f(g')$ for all $g' \in \widetilde{G}_2$ and $k \in I_\varnothing$.

While there are many characters of $I_\varnothing$, we are particularly interested in those of the following type. Let η_1, η_2 be characters of $\mathbb{F}_q^\times$. Further, let $P_\varnothing(\mathbb{F}_q)$ (resp. $M_\varnothing(\mathbb{F}_q)$) denote the $\mathbb{F}_q$ points of our standard Borel subgroup (resp. our diagonal torus). Then we have the following compositions of maps

$$I_\varnothing \to P_\varnothing(\mathbb{F}_q) \to M_\varnothing(\mathbb{F}_q) \xrightarrow{\eta_1\otimes\eta_2} \mathbb{C}^1,$$

where the first two maps are given by their respective canonical projections. Thus the relevant characters of the subgroup $I_\varnothing$ will be given by

$$\varsigma(k) = \begin{cases} 1 & \text{if } k \text{ or } {}^t k \in N_\varnothing. \\ \eta_1'(a_1)\eta_2'(a_2) & \text{if } k = m(\operatorname{diag}(a_1, a_2)), \end{cases}$$

where $\eta_i' \colon \mathcal{O}^\times \to \mathbb{C}^1$ is trivial on $1 + \mathcal{P}$ and $\eta_i'(u) = \eta_i(u + \mathcal{P})$ using the identification $\mathcal{O}^\times/(1 + \mathcal{P}) \simeq \mathbb{F}_q^\times$.

Remark 7.1 As mentioned in Remark 6.4, the only outstanding cases where we have yet to ascertain irreducibility occur when χ_1 and χ_2 are quadratic. Thus, $\chi_i|_{1+\mathcal{P}} = 1$, since $1 + \mathcal{P} \subset (\mathcal{O}^\times)^2$.

Next we would like to establish that $V_\pi^{(\mathbf{I}_\varnothing,\varsigma)} \neq 0$ for χ_i with $\chi_i|_{1+\mathcal{P}} = 1$ and $\varsigma = \chi_1|_{\mathcal{O}^\times} \otimes \chi_2|_{\mathcal{O}^\times}$. By doing this, we will also find an example of the test vectors that we will use for computing the poles and zeros of the normalized intertwining operators.

Lemma 7.2 *Let $\chi_i \colon F^\times \to \mathbb{C}^1$ be unitary characters such that $\chi_i|_{1+\mathcal{P}} = 1$. If $\pi = \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$, then $V_\pi^{(\mathbf{I}_\varnothing,\varsigma)} \neq 0$ for $\varsigma = \chi_1|_{\mathcal{O}^\times} \otimes \chi_2|_{\mathcal{O}^\times}$.*

Proof By the Iwasawa decomposition and following the arguments in Proposition 4.6, we have

$$\widetilde{G}_2 = \widetilde{P}_\varnothing \mathbf{K} = \bigcup_{w\in W_{G_2}} \widetilde{P}_\varnothing \mathbf{w} \mathbf{I}_\varnothing.$$

et us define a function $f_0 \in V_\pi$ such that $\operatorname{supp}(f_0) \subset \widetilde{P}_\varnothing \mathbf{I}_\varnothing$ and

$$f_0(\mathbf{nm}(\operatorname{diag}(a_1, a_2))k') = \delta_P^{\frac{1}{2}}(\operatorname{diag}(a_1, a_2))\chi_1(a_1)|a|^s\chi_2|a_2|^t\varsigma(k').$$

for all $\mathbf{nm}(\operatorname{diag}(a_1, a_2)) \in \widetilde{P}_\varnothing$ and $k' \in \mathbf{I}_\varnothing$. Let us show that f_0 is well defined. For any $k' \in \widetilde{P}_\varnothing \cap \mathbf{I}_\varnothing$, we have $k' = \mathbf{m}(\operatorname{diag}(a_1, a_2))\mathbf{n}$ for some $a_i \in \mathcal{O}^\times$ and $n \in N_\varnothing$. Thus, for f_0 to be well defined, we must have that

$$\chi_1(a_1)|a_1|^{s+2}\chi_2(a_2)|a_2|^{t+1} = f_0(k') = \varsigma(k') = \chi_1(a_1)\chi_2(a_2).$$

However, $a_i \in \mathcal{O}^\times$ so $|a_i| = 1$. One should also note that f_0 is fixed by $\ker(\varsigma)$, which is an index $q - 1$ compact open subgroup of $\mathbf{I}_\varnothing$, which justifies that our f_0 truly belongs in V_π. Thus we have constructed a nonzero vector in $V_\pi^{(\mathbf{I}_\varnothing,\varsigma)}$. ∎

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We are now able to compute some of the zeros and poles for the intertwining operators of interest. However, we first recall the definition of length for an element of the Weyl group. In Section 2, we fixed an identification of W_{G_2} with certain elements in G_2. Any $w \in W_{G_2}$ can be decomposed as a product of generators w_α and w_β. Let the *length* of w (denoted $\ell(w)$) be defined as

$$\ell(w) = \min\Big\{ n \in \mathbb{N} \mid w = \prod_{i=1}^{n} w_i, w_i \in \{w_\alpha, w_\beta\}\Big\}.$$

As mentioned in Section 2, the decomposition of $w \in W_{G_2}$ having the minimal number of elements is unique for all of the elements of W_{G_2} except the element

$$w = w_\alpha w_\beta w_\alpha w_\beta = w_\beta w_\alpha w_\beta w_\alpha,$$

which has two factorizations with four elements and no shorter factorizations. Finally, we can state the following proposition.

Proposition 7.3 *Let $f \in \pi^{(\mathbf{I}_\varnothing, \chi_1\otimes\chi_2)}$. If $\chi_1\chi_2^{-1}$ is unramified, then*

$$A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f)(\mathbf{w}) = \begin{cases} f(\mathbf{w}_\alpha\mathbf{w}) - \frac{(1-q^{-1})\chi_1(-1)}{1-\chi_1^{-1}\chi_2(\varpi)q^{s-t}} f(\mathbf{w}) & \text{if } \ell(w_\alpha w) > \ell(w),\\ \frac{1}{q} f(\mathbf{w}_\alpha\mathbf{w}) - \frac{(1-q^{-1})\chi_1(-1)\chi_1^{-1}\chi_2(\varpi)q^{s-t}}{1-\chi_1^{-1}\chi_2(\varpi)q^{s-t}} f(\mathbf{w}) & \text{if } \ell(w_\alpha w) < \ell(w).\end{cases}$$

If $\chi_1\chi_2^{-1}$ is ramified, then

$$A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f)(\mathbf{w}) = \begin{cases} f(\mathbf{w}_\alpha\mathbf{w}) & \text{if } \ell(w_\alpha w) > \ell(w),\\ \frac{1}{q} f(\mathbf{w}_\alpha\mathbf{w}) & \text{if } \ell(w_\alpha w) < \ell(w).\end{cases}$$

Proof The proof is by direct computation. First, let us fix the following notation: $n_0^+(x) := m\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right)$ and $n_0^-(x) = {}^t n_0^+(x)$. For our choice of f, we have

$$A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f)(\mathbf{w}) = \frac{1}{1-\chi_1^{-1}\chi_2(\varpi)q^{s-t}} \times \int_F \Big[f(\mathbf{w}_\alpha^{-1}\mathbf{n}_0^+(x)\mathbf{w}) - \chi_1^{-1}(\varpi)|\varpi|^{-s-2} f(\mathbf{w}_\alpha^{-1}\mathbf{n}_0^+(x)\mathbf{m}\left(\begin{smallmatrix} \varpi & \\ & 1\end{smallmatrix}\right)\mathbf{w})\Big]\, dx.$$

Simplifying this expression and noticing that $\mathbf{w}_\alpha^{-1} = \mathbf{w}_\alpha$, we see that

$$A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f)(\mathbf{w}) = \frac{1}{1-\chi_1^{-1}\chi_2(\varpi)q^{s-t}} \times \int_F \Big[f(\mathbf{w}_\alpha\mathbf{n}_0^+(x)\mathbf{w}) - \chi_1^{-1}\chi_2(\varpi)|\varpi|^{(s+t-1)} f(\mathbf{w}_\alpha\mathbf{n}_0^+(\varpi^{-1}x)\mathbf{w})\Big]\, dx$$

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So let us now suppose that $\ell(w_\alpha w) > \ell(w)$, then $w^{-1}n_0^+(x)w \in N_\varnothing$. Also note that

$$w_\alpha n_0^+(a) = n_0^+(a^{-1})m\begin{pmatrix} -a^{-1} & \\ & a \end{pmatrix}n_0^-(a^{-1}),$$

so we see that

$$\begin{aligned}
&A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f)(\mathbf{w}) \\
&\quad = \frac{1}{1-\chi_1^{-1}\chi_2(\varpi)q^{s-t}}\Big(\int_{\mathcal{P}}(1-\chi_1^{-1}\chi_2(\varpi)|\varpi|^{-s+t-1})f(\mathbf{w}_\alpha\mathbf{w})dx \\
&\qquad + \int_{\mathcal{O}\setminus\mathcal{P}}\Big[f(\mathbf{w}_\alpha\mathbf{w}) - \chi_1^{-1}\chi_2(\varpi)|\varpi|^{-s+t-1}f(\mathbf{m}\begin{pmatrix} -\varpi x^{-1} & \\ & \varpi^{-1}x \end{pmatrix}\mathbf{w})\Big]dx \\
&\qquad + \int_{F\setminus\mathcal{O}}\Big[f(\mathbf{m}\begin{pmatrix} -x^{-1} & \\ & x \end{pmatrix}\mathbf{w}) - \chi_1^{-1}\chi_2(\varpi)|\varpi|^{-s+t-1}f(\mathbf{m}\begin{pmatrix} -\varpi x^{-1} & \\ & \varpi^{-1}x \end{pmatrix}\mathbf{w})\Big]dx\Big).
\end{aligned}$$

We handle each integral individually. First,

$$\int_{\mathcal{P}}(1-\chi_1^{-1}\chi_2(\varpi)q^{s-t+1})f(\mathbf{w}_\alpha\mathbf{w})dx = q^{-1}(1-\chi_1^{-1}\chi_2(\varpi)q^{s-t+1})f(\mathbf{w}_\alpha\mathbf{w}). \tag{7.1}$$

Next,

$$\begin{aligned}
&\int_{\mathcal{O}\setminus\mathcal{P}}\Big[f(\mathbf{w}_\alpha) - \chi_1^{-1}\chi_2(\varpi)|\varpi|^{-s+t-1}f(\mathbf{m}\begin{pmatrix} -\varpi x^{-1} & \\ & \varpi^{-1}x \end{pmatrix}\mathbf{w})\Big]dx \\
&\quad = (1-q^{-1})f(\mathbf{w}_\alpha\mathbf{w}) - \int_{\mathcal{O}\setminus\mathcal{P}}\chi_1(-1)\chi_1^{-1}\chi_2(x)f(\mathbf{w})dx \\
&\quad = \begin{cases} (1-q^{-1})\big(f(\mathbf{w}_\alpha\mathbf{w}) - \chi_1(-1)f(\mathbf{w})\big) & \text{if } \chi_1^{-1}\chi_2 \text{ is unramified,} \\ (1-q^{-1})f(\mathbf{w}_\alpha\mathbf{w}) & \text{if } \chi_1^{-1}\chi_2 \text{ is ramified.} \end{cases}
\end{aligned} \tag{7.2}$$

Finally,

$$\begin{aligned}
&\int_{F\setminus\mathcal{O}}\Big[f(\mathbf{m}\begin{pmatrix} -x^{-1} & \\ & x \end{pmatrix}\mathbf{w}) - \frac{\chi_2(\varpi)}{\chi_1(\varpi)}|\varpi|^{(-s+t-1)}f(\mathbf{m}\begin{pmatrix} -\varpi x^{-1} & \\ & \varpi^{-1}x \end{pmatrix}\mathbf{w})\Big]dx \\
&\quad = \chi_1(-1)\int_{F\setminus\mathcal{O}}\Big[\chi_1^{-1}\chi_2(x)|x|^{-s+t-1}f(\mathbf{w}) - \chi_1^{-1}\chi_2(x)|x|^{-s+t-1}f(\mathbf{w})\Big]dx \\
&\quad = 0.
\end{aligned} \tag{7.3}$$

Summing equations (7.1)–(7.3) proves our result when $\ell(w_\alpha w) > \ell(w)$.

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The case that $\ell(w_\alpha w) < \ell(w)$ follows similarly except that $w^{-1}n_0^+(x)w \in {}^tN_\varnothing$, so we have

$$\begin{aligned}
&A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f)(\mathbf{w}) \\
&\quad = \frac{1}{1-\chi_1^{-1}\chi_2(\varpi)q^{s-t}}\Big(\int_{\mathcal{P}^2}(1-\chi_1^{-1}\chi_2(\varpi)|\varpi|^{-s+t-1})f(\mathbf{w}_\alpha\mathbf{w})dx \\
&\qquad + \int_{\mathcal{P}\setminus\mathcal{P}^2}\Big[f(\mathbf{w}_\alpha\mathbf{w}) - \chi_1^{-1}\chi_2(\varpi)|\varpi|^{-s+t-1}f(\mathbf{m}(\begin{smallmatrix}-\varpi x^{-1} & \\ & \varpi^{-1}x\end{smallmatrix})\mathbf{w})\Big]\,dx \\
&\qquad + \int_{F\setminus\mathcal{P}}\Big[f(\mathbf{m}(\begin{smallmatrix}-x^{-1} & \\ & x\end{smallmatrix})\mathbf{w}) - \frac{\chi_2(\varpi)}{\chi_1(\varpi)}|\varpi|^{-s+t-1}f(\mathbf{m}(\begin{smallmatrix}-\varpi x^{-1} & \\ & \varpi^{-1}x\end{smallmatrix})\mathbf{w})\Big]\,dx\Big),
\end{aligned}$$

which we compute as before. ■

This gives rise to the following corollary.

Corollary 7.4 *Let $\pi = \chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$. Then*

$$(7.4)\quad A_0(\mathbf{w}_\alpha, \chi_2, t, \chi_1, s, A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, f)) = \begin{cases} \frac{(1-\chi_1^{-1}\chi_2(\varpi)q^{s-t-1})(1-\chi_1\chi_2^{-1}(\varpi)q^{-s+t-1})}{(1-\chi_1^{-1}\chi_2(\varpi)q^{s-t})(1-\chi_1\chi_2^{-1}(\varpi)q^{-s+t})} f & \textit{if } \chi_1^{-1}\chi_2 \textit{ is unramified,} \\ q^{-1}f & \textit{if } \chi_1^{-1}\chi_2 \textit{ is ramified.}\end{cases}$$

Proof First, let us assume that the inducing data is regular. Then by Frobenius reciprocity, we see that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}_2}(\pi,\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{M}_\varnothing}(r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi), \chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0) = 1.$$

So we find

$$A_0(\mathbf{w}_\alpha, \chi_2, t, \chi_1, s, A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \cdot)) = C(\mathbf{w}_\alpha, \pi)\mathbb{1}_\pi,$$

where

$$C(\mathbf{w}_\alpha, \pi) = A_0(\mathbf{w}_\alpha, \chi_2, t, \chi_1, s, A_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f))(1_{\widetilde{G}_2})$$

for any $f \in \pi^{(\mathbf{I}_\varnothing, \chi_1\otimes\chi_2)}$ with $f(1_{\widetilde{G}_2}) = 1$. Once we have this result for regular data, we use meromorphic continuation to establish it for arbitrary data. ■

So let us specialize to the case that χ_1 and χ_2 are quadratic and $\chi_1\chi_2$ is unramified. Then $\chi_1\chi_2(\varpi) = -1$ implies that (7.4) has a pole when $s - t \in \frac{\pi i}{\log q}(1+2\mathbb{Z})$ and $\chi_1\chi_2(\varpi) = 1$ implies that (7.4) has a pole when $s - t \in \frac{2\pi i}{\log q}\mathbb{Z}$. In either case, we have the following.

Corollary 7.5 *For χ_1, χ_2 quadratic with $\chi_1\chi_2$ unramified, the intertwining operator $A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \cdot)$ has a pole when $\chi_1\nu^s = \chi_2\nu^t$.*

7.2 The Intertwining Operator $A_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, \cdot)$

We will now analyze the intertwining operator for the long root β. It will be rather similar to the previous section, except we must entertain a slight complication. In the case of $A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \cdot)$, we were fortunate that all matrix multiplications belonged to the Siegel parabolic $\widetilde{P}_\alpha$ where the Leray cocycle is trivial and the splitting is given by $p \mapsto [p, 1]_L$. However, we now must concern ourselves with elements outside of $\widetilde{P}_\alpha$. As we will see, the splitting of $K \to \widetilde{G}_2$ will be sufficient for the computations, but this does force us to deal with the map $\lambda\colon K \to \mathbb{C}^1$ that allows us to define our splitting $k \mapsto [k, \lambda(k)]_L$.

Let us introduce some more notation by setting $n_\pm(c) := n^\pm\left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right)$, where

$$n^+\colon \operatorname{Sym}_2(F) \to N_\alpha \qquad (\text{resp. } n^-\colon \operatorname{Sym}_2(F) \to {}^tN_\alpha)$$

identifies $\operatorname{Sym}_2(F)$ with the unipotent radical (resp. opposite unipotent radical) our Siegel Parabolic P_α. We are interested in computing $\lambda(n_-(c))$ for $c \in \mathcal{O}$.

First, notice that for any $k_1, k_2 \in K$,

$$\lambda(k_1)\lambda(k_2)c_L(k_1, k_2) = \lambda(k_1k_2).$$

We now apply this formula to $k_0 = w_\beta n_-(c) = n_+(-c)w_\beta$ in two different ways. First,

$$\lambda(k_0) = \lambda(n_+(-c))\lambda(w_\beta)c_L(n_+(-c), w_\beta) = 1$$

all of which follow straight from the definitions of λ (see [10, Remark 8.5.2]) and $c_L(\,\cdot\,,\,\cdot\,)$ (see [11, Theorem 4.1]). Secondly,

$$1 = \lambda(k_0) = \lambda(w_\beta)\lambda(n_-(c))c_L(w_\beta, n_-(c)).$$

So we ultimately see that

$$\lambda(n_-(c)) = c_L(w_\beta, n_-(c))^{-1}, \tag{7.5}$$

which we can compute using the results of Rao [11, Chapter 2 and Theorem 4.1]. In fact, we have the following result.

Proposition 7.6 *Let $c = u\varpi^t$ for some $u \in \mathcal{O}^\times$ and $t \geq 0$. Further, let $\left(\frac{\cdot}{\varpi}\right)_F$ denote the Legendre symbol of F. Then*

$$\lambda(n_-(c)) = \left[\left(\frac{-2u}{\varpi}\right)_F \dot{\gamma}(\dot{\psi})\right]^{\tau(t)},$$

where $\tau(t) = 0$ if t is even and $\tau(t) = 1$ if t is odd. Moreover, $\dot{\gamma}(\dot{\psi})$ is the Weil index of the character $\dot{\psi}\colon \mathcal{O}/\mathcal{P} \simeq \mathbb{F}_q \to \mathbb{C}^1$ given by

$$\dot{\psi}(x + \mathcal{P}) = \psi(\varpi^{-1}x).$$

Remark 7.7 Note that $\dot{\gamma}(\dot{\psi})$ is computed in [11]. It also has the property $\dot{\gamma}(\dot{\psi})^2 = \xi_\varpi(-1)$.

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Proof Let W be the symplectic space such that $\mathrm{Sp}_2(F) = \mathrm{Sp}(W)$. Then W has the complete polarization $W = X + Y$ so that $\mathrm{Stab}_{G_2}(Y) = P_\alpha$. In particular, $Y = \{(0, 0, x, y) \mid x, y \in F\}$. By [11, Theorem 4.1], we have,

$$c_L(w_\beta, n_-(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix})) = \gamma(\psi \circ \tfrac{1}{2} q_L(Y, Yn_-(\begin{smallmatrix} 0 & 0 \\ 0 & -c \end{smallmatrix}), Yw_\beta)).$$

where

$$\gamma(\psi \circ \tfrac{1}{2} q_L(Y, Yn_-(\begin{smallmatrix} 0 & 0 \\ 0 & -c \end{smallmatrix}), Yw_\beta))$$

is the Weil index of the character of the character of second degree

$$\psi \circ \tfrac{1}{2} q_L(Y, Yn_-(\begin{smallmatrix} 0 & 0 \\ 0 & -c \end{smallmatrix}), Yw_\beta)$$

and $q_L(L_1, L_2, L_3)$ is a quadratic form called the Leray invariant associated to the triple (L_1, L_2, L_3), where the L_i are maximal isotropic subspaces of V. All of these quantities are defined and computed in [11, Chapter 2 and Appendix].

In our case,

$$Yn_-(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}) = \{(0, -cy, x, y) \mid x, y \in F\}$$

and

$$Yw_\beta = \{(0, -y, x, 0) \mid x, y \in F\}.$$

Notice that for any $i \neq j \in \{1, 2, 3\}$, $Y_i \cap Y_j = \{(0, 0, x, 0) \mid x \in F\} =: M$. Thus $L_i \subset M^\perp$ for all i. Finally, the map stabilizing L_1/M that sends L_2/M to L_3/M is given by the matrix $(\begin{smallmatrix} 1 & c^{-1} \\ 0 & 1 \end{smallmatrix})$ and the quadratic form on L_2/M is given by

$$\tfrac{1}{2}\langle(-cy, y), (-cy, y)(\begin{smallmatrix} 1 & c^{-1} \\ 0 & 1 \end{smallmatrix})\rangle = \tfrac{1}{2} cy^2,$$

which is equivalent to the quadratic form $Q(y) = 2cy^2$. Finally, for our fixed additive character ψ, we find that

$$c_L(w_\beta, n_-(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix})) = \gamma_F(\psi \circ 2cx^2) = \gamma_F(\psi_{2c} \circ x^2) = \gamma_F(2c, \psi \circ x^2)\gamma_F(\psi \circ x^2).$$

However, our choice of additive character $\gamma_F(\psi \circ x^2) = 1$ and the formula from Rao's appendix in [11] give us

$$\gamma_F(2c, \psi \circ x^2) = \left[\left(\frac{2u}{\varpi} \right)_F \dot{\gamma}(\dot{\psi}) \right]^{\tau(t)}. \tag{7.6}$$

So by combining equations (7.5) and (7.6), we find

$$\lambda(n_-(c)) = c_L(w_\beta, n_-(c))^{-1} = \gamma_F(2c, \psi \circ x^2)^{-1} = \left[\left(\frac{-2u}{\varpi} \right)_F \dot{\gamma}(\dot{\psi}) \right]^{\tau(t)}.$$

Any missing details can be found in [11] and are left to the reader. ∎

Remark 7.8 In order to abbreviate notation, we denote $\gamma_F(\psi_a \circ x^2)$ simply as $\gamma_F(\psi_a)$. It is also worth noting that $\gamma_F(\psi_{ab^2}) = \gamma_F(\psi_a)$ for all $b \in F^\times$. This follows from noticing that all our computations depend only on square classes.

We can now compute the intertwining operator $A_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, \cdot\,)$ in an analogous way to $A_0(\mathbf{w}_\alpha, \chi_1, s, \chi_2, t, \cdot\,)$. Let us return to the representation $\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ where $\operatorname{Re}(t) > 0$. As before, the standard intertwining operator

$$\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0 \overset{A(\mathbf{w}_\beta,\chi_1,s,\chi_2,t,\cdot)}{\longrightarrow} \chi_1\nu^s \times \chi_2^{-1}\nu^{-t} \rtimes \tau_0$$

is associated, by Frobenius reciprocity, to the functional

$$f \overset{\Lambda(\mathbf{w}_\beta,\chi,s,\chi,t,\cdot)}{\longrightarrow} \int_{\mathbf{N}^\beta} f(\mathbf{w}_\beta^{-1}\mathbf{n})\, d\mathbf{n} = \int_{\mathbf{N}^\beta} \chi_2(-1) f(\mathbf{w}_\beta \mathbf{n})\, d\mathbf{n}.$$

Here we are using that $w_\beta^{-1} = m(\operatorname{diag}(1,-1))w_\beta$. Both the intertwining operator and the functional converge absolutely for given $\operatorname{Re}(t) > 0$.

As before, let $I(\widetilde{P}_\varnothing \mathbf{w}_\beta \widetilde{P}_\varnothing) \subset V_\pi$ denote the set of functions supported on the double coset $\widetilde{P}_\varnothing \mathbf{w}_\beta \widetilde{P}_\varnothing$. Since $f \in V_\pi$ is compactly supported mod $\widetilde{P}_\varnothing$, we see that as a distribution on $I(\widetilde{P}_\varnothing \mathbf{w}_\beta \widetilde{P}_\varnothing)$,

$$f \longrightarrow \int_{\mathbf{N}^\beta} f(\mathbf{w}_\beta^{-1}\mathbf{n})\, d\mathbf{n}$$

converges regardless of the values for s and t.

As in the previous case, we would like define a new function that is defined beyond the region $\operatorname{Re}(t) > 0$. This functional should agree with $\Lambda(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, \cdot)$ when $\operatorname{Re}(t) > 0$. In particular, we will define our functional on $I(\widetilde{P}_\varnothing \mathbf{w}_\beta \widetilde{P}_\varnothing)$, extend it to functions supported on the closure $\widetilde{P}_\varnothing \mathbf{w}_\beta \widetilde{P}_\varnothing$, which is now $\widetilde{P}_\beta$ (we will denote such functions as $I(\widetilde{P}_\beta)$) and then extend this distribution by zero to all V_π. To do this, define

$$f' := f - \chi_2^{-1}(\varpi)|\varpi|^{-t-1}\pi\left(\mathbf{m}\left(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right)\right) f.$$

Notice that for any $f \in I(\widetilde{P}_\beta)$, $f' \in I(\widetilde{P}_\varnothing \mathbf{w}_\beta \widetilde{P}_\varnothing)$. Thus we can define

$$\Lambda_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f) := \Lambda(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f').$$

Now, when $\operatorname{Re}(t) > 0$, $\Lambda(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f)$ converges absolutely for all $f \in V_\pi$. Thus, for $\operatorname{Re}(t) > 0$ and $f \in V_\pi$, we have

$$\begin{aligned}
\Lambda_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f) &= \Lambda(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f') \\
&= \Lambda(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f) - \chi_2^{-1}(\varpi)|\varpi|^{-t-1}\Lambda(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, \pi\left(\mathbf{m}\left(\begin{smallmatrix}1 & \\ & \varpi\end{smallmatrix}\right)\right) f) \\
&= \Lambda(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f) - \chi_2^{-2}(\varpi)|\varpi|^{-2t-1}\delta_{P_\varnothing}^{\frac{1}{2}}(\mathbf{m}\left(\begin{smallmatrix}1 & \\ & \varpi^{-1}\end{smallmatrix}\right))\Lambda(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f) \\
&= (1 - \chi_2^{-2}(\varpi)q^{2t})\Lambda(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f).
\end{aligned}$$

Since we have meromorphically continued the functional $\Lambda(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, \cdot)$ to a functional

$$\frac{1}{1 - \chi_2^{-2}(\varpi)q^{2t}}\Lambda_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, \cdot)$$

to arbitrary s and t, we can use Frobenius reciprocity to define an intertwining operator

$$A_0(\mathbf{w}_\beta,\chi_1,s,\chi_2,t,f)(g) := \frac{1}{1-\chi_2^{-2}(\varpi)q^{2t}}\Lambda_0(\mathbf{w}_\beta,\chi_1,s,\chi_2,t,\pi(g)f)$$

that meromorphically continues $A(\mathbf{w}_\beta,\chi_1,s,\chi_2,t,\cdot)$ to arbitrary $s,t\in\mathbb{C}$.

Also, we would like to compute $A_0(\mathbf{w}_\beta,\chi_1,s,\chi_2,t,f)$ for

$$f\in(\chi_1\nu^s\times\chi_2\nu^t\rtimes\tau_0)^{(\mathbf{I}_\varnothing,\chi_1\otimes\chi_2)}$$

analogously to the previous intertwining operator.

Proposition 7.9 *Let $f\in(\chi_1\nu^s\times\chi_2\nu^t\rtimes\tau_0)^{(\mathbf{I}_\varnothing,\chi_1\otimes\chi_2)}$. If χ_2 is unramified, we have*

$$A_0(\mathbf{w}_\beta,\chi_1,s,\chi_2,t,f)(\mathbf{w}) = \begin{cases} f(\mathbf{w}_\beta\mathbf{w}) - \frac{(1-q^{-1})}{1-\chi_2^{-2}(\varpi)q^{2t}}f(\mathbf{w}) & \text{if } \ell(w_\beta w)>\ell(w),\\ q^{-1}f(\mathbf{w}_\beta\mathbf{w}) - \frac{(1-q^{-1})\chi_2^{-2}(\varpi)q^{2t}}{1-\chi_2^{-2}(\varpi)q^{2t}}f(\mathbf{w}) & \text{if } \ell(w_\beta w)<\ell(w).\end{cases}$$

If $\chi_2^{-1}\xi_\varpi$ is unramified, we have

$$A_0(\mathbf{w}_\beta,\chi_1,s,\chi_2,t)(\mathbf{w}) = \begin{cases} f(\mathbf{w}_\beta^{-1}\mathbf{w}) - \frac{\chi_2^{-1}(2\varpi)\dot\gamma(\dot\psi)(1-q^{-1})q^{t}}{1-\chi_2^{-2}(\varpi)q^{2t}}f(\mathbf{w}) & \text{if } \ell(w_\beta w)>\ell(w),\\ q^{-1}f(\mathbf{w}_\beta^{-1}\mathbf{w}) - \frac{\chi_2^{-1}(2\varpi)\dot\gamma(\dot\psi)(1-q^{-1})q^{t}}{1-\chi_2^{-2}(\varpi)q^{2t}}f(\mathbf{w}) & \text{if } \ell(w_\beta w)<\ell(w).\end{cases}$$

If χ_2 and $\chi_2^{-1}\xi_\varpi$ are ramified, we have

$$A_0(\mathbf{w}_\beta,\chi_1,s,\chi_2,t,f)(\mathbf{w}) = \begin{cases} f(\mathbf{w}_\beta^{-1}\mathbf{w}) & \text{if } \ell(w_\beta w)>\ell(w),\\ q^{-1}f(\mathbf{w}_\beta^{-1}\mathbf{w}) & \text{if } \ell(w_\beta w)<\ell(w).\end{cases}$$

Proof The proof is done by direct computation. For our choice of f, we see that

$$\begin{aligned}&A_0(\mathbf{w}_\beta,\chi_1,s,\chi_2,t,f)(\mathbf{w})\\ &= \frac{1}{1-\chi_2^{-2}(\varpi)q^{2t}}\int_F f(\mathbf{w}_\beta^{-1}\mathbf{n}_+(x)\mathbf{w}) - \chi_2^{-1}(\varpi)|\varpi|^{-t-1}f(\mathbf{w}_\beta^{-1}\mathbf{n}_+(x)\mathbf{m}(1,\varpi)\mathbf{w})dx\\ &= \frac{1}{1-\chi_2^{-2}(\varpi)q^{2t}}\int_F f(\mathbf{w}_\beta^{-1}\mathbf{n}_+(x)\mathbf{w}) - |\varpi|^{-2(t+1)}f(\mathbf{w}_\beta^{-1}\mathbf{n}_+(\varpi^{-2}x)\mathbf{w})dx.\end{aligned}$$

Now let $\ell(w_\beta w)>\ell(w)$, so that we have $w^{-1}n_+(x)w\in N_\varnothing$. Also note that

$$\begin{aligned}&[w_\beta^{-1},1]_L[n_+(a),1]_L =\\ &\qquad [n_+(-a^{-1}),1]_L[m(1,a^{-1}),\lambda(n_-(a^{-1})^{-1}]_L[n_-(a^{-1}),\lambda(n_-(a^{-1})]_L.\end{aligned}$$

Moreover, since $\lambda(n_-(a^{-1})) \in \mathbb{C}^1$, $\lambda(n_-(a^{-1}))^{-1} = \overline{\lambda(n_-(a^{-1}))}$. Therefore, we have

$$A_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f)(\mathbf{w}) = \frac{1}{1-\chi_2^{-2}(\varpi)q^{2t}}\Bigg(\int_{\mathcal{P}^2}(1-\chi_2^{-2}(\varpi)|\varpi|^{-2t-2})f(\mathbf{w}_\beta^{-1}\mathbf{w})dx$$

$$+\int\limits_{\mathcal{O}\setminus\mathcal{P}^2}\Big[f(\mathbf{w}_\beta^{-1}\mathbf{w})-\chi_2^{-2}(\varpi)|\varpi|^{-2t-2}f([m\begin{pmatrix}1 & \\ & \varpi^2x^{-1}\end{pmatrix}, \overline{\lambda(n_-(\varpi^2x^{-1}))}]_L\mathbf{w})\Big]\,dx$$

$$+\int\limits_{F\setminus\mathcal{O}}\Big[f([m\begin{pmatrix}1 & \\ & x^{-1}\end{pmatrix}, \overline{\lambda(n_-(x^{-1}))}]_L\mathbf{w})$$

$$-\chi_2^{-2}(\varpi)|\varpi|^{-2t-2}f([m\begin{pmatrix}1 & \\ & \varpi^2x^{-1}\end{pmatrix}, \overline{\lambda(n_-(\varpi^2x^{-1}))}]_L\mathbf{w})\Big]\,dx\Bigg)$$

We handle these integrals individually. First,

$$\int_{\mathcal{P}^2}(1-\chi_2^{-2}(\varpi)|\varpi|^{-2t-2})f(\mathbf{w}_\beta^{-1}\mathbf{w})\,dx = q^{-2}(1-\chi_2^{-2}(\varpi)q^{2t+2})f(\mathbf{w}_\beta^{-1}\mathbf{w}). \tag{7.7}$$

Next,

$$\int\limits_{\mathcal{O}\setminus\mathcal{P}^2}\Big[f(\mathbf{w}_\beta^{-1}\mathbf{w})-\chi_2^{-2}(\varpi)|\varpi|^{-2t-2}f([m\begin{pmatrix}1 & \\ & \varpi^2x^{-1}\end{pmatrix}, \overline{\lambda(n_-(\varpi^2x^{-1}))}]_L\mathbf{w})\Big]\,dx$$

$$= \int\limits_{\mathcal{O}\setminus\mathcal{P}^2}\Big[f(\mathbf{w}_\beta^{-1}\mathbf{w})-\chi_2^{-1}(x)|x|^{-t-1}\gamma(\psi_{2\varpi^2x^{-1}})f(\mathbf{w})\Big]\,dx \tag{7.8}$$

$$= \sum_{i=0}^{1}\int_{\mathcal{P}^i\setminus\mathcal{P}^{i+1}}\Big[f(\mathbf{w}_\beta^{-1}\mathbf{w})-\chi_2^{-1}(x)|x|^{-t-1}\gamma(\psi_{2x})f(\mathbf{w})\Big]\,dx.$$

A simple computation from the Appendix in [11] shows that

$$\gamma(\psi_{2x}) = \begin{cases}1 & \text{if } x = u\varpi^{2j},\ j\in\mathbb{Z},\\ \xi_\varpi(2u)\dot{\gamma}(\dot{\psi}) & \text{if } x = u\varpi^{2j+1},\ j\in\mathbb{Z}.\end{cases}$$

Thus equation (7.8) becomes

$$\int_{\mathcal{O}\setminus\mathcal{P}}\big[f(\mathbf{w}_\beta^{-1}\mathbf{w})-\chi_2^{-1}(x)f(\mathbf{w})\big]\,dx$$

$$+\int_{\mathcal{P}\setminus\mathcal{P}^2}\big[f(\mathbf{w}_\beta^{-1}\mathbf{w})-\xi_\varpi(2\varpi^{-1}x)\dot{\gamma}(\dot{\psi})\chi_2^{-1}(x)q^{t+1}f(\mathbf{w})\big]\,dx$$

$$=\int_{\mathcal{O}\setminus\mathcal{P}}\big[f(\mathbf{w}_\beta^{-1}\mathbf{w})-\chi_2^{-1}(x)f(\mathbf{w})\big]\,dx$$

$$+\int_{\mathcal{P}\setminus\mathcal{P}^2}\big[f(\mathbf{w}_\beta^{-1}\mathbf{w})-\chi_2^{-1}(2\varpi)\dot{\gamma}(\dot{\psi})q^{t+1}\chi_2^{-1}\xi_\varpi(2\varpi^{-1}x)f(\mathbf{w})\big]\,dx.$$

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If χ_2 is unramified, then $\chi_2^{-1}\xi_\varpi$ is ramified and (7.8) becomes

$$(1-q^{-2})f(\mathbf{w}_\beta^{-1}\mathbf{w})-(1-q^{-1})f(\mathbf{w}).$$

If $\chi_2^{-1}\xi_\varpi$ is unramified, then (7.8) becomes

$$(1-q^{-2})f(\mathbf{w}_\beta^{-1}\mathbf{w})-\chi_2^{-1}(2\varpi)\dot{\gamma}(\dot{\psi})(1-q^{-1})q^t f(\mathbf{w}).$$

Finally, if both χ_2 and $\chi_2^{-1}\xi_\varpi$ are ramified, then (7.8) becomes $(1-q^{-2})f(\mathbf{w}_\beta^{-1}\mathbf{w})$. Next we compute the integral,

(7.9)
$$\begin{aligned}&\int_{F\setminus\mathcal{O}}\Big[f([m(\begin{smallmatrix}1&\\&x^{-1}\end{smallmatrix}),\overline{\lambda(n_-(x^{-1}))}]_L\mathbf{w})-|\varpi|^{-2t-2}f([m(\begin{smallmatrix}1&\\&\varpi^2x^{-1}\end{smallmatrix}),\overline{\lambda(\varpi^2x^{-1})}]_L\mathbf{w})\Big]\,dx\\&=\int_{F\setminus\mathcal{O}}\Big[\chi_2^{-1}(x)|x|^{-t-1}\gamma(\psi_{2x^{-1}})f(\mathbf{w})-\chi_2^{-1}(x)|x|^{-t-1}\gamma(\psi_{2\varpi^2x^{-1}})f(\mathbf{w})\Big]\,dx\\&=0,\end{aligned}$$

since $\gamma(\psi_{ab^2})=\gamma(\psi_a)$ for any $a,b\in F^\times$. Summing equations (7.7)–(7.9) proves the result when $\ell(w_\beta w)>\ell(w)$.

The case that $\ell(w_\beta w)<\ell(w)$ is similar except for a shifting of integrands. In this case, $w^{-1}n_+(x)w\in\overline{N}_\varnothing$, and

$$\begin{aligned}&A_0(\mathbf{w}_\beta,\chi_1,s,\chi_2,t,f)(\mathbf{w})\\&=\frac{1}{1-\chi_2^{-2}(\varpi)q^{2t}}\Bigg(\int_{\mathcal{P}^3}(1-\chi_2^{-2}(\varpi)|\varpi|^{-2t-2})f(\mathbf{w}_\beta^{-1}\mathbf{w})dx\\&\quad+\int_{\mathcal{P}\setminus\mathcal{P}^3}\Big[f(\mathbf{w}_\beta^{-1}\mathbf{w})-\chi_2^{-2}(\varpi)|\varpi|^{-2t-2}f([m(\begin{smallmatrix}1&\\&\varpi^2x^{-1}\end{smallmatrix}),\overline{\lambda(n_-(\varpi^2x^{-1}))}]_L\mathbf{w})\Big]\,dx\\&\quad+\int_{F\setminus\mathcal{P}}\Big[f([m(\begin{smallmatrix}1&\\&x^{-1}\end{smallmatrix}),\overline{\lambda(n_-(x^{-1}))}]_L\mathbf{w})\\&\qquad-\chi_2^{-2}(\varpi)|\varpi|^{-2t-2}f([m(\begin{smallmatrix}1&\\&\varpi^2x^{-1}\end{smallmatrix}),\overline{\lambda(n_-(\varpi^2x^{-1})}]_L\mathbf{w}))\Big]\,dx\Bigg).\end{aligned}$$

We then compute the results as in the previous case. ∎

This also gives rise the the following corollary.

Corollary 7.10 *Let $\pi=\chi_1\nu^s\times\chi_2\nu^t\rtimes\tau_0$ with χ_2 quadratic. Then we have*
(7.10)
$$A_0(\mathbf{w}_\beta,\chi_1,s,\chi_2,-t,A_0(\mathbf{w}_\beta,\chi_1,s,\chi_2,t,f))=\chi_2(-1)\frac{(1-q^{2t+1})(1-q^{-2t+1})}{(1-q^{2t})(1-q^{-2t})}f.$$

Proof Let us first assume that $\chi_1\nu^s \otimes \chi_2\nu^t$ is regular. Then by Frobenius reciprocity, we have that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\widetilde{G}_n}(\pi, \pi) = \dim_{\mathbb{C}} \operatorname{Hom}_{\widetilde{M}_\varnothing}(r^{\widetilde{G}_2}_{\widetilde{P}_\varnothing}(\pi), \chi_1\nu^s \otimes \chi_2\nu^t \otimes \tau_0) = 1.$$

So we find that

$$A_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, -t, A_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, \cdot)) = C(\mathbf{w}_\beta, \pi)\mathbb{1}_\pi,$$

where

$$C(\mathbf{w}_\beta, \pi) = A_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, -t, A_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, f))(1_{\widetilde{G}_2})$$

for any $f \in \pi^{(\mathbf{I}_\varnothing, \chi_1 \otimes \chi_2)}$ with $f(1_{\widetilde{G}_2}) = 1$. Once we have established this result for regular inducing data, we use analytic continuation to extend it to arbitrary s and t. ■

Notice that for our special case where χ_2 is quadratic, (7.10) yields the following result.

Corollary 7.11 *For χ_2 quadratic, the intertwining operator $A_0(\mathbf{w}_\beta, \chi_1, s, \chi_2, t, \cdot)$ has a pole when $t \in \frac{\pi i}{\log q}\mathbb{Z}$.*

Now that we have studied the intertwining operators corresponding to the simple roots, we want to use some factorization results of Shahidi and Ban to reduce arbitrary intertwining operators to products of these.

7.3 Factoring Arbitrary Intertwining Operators

In [12], Shahidi gives a theorem for factoring intertwining operators for split, reductive, connected algebraic groups G. However, his techniques rely only on the most basic structures of such groups like split tori, Weyl groups, and Jacquet modules. In particular, all of the tools that Shahidi employs exist for our covering groups $\widetilde{G}_n$ in essentially the identical form. As such, we would like to employ [12, Theorem 2.1.1] in the specific case of $\theta = \theta' = \varnothing \subset \Delta$. In particular, we have the following lemma based on the theorem in Shahidi.

Lemma 7.12 *Let $w \in W_{G_2}$ and let $w = \prod_{i=1}^m w_i$ be a minimal length expression for w (with $i \in \{\alpha, \beta\}$) and let $\operatorname{Re}(s) > \operatorname{Re}(t) > 0$. Then*

$$A(w, \pi) = \prod_{i=1}^m A(w_i, \pi_i),$$

where $\pi_i = \pi_{i-1}^{w_{i-1}}$ for $2 \leq i \leq m$ and $\pi_1 = \pi$.

It is also worth noting that Ban has similar factorization lemmas in [1, 2] that demonstrate the same factorization results beyond the region $\operatorname{Re}(s) > \operatorname{Re}(t) > 0$. We would like to use this lemma to establish the following related result regarding normalized intertwining operators.

Lemma 7.13 *Let $w \in W_{G_2}$ and let $w = \prod_{i=1}^{m} w_i$ be a minimal length expression for w (with $i \in \{\alpha, \beta\}$). Define*

$$A_0(w, \pi) := \prod_{i=1}^{m} A_0(w_i, \pi_i)$$

where $\pi_i = \pi_{i-1}^{w_{i-1}}$ for $2 \le i \le m$ and $\pi_1 = \pi$. Then $A_0(w, \pi)$ meromorphically continues the $A(w, \pi)$ to arbitrary $s, t \in \mathbb{C}$.

Proof Let $\pi \subset \chi_1 \nu^s \times \chi_2 \nu^t \rtimes$ for some unitary characters χ_1 and χ_2 and $s, t \in \mathbb{C}$. One can verify directly using Propositions 7.3 and 7.9 that $A_0(w_i, \pi_i)$ contains no poles for $\mathrm{Re}(s) > \mathrm{Re}(t) > 0$. Furthermore, by our earlier constructions, we have that $A_0(w_i, \pi_i) = A(w_i, \pi_i)$ for the values of $s, t \in \mathbb{C}$ where $A(w_i, \pi_i)$ converges absolutely. These regions all contain the $s, t \in \mathbb{C}$ with $\mathrm{Re}(s) > \mathrm{Re}(t) > 0$. Consequently, we have that $A_0(w, \pi) = A(w, \pi)$ for $\mathrm{Re}(s) > \mathrm{Re}(t) > 0$. Moreover, the poles of $A_0(w, \pi)$ are some subset of the union of poles coming from all $A_0(w_i, \pi_i)$. However, each of these contributes poles along some hyperplanes that are explicitly computable via Propositions 7.3 and 7.9. Thus $A_0(w, \pi)$ meromorphically continues $A(w, \pi)$ to arbitrary $s, t \in \mathbb{C}$ with possible poles along predictable hyperplanes. ■

Remark 7.14 While we never defined the meromorphic continuation of an arbitrary family intertwining operators $A(w, \pi)$, Lemma 7.13 defines such an extension by taking the factorization of $A(w, \pi)$ and continuing each family of operators corresponding to a simple reflection. The key point is the uniqueness of meromorphic continuation. Since our $A_0(w, \pi)$ agrees with $A(w, \pi)$ in the region $\mathrm{Re}(s) > \mathrm{Re}(t) > 0$, any other continuation of $A(w, \pi)$ should agree with $A_0(w, \pi)$ on the region where it is defined. As such, any such continuation should also have a factorization along the lines of Lemma 7.12.

With this factorization lemma, we can finally prove the following statement, which will suffice to determine the remaining cases of irreducibility of the unramified principal series of $\widetilde{G}_2$.

Lemma 7.15 *Let $\pi = \xi_a \times \xi_b \rtimes \tau_0$ for $a, b \in F^\times$ (thus π is unitary). Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}_2}(\pi, \pi) = 1,$$

and consequently, π is irreducible.

Proof The proof can be reduced to defining and then factoring the various intertwining operators away from the unitary inducing data and then showing that they have poles at the unitary quadratic inducing data. We will show this explicitly in one case since the others are similar.

Let $\pi_{s,t} = \xi_a \nu^s \times \xi_b \nu^t \rtimes \tau_0$ with $s, t \in \mathbb{C}$ chosen so that the inducing data is regular. Let us prove this result for a $w \in W_{G_2}$ with $\ell(w) = 2$. Other w's will follow similarly. Thus, we want $a = b$ and let $w_{\beta\alpha} = w_\beta w_\alpha$, which is a reduced expression for $w_{\alpha\beta}$.

Now by Lemma 7.13, we have that

$$\begin{aligned}
&A_0(\mathbf{w}_{\alpha\beta}, \xi_a, t, \xi_a, -s, A_0(\mathbf{w}_{\beta\alpha}, \xi_a, s, \xi_a, t, f)) \\
&\quad = A_0(\mathbf{w}_{\alpha}, \xi_a, t, \xi_a, s, A_0(\mathbf{w}_{\beta}, \xi_a, t, \xi_a, -s, A_0(\mathbf{w}_{\beta}, \xi_a, t, \xi_a, s, A_0(\mathbf{w}_{\alpha}, \xi_a, s, \xi_a, t, f)))) \\
&\quad = \xi_a(-1)\frac{(1-q^{-2s-1})(1-q^{2s-1})}{(1-q^{-2s})(1-q^{2s})} A_0(\mathbf{w}_{\alpha}, \xi_a, t, \xi_a, s, A_0(\mathbf{w}_{\alpha}, \xi_a, s, \xi_a, t, f)) \\
&\quad = \xi_a(-1)\frac{(1-q^{-2s-1})(1-q^{2s-1})}{(1-q^{-2s})(1-q^{2s})}\,\frac{(1-q^{s-t-1})(1-q^{-s+t-1})}{(1-q^{s-t})(1-q^{-s+t})} f
\end{aligned}$$

Notice that this expression has a pole when $s \in \frac{\pi i}{\log q}\mathbb{Z}$ or $s - t \in \frac{2\pi i}{\log q}$. In fact, there is no cancellation of numerators and denominators. Consequently, $A_0(\mathbf{w}_{\beta\alpha}, \xi_a, s, \xi_a, t, \cdot)$ has a pole for $\pi_{0,0} = \xi_a \times \xi_a \rtimes \tau_0$ as above, and, therefore, the intertwining map $A_0(\mathbf{w}_{\beta\alpha}, \xi_a, s\chi_a, t\cdot)$ does not extend to an element of $\mathrm{Hom}_{\widetilde{G}_2}(\pi_{0,0}, \pi_{0,0})$.

The other cases are handled similarly. Consequently, none of our standard intertwining operators extend to the case where $\pi = \pi_{0,0} = \xi_a \times \xi_a \rtimes \tau_0$. These computations would verify that

$$\mathrm{Hom}_{\widetilde{G}_2}(\pi, \pi) = \mathbb{C} \cdot \mathrm{id}_\pi \tag{7.11}$$

if there exists a version of Harish–Chandra's completeness theorem for metaplectic groups (see [13, Theorem 5.5.3.2] for the p-adic group case). This theorem characterizes the commuting algebra of π in terms of normalized intertwining operators. Fortunately, this theorem's metaplectic analog is asserted in [7, Section 27], so (7.11) holds. The case that $\pi = \xi_b \times \xi_a \rtimes \tau_0$ follows as above but must be verified for fewer Weyl group elements. As mentioned before, because these representations are unitary, they are completely reducible and the equality in (7.11) ensures the irreducibility of π. ∎

With this lemma, we have finally ascertained the irreducibility of the representations whose irreducibility could not be verified using Jacquet module graphs. Consequently, we have finished establishing all of the reducibility points of the principal series for $\widetilde{G}_2$ and characterizing the irreducible constituents of the reducible principal series representations.

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8 Appendix

We now include some tables that both summarize the results of this paper as well as contain the Jacquet modules with respect to the maximal parabolic subgroups alluded to previously. Note that most of the notation comes straight from the previous sections. In the first table, we also compute the dimension of the parahoric invariance of a constituent π if the inducing data happens to be unramified.

Table 1: Constituents of the Principal Series for $\widetilde{G}_2$ and the Dimension of Parahoric Invariants for Unramified Inducing Data

		Representation	Constituents	$\mathbf{I}_\varnothing$	$\mathbf{I}_\alpha$	$\mathbf{I}_\beta$	$\mathbf{K}$
I		$\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ (irreducible)		8	4	4	1
II	a	$\chi\nu^{s+\frac{1}{2}} \times \chi\nu^{s-\frac{1}{2}} \rtimes \tau_0$	$\chi\nu^s St_{\mathrm{GL}_2} \rtimes \tau_0$	4	1	2	0
	b	$\chi\nu^s \notin \{\xi_a, \xi_a\nu^{\pm 1}\}$	$\chi\nu^s \mathbb{1}_{GL_2} \rtimes \tau_0$	4	3	2	1
III	a	$\chi\nu^s \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0$	$\chi\nu^s \rtimes sp_a$	4	2	1	0
	b	$\chi\nu^s \notin \{\xi_a\nu^{\pm\frac{1}{2}}, \xi_a\nu^{\pm\frac{3}{2}}, \xi_b\nu^{\pm\frac{1}{2}}\}$	$\chi\nu^s \rtimes \omega_a^+$	4	2	3	1
IV	a	$\xi_a\nu^{\frac{3}{2}} \times \xi_a\nu^{\frac{1}{2}} \rtimes \tau_0$	$sp_{a,2}$	1	0	0	0
	b		$Q(\xi_a\nu St_{\mathrm{GL}_2}, \tau_0)$	3	1	2	0
	c		$Q(\xi_a\nu^{\frac{3}{2}}, sp_a)$	3	2	1	0
	d		$\omega_{a,2}^+$	1	1	1	1
V	a	$\xi_a\nu^{\frac{1}{2}} \times \xi_a\nu^{-\frac{1}{2}} \rtimes \tau_0$	$T_2(\xi_a St_{\mathrm{GL}_2}, \tau_0)$	3	1	1	0
	b		$T_1(\xi_a St_{\mathrm{GL}_2}, \tau_0)$	1	0	1	0
	c		$Q(\xi_a\nu^{\frac{1}{2}}, sp_a)$	1	1	0	0
	d		$Q(\xi_a\nu^{\frac{1}{2}}, \xi_a\nu^{-\frac{1}{2}}, \tau_0)$	3	2	2	1
VI	a	$\xi_a\nu^{\frac{1}{2}} \times \xi_b\nu^{-\frac{1}{2}} \rtimes \tau_0$	$T(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0)$	2	1	0	0
	b	$ab^{-1} \in F^\times \setminus (F^\times)^2$	$Q(\xi_b\nu^{\frac{1}{2}}, sp_a)$	2	1	1	0
	c		$Q(\xi_a\nu^{\frac{1}{2}}, sp_b)$	2	1	1	0
	d		$Q(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0)$	2	1	2	1

Table 2: Jacquet Modules-$\widetilde{G}_2$-Siegel Parabolic

		Representation	$\mathbf{r}^{\widetilde{\mathbf{G}}_2}_{\widetilde{\mathbf{P}}_\alpha}(\pi) \in \mathfrak{R}(\widetilde{\mathbf{M}}_\alpha)$	#
I		$\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ (irred.)	$\chi_1\nu^s \times \chi_2\nu^t \otimes \tau_0 + \chi_1\nu^s \times \chi_2^{-1}\nu^{-t} \otimes \tau_0$ $+\chi_1^{-1}\nu^{-s} \times \chi_2\nu^t \otimes \tau_0$ $+\chi_1^{-1}\nu^{-s} \rtimes \chi_2^{-1}\nu^{-t} \otimes \tau_0$	4
II	a	$\chi\nu^s St_{\mathrm{GL}_2} \rtimes \tau_0$	$\chi\nu^s St_{\mathrm{GL}_2} \otimes \tau_0 + \chi^{-1}\nu^{-s} St_{\mathrm{GL}_2} \otimes \tau_0$ $+\chi\nu^{s+\frac{1}{2}} \times \chi^{-1}\nu^{-s+\frac{1}{2}} \otimes \tau_0$	3
	b	$\chi\nu^s \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0$	$\chi\nu^s \mathbb{1}_{\mathrm{GL}_2} \otimes \tau_0 + \chi^{-1}\nu^{-s} \mathbb{1}_{\mathrm{GL}_2} \otimes \tau_0$ $+\chi\nu^{s-\frac{1}{2}} \times \chi^{-1}\nu^{-s-\frac{1}{2}} \otimes \tau_0$	3
III	a	$\chi\nu^s \rtimes \rtimes sp_a$	$\chi\nu^s \times \xi_a\nu^{\frac{1}{2}} \otimes \tau_0 + \chi^{-1}\nu^{-s} \times \xi_a\nu^{\frac{1}{2}} \otimes \tau_0$	2
	b	$\chi \rtimes \omega_a^+$	$\chi\nu^s \times \xi_a\nu^{-\frac{1}{2}} \otimes \tau_0 + \chi^{-1}\nu^{-s} \times \xi_a\nu^{-\frac{1}{2}} \otimes \tau_0$	2
IV	a	$sp_{a,2}$	$\xi_a\nu St_{\mathrm{GL}_2} \otimes \tau_0$	1
	b	$Q(\xi_a\nu St_{\mathrm{GL}_2}, \tau_0)$	$\xi_a\nu^{-1} St_{\mathrm{GL}_2} \otimes \tau_0 + \xi_a\nu^{\frac{3}{2}} \times \xi_a\nu^{-\frac{1}{2}} \otimes \tau_0$	2
	c	$Q(\xi_a\nu^{\frac{3}{2}}, sp_a)$	$\xi_a\nu \mathbb{1}_{\mathrm{GL}_2} \otimes \tau_0 + \xi_a\nu^{\frac{1}{2}} \times \varsigma\nu^{-\frac{3}{2}} \otimes \tau_0$	2
	d	$\omega_{a,2}^+$	$\varsigma\nu^{-1} \mathbb{1}_{\mathrm{GL}_2} \otimes \tau_0$	1
V	a	$T_2(\xi_a St_{\mathrm{GL}_2}, \tau_0)$	$\xi_a St_{\mathrm{GL}_2} \otimes \tau_0 + \xi_a\nu^{\frac{1}{2}} \times \xi_a\nu^{\frac{1}{2}} \otimes \tau_0$	2
	b	$T_1(\xi_a St_{\mathrm{GL}_2}, \tau_0)$	$\xi_a St_{\mathrm{GL}_2} \otimes \tau_0$	1
	c	$Q(\xi_a\nu^{\frac{1}{2}}, sp_a)$	$\xi_a \mathbb{1}_{\mathrm{GL}_2} \otimes \tau_0$	1
	d	$Q(\xi_a\nu^{\frac{1}{2}}, \xi_a\nu^{-\frac{1}{2}}, \tau_0)$	$\xi_a \mathbb{1}_{\mathrm{GL}_2} \otimes \tau_0 + \xi_a\nu^{-\frac{1}{2}} \times \xi_a\nu^{-\frac{1}{2}} \otimes \tau_0$	2
VI	a	$T(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0)$	$\xi_a\nu^{\frac{1}{2}} \times \xi_b\nu^{\frac{1}{2}} \otimes \tau_0$	1
	b	$Q(\xi_a\nu^{\frac{1}{2}}, sp_b)$	$\xi_a\nu^{-\frac{1}{2}} \times \xi_b\nu^{\frac{1}{2}} \otimes \tau_0$	1
	c	$Q(\xi_b\nu^{\frac{1}{2}}, sp_a)$	$\xi_a\nu^{\frac{1}{2}} \times \xi_b\nu^{-\frac{1}{2}} \otimes \tau_0$	1
	d	$Q(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0)$	$\xi_a\nu^{-\frac{1}{2}} \times \xi_b\nu^{-\frac{1}{2}} \otimes \tau_0$	1

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Table 3: Jacquet Modules-$\widetilde{G}_2$-Long Root Parabolic

		Representation	$\mathbf{r}_{\widetilde{\mathbf{P}}}^{\widetilde{G}_2}\beta(\pi) \in \mathfrak{R}(\widetilde{M}_\beta)$	#
I		$\chi_1\nu^s \times \chi_2\nu^t \rtimes \tau_0$ (irred.)	$\chi_1\nu^s \otimes (\chi_2\nu^t \rtimes \tau_0) + \chi_1^{-1}\nu^{-s} \otimes (\chi_2\nu^t \rtimes \tau_0)$ $+\chi_2\nu^t \otimes (\chi_1\nu^s \rtimes \tau_0) + \chi_2^{-1}\nu^{-t} \otimes (\chi_1\nu^s \rtimes \tau_0)$	4
II	a	$\chi\nu^s St_{\mathrm{GL}_2} \rtimes \tau_0$	$\chi\nu^{s+\frac{1}{2}} \otimes (\chi\nu^{s-\frac{1}{2}} \rtimes \tau_0)$ $+\chi^{-1}\nu^{-s+\frac{1}{2}} \otimes (\chi\nu^{s+\frac{1}{2}} \rtimes \tau_0)$	2
	b	$\chi\nu^s \mathbb{1}_{\mathrm{GL}_2} \rtimes \tau_0$	$\chi\nu^{s-\frac{1}{2}} \otimes (\chi\nu^{s+\frac{1}{2}} \rtimes \tau_0)$ $+\chi^{-1}\nu^{-s-\frac{1}{2}} \otimes (\chi\nu^{s-\frac{1}{2}} \rtimes \tau_0)$	2
III	a	$\chi\nu^s \rtimes sp_a$	$\chi\nu^s \otimes sp_a + \chi^{-1}\nu^{-s} \otimes sp_a$ $+\xi_a\nu^{\frac{1}{2}} \otimes (\chi\nu^s \rtimes \tau_0)$	3
	b	$\chi\nu^s \rtimes \omega_a^+$	$\chi\nu^s \otimes \omega_a^+ + \chi^{-1}\nu^{-s} \otimes \omega_a^+$ $+\xi_a\nu^{-\frac{1}{2}} \otimes (\chi\nu^s \rtimes \tau_0)$	3
IV	a	$sp_{a,2}$	$\xi_a\nu^{\frac{3}{2}} \otimes sp_a$	1
	b	$Q(\xi_a\nu St_{\mathrm{GL}_2}, \tau_0)$	$\xi_a\nu^{\frac{3}{2}} \otimes \omega_a^+ + \xi_a\nu^{\frac{1}{2}} \otimes (\xi_a\nu^{\frac{3}{2}} \rtimes \tau_0)$	2
	c	$Q(\varsigma\nu^{\frac{3}{2}}, sp_a)$	$\xi_a\nu^{-\frac{3}{2}} \otimes sp_a + \xi_a\nu^{-\frac{1}{2}} \otimes (\xi_a\nu^{\frac{3}{2}} \rtimes \tau_0)$	2
	d	$\omega_{a,2}^+$	$\xi_a\nu^{-\frac{3}{2}} \otimes \omega_a^+$	1
V	a	$T_2(\xi_a St_{\mathrm{GL}_2}, \tau_0)$	$\xi_a\nu^{\frac{1}{2}} \otimes sp_a + 2 \cdot \xi_a\nu^{\frac{1}{2}} \otimes \omega_a^+$	3
	b	$T_1(\xi_a St_{\mathrm{GL}_2}, \tau_0)$	$\xi_a\nu^{\frac{1}{2}} \otimes \omega_a^+$	1
	c	$Q(\xi_a\nu^{\frac{1}{2}}, sp_a)$	$\xi_a\nu^{-\frac{1}{2}} \otimes sp_a$	1
	d	$Q(\xi_a\nu^{\frac{1}{2}}, \xi_a\nu^{-\frac{1}{2}}, \tau_0)$	$\xi_a\nu^{-\frac{1}{2}} \otimes sp_a + 2 \cdot \xi_a\nu^{-\frac{1}{2}} \otimes \omega_a^+$	3
VI	a	$T(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0)$	$\xi_a\nu^{\frac{1}{2}} \otimes sp_b + \xi_b\nu^{\frac{1}{2}} \otimes sp_a$	2
	b	$Q(\xi_a\nu^{\frac{1}{2}}, sp_b)$	$\xi_a\nu^{-\frac{1}{2}} \otimes sp_b + \xi_b\nu^{\frac{1}{2}} \otimes \omega_a^+$	2
	c	$Q(\xi_b\nu^{\frac{1}{2}}, sp_a)$	$\xi_a\nu^{\frac{1}{2}} \otimes \omega_b^+ + \xi_b\nu^{-\frac{1}{2}} \otimes sp_a$	2
	d	$Q(\xi_a\nu^{\frac{1}{2}}, \xi_b\nu^{\frac{1}{2}}, \tau_0)$	$\xi_a\nu^{-\frac{1}{2}} \otimes \omega_b^+ + \xi_b\nu^{-\frac{1}{2}} \otimes \omega_b^+$	2

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References

[1] D. Ban, *The Aubert involution and R-groups.* Ann. Sci. École Norm. Sup. (4) **35**(2002), no. 5, 673–693. doi=10.1016/S0012-9593(02)01105-9

[2] ———, *Linear independence of intertwining operators.* J. Algebra **271**(2004), no. 2, 749–767. doi=10.1016/j.jalgebra.2002.11.004 doi:10.1016/j.jalgebra.2002.11.004

[3] I. N. Bernstein and A. V. Zelevinsky, *Representations of the group* $\mathrm{GL}_n(F)$*, where F is a local nonArchimedean field.* Uspehi Mat. Nauk **31**(1976), no. 3(189), 5–70.

[4] ———, *Induced representations of p-adic groups. I.* Ann. Sci. École Norm. Sup. (4) **10**(1997), no. 4, 441–472.

[5] D. Bump, *Automorphic forms and representations.* Cambride Studies in Advanced Mathematics, 55, Cambridge University Press, Cambridge, 1997.

[6] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups.* http://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf

[7] Y. Flicker and D. Kazhdan, *Metaplectic correspondence.* Inst. Hautes Études Sci. Publ. Math. **64**(1986), 53–110.

[8] S. Kudla, *On the local theta-correspondence.* Invent. Math. **83**(1986), no. 2, 229–255. doi=10.1007/BF01388961 doi:10.1007/BF01388961

[9] ———, *On the local theta correspondence.* Notes from the European School of Group Theory, Beilngries (1996). http://www.math.toronto.edu/~skudla/castle.pdf

[10] S. Kudla, M. Rapoport, and T. Yang, *Modular forms and special cycles on Shimura curves.* Annals of Mathematics Studies, 161, Princeton University Press, Princeton, NJ, 2006.

[11] R. Ranga Rao, *On some explicit formulas in the theory of Weil representation.* Pacific J. Math. **157**(1993), no. 2, 335–371.

[12] F. Shahidi, *On certain L-functions.* Amer. J. Math. **103**(1981), no. 2, 297–355. doi:10.2307/2374219

[13] A. J. Silberger, *Introduction to harmonic analysis on reductive p-adic groups.* Mathematical Notes, 23, Princeton University Press, Princeton, NJ, 1979.

[14] T. A. Springer, *Reductive groups.* In: Automorphic forms, representations and L-functions Part 1, Proc. Sympos. Pure Math., XXXIII, American Mathematical Society, Providence, RI, pp. 3–27.

[15] M. Tadić, *Representations of classical p-adic groups.* In: Representations of Lie groups and quantum groups, Pitman Research Notes in Mathematics, 311, Longman Sci. Tech., Harlow, 1994, pp. 129–204.

[16] ———, *Jaquet modules and induced representations.* Math. Commun. **3**(1998), no. 1, 1–17.

[17] ———, *On reducibility of parabolic induction.* Israel J. Math. **107**(1998), 29–91. doi=10.1007/BF02764004 doi:10.1007/BF02764004

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