THE SEMI-ALGEBRA GENERATED BY A COMPACT LINEAR OPERATOR

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1. Introduction

We prove that if t is a compact linear operator that is not quasi-nilpotent and is appropriately normalised, then the closed semi-algebra $A(t)$ generated by t is locally compact. The theory of locally compact semi-algebras (**2**) is therefore applicable to $A(t)$, and we show that it can be used to obtain spectral properties of t.

We collect together in Section **2** the properties of locally compact semi-algebras that we need. These are mainly properties of a prime strict commutative locally compact semi-algebra and its unique minimal idempotent, which are given in (**2**), together with some simple deductions from them. A new result relates a certain compact group $G(a)$ with each element a with spectral radius 1.

In Section **3**, we consider a compact linear operator t such that the spectral radius of t is 1 and belongs to the spectrum of t. We prove that the closed semi-algebra $A(t)$ generated by t is locally compact, and that it is prime and strict if and only if t has equibounded iterates, i.e. the sequence $\{\| t^n \|\}$ is bounded. The rest of Section **3** is concerned with an operator t that satisfies these conditions. If ζ is an eigenvalue of t on the unit circle, then $\zeta^{-1}t$ also satisfies the conditions, and so the results in Section **2** are also applicable to $A(\zeta^{-1}t)$. It follows in particular that t has equibounded iterates if and only if all eigenvalues on the unit circle have index 1.

Two projections belonging to $A(t)$ are of special interest. One is the unique minimal projection p in $A(t)$, which is also the spectral projection corresponding to the eigenvalue 1. The other is the identity u of the compact group $G(t)$ associated with t. This is the greatest projection in $A(t)$, and also satisfies

$$u = p_1 + \ldots + p_m,$$

where p_i is the minimal projection in $A(\zeta_i^{-1}t)$, and $\zeta_1, \ldots, \zeta_m$ are the eigenvalues of t on the unit circle.

Using a technique developed in (**3**), we prove that $A(t)$ is the set of all operators a of the form

$$a = \sum_{k=1}^{\infty} \alpha_k t^k + b,$$

where $\alpha_k \geqq 0$, $\sum_{k=1}^{\infty} \alpha_k < \infty$, and b belongs to the least convex cone containing the group $G(t)$.

In Section **4**, we consider positive operators in a partially ordered Banach space, and introduce a generalisation of the concept of irreducible non-negative matrix that is available in this setting. The theory in Section **3** is applicable to a compact irreducible positive operator t, and indeed to any compact positive operator that commutes with such an operator t. A special feature of an irreducible t is that the minimal projection p in $A(t)$ has rank 1.

Finally in Section **5**, we specialise our considerations by studying irreducible positive operators in a complex Banach lattice. A complex Banach lattice is said to be alignable if it satisfies a certain condition that is obviously satisfied by the sequence spaces l^p and m. For a compact irreducible positive operator with spectral radius 1 in an alignable space, we obtain almost the whole of the results proved by Frobenius for an irreducible non-negative matrix. In particular, the eigenvalues on the unit circle are natural roots of unity, and the entire spectrum is invariant under the corresponding rotations of the complex plane. These results go considerably further than the corresponding results in the classical paper by Krein and Rutman (**6**).

The authors owe several simplifications to members of the functional analysis seminar at Newcastle.

2. Locally compact semi-algebras

We collect together here the properties of locally compact semi-algebras that we need. They are either known theorems or simple consequences of known theorems.

Let B denote a Banach algebra over the real field $\boldsymbol{R}$, and let $\boldsymbol{R}^+$ denote the set of all non-negative real numbers. A non-empty subset A of B is called a *semi-algebra* if and only if $x+y$, xy, $\alpha x \in A$ whenever $x, y \in A$ and $\alpha \in \boldsymbol{R}^+$. A semi-algebra A is said to be a *locally compact semi-algebra* if and only if it satisfies two additional axioms:

(i) A contains non-zero vectors.

(ii) The set of vectors x in A with $\| x \| \leqq 1$ is a compact subset of B (with respect to the norm topology).

It is elementary that a locally compact semi-algebra is a closed subset of the Banach algebra B and is a locally compact space in the induced topology. A semi-algebra A is said to be *strict* if and only if $A \cap (-A) = (0)$.

We shall be concerned only with commutative semi-algebras. A commutative semi-algebra A is *semi-simple* if and only if $a^2 \neq 0$ for all non-zero $a \in A$, and is *prime* if and only if there are no divisors of zero, i.e. if and only if $ab \neq 0$ whenever $a, b \in A$, $a \neq 0$, $b \neq 0$. An *ideal* in a commutative semi-algebra A is a semi-algebra J contained in A such that $ja \in J$ whenever $j \in J$ and $a \in A$. An *idempotent* is a non-zero vector e such that $e^2 = e$, and a *minimal idempotent* is an idempotent e in A such that eA is a minimal closed ideal in A.

The *spectral radius* of an element a of a Banach algebra is denoted by $r(a)$, i.e.

$$r(a) = \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}}.$$

The following theorem is (essentially) Theorems 11 and 12 in (**2**).

Theorem 1. *Let A be a semi-simple strict commutative locally compact semi-algebra. Then the set of minimal idempotents of A is a finite non-empty set $e_1, e_2, \ldots, e_m$, and $e_ie_j = 0$ $(i \neq j)$. For each non-zero element a of A, $r(a) > 0$, and there exist non-negative real numbers $\lambda_1, \ldots, \lambda_m$ such that*

$$ae_i = \lambda_i e_i \quad (i = 1, \ldots, m),$$

and $\max\{\lambda_i : 1 \leqq i \leqq m\} = r(a)$.

If also A is prime, then there exists exactly one minimal idempotent p, and

$$ap = r(a)p \quad (a \in A).$$

The two theorems that follow are simple consequences of Theorem 1.

Theorem 2. *Let A be a semi-simple strict commutative locally compact semi-algebra. Then there exists a positive real constant M such that*

$$\| a \| \leqq Mr(a) \quad (a \in A).$$

Proof. Let $e_1, \ldots, e_m$ be the minimal idempotents in A, let

$$u = e_1 + e_2 + \ldots + e_m,$$

and let

$$\kappa = \inf\{\| au \| : a \in A \text{ and } \| a \| = 1\}.$$

By the local compactness of A, κ is attained. But, by Theorem 1, $au \neq 0$ whenever $a \neq 0$. Therefore $\kappa > 0$. Using Theorem 1 again, we have

$$\begin{aligned}\kappa \| a \| \leqq \| au \| &= \| ae_1 + \ldots + ae_m \| \\ &= \| \lambda_1 e_1 + \ldots + \lambda_m e_m \| \leqq r(a)\{\| e_1 \| + \ldots + \| e_m \|\},\end{aligned}$$

and the proof is complete.

Theorem 3. *Let A be a prime strict commutative locally compact semi-algebra, and let $S = \{a : a \in A \text{ and } r(a) = 1\}$. Then*

(i) *$S = \{a : a \in A$ and $ap = p\}$, where p is the unique minimal idempotent in A;*

(ii) *S is a compact convex set;*

(iii) *S is a semi-group with respect to the given multiplication of vectors;*

(iv) *S is a base for A in the sense that every non-zero element of A has a unique representation in the form $a = \lambda s$ with $\lambda > 0$ and $s \in S$.*

Proof. (i) By Theorem 1, $ap = r(a)p$ for all a in A, and so $ap = p$ if and only if $r(a) = 1$.

(ii) By (i), S is a closed convex subset of A and hence of B. By Theorem 2, S is bounded, and it is therefore compact.

(iii) That S is a semi-group is clear from (i).

(iv) Given $a \in A$ with $a \neq 0$, we have $r(a) > 0$, and so $a = \lambda s$ with $\lambda = r(a)$, $s = \lambda^{-1} a \in S$. The uniqueness of the representation is easily verified.

Theorem 4. *Let A be a prime strict commutative locally compact semi-algebra, and let S be the compact convex semi-group of all $a \in A$ with $r(a) = 1$. Let $a \in S$, and let $G(a)$ denote the set of all cluster points of the sequence $\{a^n\}$. Then $G(a)$ is a non-empty compact Abelian group contained in the semi-group S.*

Proof. That $G(a)$ is a group has been proved for an element a of any compact topological semi-group by K. Numakura (**8**). We give here the very simple proof that is available in our case. We recall that a cluster point of the sequence $\{a^n\}$ is a point x such that every neighbourhood of x contains some a^n for arbitrarily large n. Since we are concerned with a metric topology, the cluster points of $\{a^n\}$ are the limits of convergent subsequences of $\{a^n\}$.

Since $a \in S$ and S is a semi-group, we have $a^n \in S$ for all n. Since S is compact, it follows that $G(a)$ is a non-empty compact subset of S. Moreover, every subsequence of $\{a^n\}$ has a subsequence that converges to an element of $G(a)$. Let $b, c \in G(a)$. It is easily verified that $bc \in G(a)$, and we prove that there exists $g \in G(a)$ such that $b = cg$. There exist strictly increasing sequences $\{m(k)\}$, $\{n(k)\}$ of positive integers such that $b = \lim_{k\to\infty} a^{m(k)}$, $c = \lim_{k\to\infty} a^{n(k)}$. Then we can construct a strictly increasing sequence $\{k(j)\}$ of positive integers such that

$$r(j) = m(k(j)) - n(j) > r(j-1) > 0 \quad (j = 2, 3, \ldots).$$

Then the sequence $\{a^{r(j)}\}$ has a subsequence that converges to an element g of $G(a)$, and we have $b = cg$.

We have now proved that $G(a)$ is an Abelian semi-group in which every group equation has a solution, and it follows that $G(a)$ is a group (**1**, Theorem 5, p. 128).

3. The semi-algebra generated by a compact linear operator

Let X be a Banach space over the complex field $\boldsymbol{C}$, and let B denote the Banach algebra $B(X, X)$ of all bounded linear operators in X, with the usual operator norm. Let $t \in B$. We denote by $P(t)$ the least semi-algebra containing t, i.e. the set of all operators of the form $\sum_{k=1}^{n} \alpha_k t^k$ with $\alpha_k \in \boldsymbol{R}^+$ $(k = 1, \ldots, n)$. We denote by $A(t)$ the least closed semi-algebra in B containing t, i.e. the closure of $P(t)$ in B. We denote the spectrum of t by spec(t), and the identity operator by e.

Theorem 5. *Let t be a compact linear operator in X such that $r(t) \neq 0$ and $r(t) \in$ spec (t). Then $A(t)$ is a locally compact semi-algebra.*

Proof. It is clear that $A(\lambda t) = A(t)$ whenever $\lambda > 0$. Therefore there is no loss of generality in supposing that $r(t) = 1$. Then $1 \in$ spec (t), and, since t is compact, 1 is an eigenvalue of t, i.e. there exists $x \in X$ with $x \neq 0$ and $tx = x$.

Given $a = \alpha_1 t + \ldots + \alpha_n t^n \in P(t)$, we have

$$ax = (\alpha_1 + \ldots + \alpha_n)x,$$

and therefore

$$(\alpha_1 + \ldots + \alpha_n) \leqq \| a \|. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Let $a_n \in A(t)$ with $\| a_n \| \leqq 1$ $(n = 1, 2, \ldots)$. Since $A(t)$ is the closure of $P(t)$, there exist $b_n \in P(t)$ such that

$$\| a_n - b_n \| < n^{-1}, \ \| b_n \| \leqq 1 \quad (n = 1, 2, \ldots).$$

We have, for each n,

$$b_n = \sum_{k=1}^{\infty} \alpha(n, k) t^k,$$

where $\alpha(n, k) \geqq 0$ (k $= 1, 2, \ldots$) and $\alpha(n, k) = 0$ for sufficiently large k. By (1),

$$\sum_{k=1}^{\infty} \alpha(n, k) \leqq \| b_n \| \leqq 1.$$

For each fixed k, $\{\alpha(n, k)\}$ is a bounded sequence in $\boldsymbol{R}^+$. Therefore, by the diagonal process, there exists a strictly increasing sequence $\{n_j)$ of positive integers such that

$$\lim_{j \to \infty} \alpha(n_j, k) = \alpha_k \in \boldsymbol{R}^+ (k = 1, 2, \ldots). \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

With an arbitrary positive integer N, we have

$$\sum_{k=1}^{N} \alpha_k = \lim_{j \to \infty} \sum_{k=1}^{N} \alpha(n_j, k) \leqq 1,$$

and therefore

$$\sum_{k=1}^{\infty} \alpha_k \leqq 1. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

By the Riesz-Schauder theory for compact operators, there exists a bounded linear projection q of finite rank such that $tq = qt$ and $r(t(e-q)) < 1$. This inequality shows that $\lim_{n \to \infty} \| t^n(e-q) \| = 0$, and this with (3) shows that

$$b = \sum_{k=1}^{\infty} \alpha_k t^k (e-q)$$

is a well defined bounded linear operator.

Given $\varepsilon > 0$, there exists N such that

$$\| t^k(e-q) \| < \varepsilon/8 \quad (k \geqq N),$$

and therefore, for all n,

$$\left\| \sum_{k=N}^{\infty} \alpha(n, k) t^k (e-q) - \sum_{k=N}^{\infty} \alpha_k t^k (e-q) \right\| < \varepsilon/4.$$

Also, by (2), there exists j_0 such that

$$\left\| \sum_{k=1}^{N-1} \alpha(n_j, k) t^k (e-q) - \sum_{k=1}^{N-1} \alpha_k t^k (e-q) \right\| < \varepsilon/4 \quad (j \geqq j_0).$$

From these two inequalities, we obtain

$$\| b_{n_j}(e-q) - b \| < \varepsilon/2 \quad (j \geqq j_0). \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Let $Y = qX$. Then Y is a finite dimensional linear subspace of X, and if c_n denotes the restriction of b_n to Y, then $c_n \in B(Y, Y)$ (since $b_n q = q b_n$), and

$$\| c_n \| \leqq \| b_n \| \leqq 1.$$

The unit ball in $B(Y, Y)$ is compact, and so, by replacing the sequence $\{b_{n_j}\}$ by a subsequence if necessary, we may suppose that $\{c_{n_j}\}$ converges in operator norm to $c \in B(Y, Y)$. Then $cq \in B(X, X)$, and $\lim_{j\to\infty} \| b_{n_j} q - cq \| = 0$, for

$$\| b_{n_j} q - cq \| = \| c_{n_j} q - cq \| \leqq \| c_{n_j} - c \| \, . \, \| q \|.$$

We may now suppose that j_0 is chosen so large that also

$$\| b_{n_j} q - cq \| < \varepsilon/4 \quad (j \geqq j_0), \qquad \text{(5)}$$

$$\| a_{n_j} - b_{n_j} \| < \varepsilon/4 \quad (j \geqq j_0). \qquad \text{(6)}$$

Then we have, from (4), (5) and (6),

$$\| a_{n_j} - (b + cq) \| < \varepsilon \quad (j \geqq j_0).$$

It follows that $\lim_{j\to\infty} a_{n_j} = b + cq$, and, since $A(t)$ is closed, $b + cq \in A(t)$. Therefore $A(t)$ is locally compact.

Remarks 1. We have not used the full force of the compactness of the operator t, but only that t has certain spectral properties, namely that 1 is an eigenvalue, and that there exists a bounded linear projection q of finite rank that commutes with t and satisfies $r(t(e-q)) < 1$.

(2). Given any compact linear operator a in X such that $r(a) \neq 0$, there exists a complex number λ such that $t = \lambda a$ has the properties assumed in the statement of the theorem.

Definition. Let t be a bounded linear operator with $r(t) = 1$. We say that t has *equibounded iterates* if and only if the sequence $\{\| t^n \|\}$ is bounded.

Lemma 1. *Let t be a bounded linear operator such that $tx = x$ for some non-zero vector x and $\| t^n \| \leqq M$ $(n = 1, 2, \ldots)$. Then for each $a \in A(t)$ there exists $\alpha \in \mathbf{R}^+$ such that $ax = \alpha x$ and $\alpha \geqq M^{-1} \| a \|$.*

Proof. Given $a \in A(t)$ and $\varepsilon > 0$, there exists $b \in P(t)$ such that

$$\| b - a \| \leqq \varepsilon \| a \|,$$

and it is clear that $ax = \alpha x$ for some $\alpha \in \mathbf{R}^+$. We have $b = \alpha_1 t + \ldots + \alpha_m t^m$, and so

$$(b-a)x = (\alpha_1 + \ldots + \alpha_m - \alpha)x,$$

$$| \alpha_1 + \ldots + \alpha_m - \alpha | \leqq \| b - a \| \leqq \varepsilon \| a \|,$$

$$\alpha \geqq (\alpha_1 + \ldots + \alpha_m) - \varepsilon \| a \|.$$

Also

$$(1-\varepsilon) \| a \| \leqq \| b \| \leqq (\alpha_1 \| t \| + \ldots + \alpha_m \| t^m \|) \leqq (\alpha_1 + \ldots + \alpha_m) M.$$

Therefore

$$\alpha \geqq M^{-1}(1-\varepsilon) \| a \| - \varepsilon \| a \|.$$

Since ε is arbitrary, this completes the proof.

Theorem 6. *Let t be a compact linear operator such that $r(t) = 1$. Then $A(t)$ is a prime strict commutative locally compact semi-algebra if and only if $1 \in \operatorname{spec}(t)$ and t has equibounded iterates.*

Proof. We have $r(t^n) = 1$ for all positive integers n. If $A(t)$ is a prime strict commutative locally compact semi-algebra, Theorem 2 therefore shows that the sequence $\{\| t^n \|\}$ is bounded. Also, by Theorem 1, $A(t)$ contains a minimal idempotent p, and we have $tp = r(t)p = p$. This shows that $1 \in \operatorname{spec}(t)$.

Suppose on the other hand that $1 \in \operatorname{spec}(t)$ and that $\| t^n \| \leqq M$ $(n = 1, 2, \ldots)$. Since t is a compact linear operator and $1 \in \operatorname{spec}(t)$, there exists a non-zero vector x with $tx = x$. By Lemma 1, if $a, b \in A(t)$, there exist $\alpha, \beta \in \boldsymbol{R}^+$ such that $ax = \alpha x, bx = \beta x, \alpha \geqq M^{-1} \| a \|, \beta \geqq M^{-1} \| b \|$. Then $(a+b)x = (\alpha+\beta)x$, $(ab)x = (\alpha\beta)x$, from which it follows that $a+b \neq 0$ unless $a = b = 0$, and $ab \neq 0$ unless $a = 0$ or $b = 0$. Thus $A(t)$ is strict and prime. That $A(t)$ is locally compact has been proved in Theorem 5, and it is obviously commutative.

Corollary. *If t is a compact linear operator with equibounded iterates and $r(t) = 1 \in \operatorname{spec}(t)$, then $A(t)$ has a unique minimal idempotent p, and*

$$ap = r(a)p \ (a \in A(t)).$$

Theorem 7. *Let t be a compact linear operator with equibounded iterates, let $r(t) = 1 \in \operatorname{spec}(t)$, and let p be the unique minimal idempotent in $A(t)$. Then*

(i) *$tx = x$ if and only if $px = x$;*

(*ii*) *the eigenvalue* 1 *has index* 1*;*

(iii) *p is the spectral projection for the eigenvalue* 1, *i.e. the range and null-space of p are the null-space and range respectively of $t-e$.*

Proof. Let R be the range and N the null-space of $t-e$. Since $r(t) = 1$, we have

$$tp = pt = p. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Given $x \in N$, Lemma 1 shows that $px = \alpha x$ with $\alpha > 0$. Since $p^2 = p$, we have $\alpha = 1$, $px = x$. Conversely, if $px = x$, then, by (1), $tx = tpx = px = x$, and so $x \in N$. This proves (i) and shows that N is the range of p.

By (1), $p(t-e) = 0$, which shows that R is contained in the null-space of p. But the range and null-space of a projection have zero intersection, and so $N \cap R = (0)$. It follows at once that the eigenvalue 1 has index 1. For if $(t-e)^2 x = 0$, then $(t-e)x \in N \cap R$.

Since the index of 1 is 1, we have $X = N \oplus R$, and so R is the null-space of p.

Theorem 8. *Let t be a compact linear operator with equibounded iterates, let $r(t) = 1 \in \operatorname{spec}(t)$, let p be the minimal idempotent in $A(t)$, let $S(t) = \{a\colon a \in A(t) \text{ and } r(a) = 1\}$, and let $G(t)$ be the set of cluster points of the sequence $\{t^n\}$. Then the following conclusions hold.*

(i) *$S(t)$ is a compact convex base for $A(t)$, and is a semi-group under operator multiplication.*

(ii) $p = \lim_{n\to\infty} n^{-1}(t+t^2+\ldots+t^n)$.

(iii) *$G(t)$ is a non-empty subset of $S(t)$, and is a compact abelian group.*

(iv) *The identity of the group $G(t)$ is a projection u with finite rank.*

(v) *$G(t)$ is the closure of the set $\{t^k u: k = 1, 2, \ldots\}$.*

Proof. We have proved that $A(t)$ is a prime strict commutative locally compact semi-algebra. Thus (i) and (iii) follow at once from Theorems 3 and 4.

Let $a_n = n^{-1}(t+t^2+\ldots+t^n)$. Since $t \in S(t)$ and $S(t)$ is a convex semi-group, $a_n \in S(t)$ $(n = 1, 2, \ldots)$. Since $S(t)$ is compact, the sequence $\{a_n\}$ has at least one cluster point. Let q be any cluster point of $\{a_n\}$. We have

$$ta_n - a_n = n^{-1}(t^{n+1}-t),$$

and so $\lim_{n\to\infty} (ta_n - a_n) = 0$. Therefore $tq = q$. By Theorem 7 (i), it follows that $pq = q$. But $pq = r(q)p$, and, since $q \in S(t)$, $r(q) = 1$. Therefore $q = p$. We have now proved that $\{a_n\}$ has a unique cluster point p, and so $\lim_{n\to\infty} a_n = p$. This proves (ii).

Let u be the identity of the group $G(t)$. Then $u^2 = u$, and so u is a projection. Since $u \in A(t)$, u is also a compact linear operator, and therefore has finite rank.

By definition of $G(t)$, it is clear that $G(t)$ contains all operators of the form $t^k u$ $(k = 1, 2, \ldots)$, and therefore contains the closure of this set of operators. On the other hand, each element g of $G(t)$ is a cluster point of the sequence $\{t^k\}$, and, since $g = gu$, it is a cluster point of the sequence $\{t^k u\}$.

Theorem 9. *Let t be a compact linear operator such that $r(t) = 1$. Then t has equibounded iterates if and only if every eigenvalue of modulus* 1 *has index* 1.

Proof. Suppose first that t has equibounded iterates and that ζ is an eigenvalue with $|\zeta| = 1$. Let $s = \zeta^{-1}t$. Then s has equibounded iterates and $r(s) = 1 \in \operatorname{spec}(s)$. Therefore, by Theorem 7, 1 is an eigenvalue of s of index 1. It follows that ζ is an eigenvalue of t of index 1.

Suppose, on the other hand, that each of the (distinct) eigenvalues $\zeta_1, \zeta_2, \ldots, \zeta_m$ of modulus 1 has index 1, and let $t_i = t - \zeta_i e$. Then, for each i, there exists a spectral projection p_i for ζ_i, i.e. a bounded linear projection p_i the range and null-space of which coincide with the null-space and range of t_i. Let $q = p_1 + p_2 + \ldots + p_m$ and $s = t(e-q)$. Then $p_i p_j = 0$ $(i \neq j)$, q is a projection, and $r(s) < 1$. We have $tp_i = \zeta_i p_i$, and so

$$t^n = t^n q + s^n = \zeta_1^n p_1 + \ldots + \zeta_m^n p_m + s^n \quad (n = 1, 2, \ldots).$$

It follows that t has equibounded iterates.

Theorem 10. *Let t be a compact linear operator such that $r(t) = 1$ and t has equibounded iterates. Let $\zeta_1, \ldots, \zeta_m$ be the eigenvalues of t on the unit circle* $|\zeta| = 1$, and let $t_i = \zeta_i^{-1} t$. *Then the following conclusions hold.*

(i) *$A(t_i)$ is a prime strict commutative locally compact semi-algebra.*

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(ii) *$A(t_i)$ contains a unique minimal idempotent p_i, p_i is a projection of finite rank, and is the spectral projection for the eigenvalue ζ_i which has index* 1.

(iii) *All the groups $G(t_i)$ $(i = 1, \ldots, m)$ have the same identity u, and $u = p_1 + \ldots + p_m$.*

(iv) *$G(t_i)$ is the set of all operators of the form $\lambda_1 p_1 + \ldots + \lambda_m p_m$, where $(\lambda_1, \ldots, \lambda_m)$ belongs to the closure in C^m of the set*

$$\{(\eta_1^k, \eta_2^k, \ldots, \eta_m^k) : k = 1, 2, \ldots\}, \text{ and } \eta_j = \zeta_i^{-1}\zeta_j \quad (j = 1, \ldots, m).$$

Proof. We have $r(t_i) = 1 \in \operatorname{spec}(t_i)$ and t_i has equibounded iterates. Thus (i) and (ii) are already clear.

Let $q = p_1 + \ldots + p_m$. Since p_i is the spectral projection for ζ_i, we have $p_i p_j = 0$ $(i \neq j)$, and q is a projection reducing t. Moreover $r(t(e-q)) < 1$. Let u_i be the identity of the group $G(t_i)$. We have $t_j p_j = p_j$, and so $t_i p_j = \lambda p_j$, where $\lambda = \zeta_j \zeta_i^{-1}$. Since u_i is a cluster point of the sequence $\{t_i^n\}$, there exists a strictly increasing sequence $\{n(k)\}$ of positive integers such that $u_i = \lim_{k \to \infty} t_i^{n(k)}$, and also $\{\lambda^{n(k)}\}$ converges, to μ say. Then $u_i p_j = \mu p_j$. But $\mu \neq 0$, and u_i is an idempotent. Therefore $u_i p_j = p_j$. Since this holds for $j = 1, \ldots, m$, we have $u_i q = q$. Also, since $r(t(e-q)) < 1$, we have

$$\lim_{n \to \infty} t_i^n(e-q) = 0,$$

and therefore $u_i(e-q) = 0$, $u_i = u_i q = q$. This proves (iii).

We have

$$t_i^k u = t_i^k(p_1 + \ldots + p_m) = \eta_1^k p_1 + \ldots + \eta_m^k p_m,$$

with $\eta_j = \zeta_i^{-1}\zeta_j$. Therefore (iv) now follows from Theorem 8(v).

Corollary 1. *The following statements are equivalent to each other.*

(i) *ζ_i is the only eigenvalue of t of modulus* 1.

(ii) $u = p_i$.

(iii) *u is the only element of $G(t_i)$.*

(iv) $\lim_{n \to \infty} t_i^n = u$.

Proof. (i)$\Leftrightarrow$(ii) We have $u = p_1 + \ldots + p_m$ and $p_i p_j = 0$ (i $\neq j$).

(ii)$\Rightarrow$(iii) If $u = p_i$ and $g \in G(t_i)$, then, since $G(t_i) \subset S(t_i)$, we have $g = gu = gp_i = r(g)p_i = p_i = u$.

(iii)$\Rightarrow$(ii) Suppose that u is the only element of $G(t_i)$. Then, since $t_i u \in G(t_i)$, we have $t_i u = u$. Therefore $p_i u = u$. But $p_i u = p_i$.

(iii)$\Rightarrow$(iv) If u is the only element of $G(t_i)$, then every subsequence of $\{t_i^n\}$ has a subsequence that converges to u.

(iv)$\Rightarrow$(iii) Clear.

Corollary 2. *The closed convex hull of $G(t_i)$ is the set of all operators of the form $\alpha_1 g_1 + \ldots + \alpha_r g_r$ with $r = 2m+1$, $g_j \in G(t_i)$, $\alpha_j \geq 0$ $(j = 1, \ldots, r)$,*

$$\sum_{j=1}^{r} \alpha_j = 1.$$

Proof. By Theorem 10 (iv), $G = G(t_i)$ lies in a subspace of $B(X, X)$ of dimension $2m$ over $\boldsymbol{R}$. Therefore its convex hull $co(G)$ is the set of operators of the stated form. Since G is compact and the mapping

$$(g_1, \ldots, g_r, \alpha_1, \ldots, \alpha_r) \to \alpha_1 g_1 + \ldots + \alpha_r g_r$$

is continuous, $co(G)$ is compact, and therefore closed.

Theorem 11. *Let t be a compact linear operator with equibounded iterates, let $r(t) = 1 \in \operatorname{spec}(t)$, and let t have m eigenvalues of modulus 1. Then $A(t)$ is the set of all operators a of the form*

$$a = \sum_{k=1}^{\infty} \alpha_k t^k + \beta_1 g_1 + \ldots + \beta_r g_r,$$

where $r = 2m+1$, $\alpha_k \geqq 0$ $(k = 1, 2, \ldots)$, $\beta_k \geqq 0$, $g_k \in G(t)$ $(k = 1, 2, \ldots, r)$, and

$$\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{r} \beta_k = r(a).$$

Proof. It is clear that each operator a of the stated form belongs to $A(t)$. Suppose, on the other hand, that $a \in A(t)$. Then there exist $a_n \in P(t)$ with $\| a_n \| \leqq \| a \|$ such that $a = \lim_{n \to \infty} a_n$. For each n,

$$a_n = \sum_{k=1}^{\infty} \alpha(n, k) t^k,$$

where $\alpha(n, k) \geqq 0$ $(n, k = 1, 2, \ldots)$, and $\alpha(n, k) = 0$ for sufficiently large k (depending on n). We have $tp = p$, where p is the minimal idempotent in $A(t)$, and $a_n p = r(a_n)p$. Therefore

$$\sum_{k=1}^{\infty} \alpha(n, k) = r(a_n) \leqq \| a_n \| \leqq \| a \|.$$

Using the diagonal process and then throwing away all but a suitable subsequence, we may suppose that for each fixed k the sequence $\{\alpha(n, k)\}$ converges to α_k, say. Then

$$\sum_{k=1}^{\infty} \alpha_k \leqq \| a \|.$$

Let $b = \sum_{k=1}^{\infty} \alpha_k t^k$, which is well defined since t has equibounded iterates. Then, just as in the proof of Theorem 5, using the fact that $r(t(e-u)) < 1$, where u is the identity of the group $G(t)$, we obtain

$$a(e-u) = b(e-u). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Given a positive integer N, let

$$a(N, n) = \sum_{k=1}^{N} \alpha(n, k) t^k, \; b_N = \sum_{k=1}^{N} \alpha_k t^k, \; c(N, n) = a_n - a(N, n).$$

Plainly $c(N, n) \in A(t)$, $\lim_{n \to \infty} a(N, n) = b_N$, $\lim_{n \to \infty} a_n = a$. Therefore

$$\lim_{n \to \infty} c(N, n) = c_N,$$

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say, and, since $A(t)$ is closed, $c_N \in A(t)$. Also $c_N = a - b_N$. We have $\lim_{N\to\infty} b_N = b$, and therefore $\lim_{N\to\infty} c_N = c$, say, $c \in A(t)$, and $c = a - b$.

We have now proved that $a = b + c$, and therefore, by (1),

$$a = b + cu.$$

Since $t^k u \in G(t)$, we have

$$P(t)u \subset \mathbf{R}^+ co(G(t)).$$

Also, since $\overline{co}(G(t)) \subset S(t)$, $0 \notin \overline{co}(G(t))$. Therefore

$$A(t)u \subset \mathbf{R}^+ \overline{co}(G(t)).$$

By Theorem 10, Corollary 2, it follows that

$$cu = \beta_1 g_1 + \ldots + \beta_r g_r$$

with $r = 2m+1$, $\beta_k \geqq 0$, $g_k \in G(t)$ $(k = 1, \ldots, r)$.

Finally, we have $ap = r(a)p$, $t^k p = p$, $g_k p = p$. Therefore

$$r(a) = \sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{r} \beta_k.$$

Corollary. *Let p be the minimal idempotent in $A(t)$ and u the identity of the group $G(t)$. Then p is the least and u is the greatest projection in $A(t)$ in the sense that*

$$qp = p, \; qu = q$$

for every non-zero projection q belonging to $A(t)$.

Proof. Let q be a non-zero projection belonging to $A(t)$. Then $r(q) = 1$, and so $qp = p$. Also, by Theorem 11,

$$q = \sum_{k=1}^{\infty} \alpha_k t^k + \beta_1 g_1 + \ldots + \beta_r g_r,$$

with $r = 2m+1$, $\alpha_k \geqq 0$ $(k = 1, 2, \ldots)$, $\beta_k \geqq 0$, $g_k \in G(t)$ $(k = 1, \ldots, r)$, and

$$\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{r} \beta_k = r(q) = 1.$$

For each $g \in G(t)$, we have $gu = g$, i.e. $g(e-u) = 0$. Let $s = t(e-u)$. Then

$$q(e-u) = \sum_{k=1}^{\infty} \alpha_k t^k (e-u)$$

$$= \{\alpha_1 e + \sum_{k=2}^{\infty} \alpha_k t^{k-1}\} s.$$

Since $r\left(\sum_{k=2}^{\infty} \alpha_k t^{k-1}\right) = \sum_{k=2}^{\infty} \alpha_k$, we therefore have

$$r(q(e-u)) \leqq \left\{\sum_{k=1}^{\infty} \alpha_k\right\} r(s) \leqq r(s) < 1.$$

But $q(e-u)$ is a projection, and therefore $q(e-u) = 0$, $q = qu$.

That $A(t)$ may contain a projection different from p and u is shown by the following simple example.

Example. Let $[\delta_1, \delta_2, \delta_3, \delta_4]$ denote the 4×4 diagonal matrix with diagonal elements $\delta_1, \delta_2, \delta_3, \delta_4$. Let $t = [1, i, -1, -i]$ $(i^2 = -1)$. Then, as an operator in C^4, t has equibounded iterates, and $r(t) = 1 \in \operatorname{spec}(t)$. We have

$$\tfrac{1}{4}(t+t^2+t^3+t^4) = [1, 0, 0, 0],\ \tfrac{1}{2}(t^2+t^4) = [1, 0, 1, 0],\ t^4 = [1, 1, 1, 1].$$

Thus $A(t)$ contains the three projections $p = [1, 0, 0, 0]$, $q = [1, 0, 1, 0]$, $u = [1, 1, 1, 1]$.

4. Irreducible positive operators in a partially ordered Banach space

Let X be a Banach space over C such that $X = X_R \oplus iX_R$, where X_R is a closed real subspace of X. By a *cone* in X_R we mean a subset K of X_R such that

$$K+K\subset K,\ \boldsymbol{R}^+K\subset K,\ K\cap(-K) = (0).$$

We suppose that we are given a closed cone X^+ in X_R such that X_R is the closure of X^+-X^+, and consequently, the complex linear hull of X^+ is dense in X. We call X^+ the *positive cone* in X, and regard X_R as a partially ordered linear space with the relation $\leqq$ of partial order given by $x\leqq y$ (or $y\geqq x$) meaning $y-x\in X^+$. We exclude the trivial case $X = (0)$, and hence exclude also $X^+ = (0)$. We call X with such X_R and X^+ a *partially ordered Banach space.*

Following Namioka (7), we shall say that a cone K is *full* if and only if it satisfies the two extra conditions: (i) $K\subset X^+$, (ii) $X^+\cap(K-X^+)\subset K$. In terms of the partial order relation, (ii) states that $x\in K$ whenever $0\leqq x\leqq k$ for some $k\in K$.

A *non-negative operator* is a bounded linear operator a in X such that $aX^+\subset X^+$, a *positive operator* is a non-zero non-negative operator, and a *strictly positive operator* is a positive operator that maps all non-zero points of X on to non-zero points. Similarly, a continuous linear functional f on X is said to be *non-negative*, *positive*, *strictly positive* if and only if $f(X^+)\subset\boldsymbol{R}^+$, $(0) \neq f(X^+)\subset\boldsymbol{R}^+$, $f(X^+\sim(0))\subset\boldsymbol{R}^+\sim(0)$, respectively.

A positive operator a is said to be *reducible* if and only if there exists a full cone K such that $aK\subset K$, $K\neq(0)$, and K is not dense in X^+. An *irreducible positive operator* is a positive operator that is not reducible. This definition is derived by abstraction from the concept of irreducible non-negative matrix introduced by Frobenius; see Gantmacher (**4**, Chapter 3) and Wielandt (**9**).

It is clear that if t is a positive operator that maps each non-zero point of X^+ on to an interior point of X^+ relative to X_R, then a full cone K with $K\neq(0)$ and $tK\subset K$ contains interior points of X^+, and hence contains X^+. Thus all such operators are irreducible. However, it is easy to construct irreducible positive operators that are not of this kind.

Lemma 2. *Let a be an irreducible positive operator, and let b be a positive operator such that $ab-ba$ is non-negative. Then b is strictly positive.*

Proof. Let $K = \{x: x \in X^+ \text{ and } bx = 0\}$. Then K is a closed full cone. Since $b \neq 0$ and X is the closed linear hull of X^+, $K \neq X^+$. Since $0 \leqq bax \leqq abx = 0$ $(x \in K)$, we have $bax = 0$ $(x \in K)$, and so $aK \subset K$. Therefore $K = (0)$ and b is strictly positive.

Lemma 3. *If X^+ is locally compact* (*in the norm topology*) *and every positive operator is strictly positive, then X has dimension* 1.

Proof. If there exists a positive continuous linear functional f that is not strictly positive, then with any non-zero point u of X^+, the bounded linear operator $u \otimes f$, given by $(u \otimes f)(x) = f(x)u$, is positive, but not strictly positive. Thus we may suppose that X^+ is locally compact and that every positive continuous linear functional is strictly positive. Let X^{*+} denote the set of all non-negative continuous linear functionals on X. Then, since X^+ is a closed convex set, and therefore weakly closed, $X^+ = \{x: f(x) \geqq 0\ (f \in X^{*+})\}$. If X^{*+} does not contain two linearly independent elements, then there exists a single $f \in X^{*+}$ such that $X^+ = \{x: f(x) \geqq 0\}$. But then $f(x) = 0$ implies $x \in X^+ \cap (-X^+)$, $x = 0$. Thus f is a linear isomorphism of X on to $\boldsymbol{C}$, and X has dimension 1.

Suppose then that X^{*+} contains two linearly independent elements f_1, f_2. Since X^+ is locally compact, there exists a compact convex base K for X^+ (**5**, 2.4), i.e. a compact convex subset of $X^+ \sim (0)$ such that $X^+ = \boldsymbol{R}^+ K$. Each positive continuous linear functional is bounded, and bounded away from zero on K, and so μ defined by

$$\mu = \sup\{\lambda: \lambda \geqq 0, f_1 - \lambda f_2 \in X^{*+}\}$$

is finite and non-zero. Then $f = f_1 - \mu f_2$ is positive. But this is absurd, for it implies that there exists $\varepsilon > 0$ such that $f - \varepsilon f_2 \in X^{*+}$.

Theorem 12. *Let p be a compact irreducible positive projection. Then the rank of p is* 1.

Proof. Let $Y = pX$, $Y_R = pX_R$, $Y^+ = pX^+$. Then (Y, Y_R, Y^+) is a partially ordered Banach space in the sense of the present Section. Y has finite dimension, and therefore Y^+ is locally compact. If the rank of p is greater than 1, then, by Lemma 3, there exists a positive operator T on Y that is not strictly positive. Then the operator t on X given by $t = T \bigcirc p$ is a positive operator that is not strictly positive, and it commutes with p, since p is the identity operator on Y. But this is impossible, by Lemma 2.

Lemma 4. *Let A be a semi-algebra of non-negative operators. Then A is strict, and if every non-zero element of A is strictly positive, then A is prime.*

Proof. Let $a, b \in A$ with $a \neq 0$. Then there exists $x \in X^+$ with $ax > 0$, and so $(a+b)x = ax + bx \geqq ax > 0$, $a + b \neq 0$. If a and b are strictly positive, then for every $x > 0$, $(ab)x = a(bx) > 0$.

Lemma 5. *Let a be an irreducible positive operator, and let b be a strictly positive operator such that $ab = b$. Then b is irreducible.*

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Proof. Let J be a full cone that is not dense in X^+ such that $bJ \subset J$; and let

$$K = \{x: 0 \leqq x \leqq bj \text{ for some } j \in J\}.$$

Then K is a full cone. Since $bJ \subset J$ and J is full, we have $K \subset J$, and so K is not dense in X^+. If $x \in K$, then $0 \leqq x \leqq bj$ for some $j \in J$. Therefore $0 \leqq ax \leqq abj = bj$, and so $ax \in K$. Thus $aK \subset K$, and so $K = (0)$. But $bJ \subset K$, and so $bJ = (0)$. Since b is strictly positive, this gives $J = (0)$, b is irreducible.

Theorem 13. *Let t be a compact irreducible positive operator with non-zero spectral radius. Then $A(t)$ is a prime strict commutative locally compact semi-algebra, and the minimal idempotent p in $A(t)$ is an irreducible operator and has rank* 1.

Proof. By the Krein-Rutman theorem (**6**, Theorem 6.1), $r(t)$ is an eigenvalue of t. Therefore, by Theorem 5, $A(t)$ is a locally compact semi-algebra. It is obviously commutative, and by Lemma 4 it is prime and strict. We have

$$tp = pt = r(t)p.$$

Therefore, by Lemma 2, p is strictly positive, and then by Lemma 5, p is irreducible. Finally, by Theorem 12, p has rank 1.

Theorem 14. *Let t_0 be a compact irreducible positive operator with non-zero spectral radius, let p_0 be the minimal idempotent in $A(t_0)$, and let t be a compact positive operator that commutes with t_0. Then the following conclusions hold.*

(i) *$A(t)$ is a prime strict commutative locally compact semi-algebra.*

(ii) $r(t) \neq 0$.

(iii) *$t' = (r(t))^{-1}t$ has equibounded iterates, and* $r(t') = 1 \in$ spec (t').

(iv) $tp_0 = r(t)p_0$.

Proof. By Theorem 13, there exists $u \in X^+$, $u \neq 0$, such that $p_0X = Cu$. Thus there exists a strictly positive continuous linear functional f such that

$$p_0x = f(x)u \quad (x \in X).$$

Since t commutes with t_0 and $p_0 \in A(t_0)$, t commutes with p_0, and so

$$f(x)tu = f(tx)u \quad (x \in X).$$

By Lemma 2, t is strictly positive, and therefore $tx > 0$ and $f(tx) > 0$ whenever $x > 0$. Therefore $tu = \lambda u$ with $\lambda > 0$. It follows that $r(t) > 0$. We can therefore apply the argument used in the proof of Theorem 13, and we see that $A(t)$ is a prime strict commutative locally compact semi-algebra. We have now proved (i) and (ii), and clearly $r(t') = 1$. Therefore, by Theorem 6, $1 \in$ spec (t') and t' has equibounded iterates.

Since $tu = \lambda u$, we have $tp_0 = \lambda p_0$. Let p be the minimal idempotent in $A(t)$. Then $pt = tp = r(t)p$, and so

$$r(t)pp_0 = tpp_0 = ptp_0 = p(\lambda p_0) = \lambda pp_0.$$

Also, since p and p_0 are strictly positive, $pp_0 \neq 0$, $\lambda = r(t)$.

5. Irreducible positive operators in a complex Banach lattice

Let X be a linear space over C, and X_R a real linear subspace of X such that $X = X_R \oplus iX_R$, i.e. such that each $x \in X$ has a unique expression in the form $x = x_1 + ix_2$ with $x_1, x_2 \in X_R$. We define $\mathrm{Re}\,(x)$ for such x by

$$\mathrm{Re}\,(x) = x_1.$$

Clearly the mapping $x \to \mathrm{Re}\,(x)$ is a real linear projection of X on to X_R.

Let X^+ be a cone in X_R, and let X_R be given the corresponding partial ordering. We say that (X, X_R, X^+) is a *complex linear lattice* if and only if there exists a mapping $x \to |x|$ of X into X^+ that satisfies the following axioms.

L1. $\mathrm{Re}\,(e^{i\theta}x) \leqq |x|$ for all real θ.

L2. If $\mathrm{Re}\,(e^{i\theta}x) \leqq y$ for all real θ, then $|x| \leqq y$.

The axioms state that the set $\{\mathrm{Re}\,(e^{i\theta}x) : \theta \in R\}$ has a least upper bound $|x|$. We list in the following lemma some elementary properties of a complex linear lattice.

Lemma 6. *Let (X, X_R, X^+) be a complex linear lattice. Then for all $x, y \in X$ and $\alpha \in C$, we have:*

(i) $|x+y| \leqq |x| + |y|$;

(ii) $|\alpha x| = |\alpha|\,|x|$;

(iii) $||x| - |y|| \leqq |x-y|$;

(iv) $||x|| = |x|$;

(v) $|\mathrm{Re}\,(x)| \leqq |x|$;

(vi) $X^+ = \{x : x = |x|\}$;

(vii) $x = 0$ *if and only if* $|x| = 0$.

Proof. (i) We have $\mathrm{Re}\,(e^{i\theta}x) \leqq |x|$ and $\mathrm{Re}\,(e^{i\theta}y) \leqq |y|$ for all real θ. Therefore

$$\mathrm{Re}\,(e^{i\theta}(x+y)) \leqq |x| + |y| \quad (\theta \in R),$$

and so, by L.2, $|x+y| \leqq |x| + |y|$.

(ii) Straightforward.

(iii) By (i), $|x| \leqq |x-y| + |y|$, and so $|x| - |y| \leqq |x-y|$. Interchanging x and y and using (ii), we have

$$|y| - |x| \leqq |y-x| = |-1(x-y)| = |x-y|.$$

Since $-1 \leqq \cos\theta \leqq 1$, we now have

$$\mathrm{Re}\,(e^{i\theta}(|x| - |y|)) = (\cos\theta)(|x| - |y|) \leqq |x-y| \quad (\theta \in R),$$

and therefore $||x| - |y|| \leqq |x-y|$.

(iv) We have $\mathrm{Re}\,(e^{i\theta}|x|) = (\cos\theta)|x| \leqq |x|$ $(\theta \in R)$, and so $||x|| \leqq |x|$. On the other hand, $|x| = \mathrm{Re}\,(e^{i0}|x|) \leqq ||x||$.

(v) $\mathrm{Re}\,(e^{i\theta}\,\mathrm{Re}\,(x)) = (\cos\theta)\,\mathrm{Re}\,(x) \leqq |x|$, since $\pm\mathrm{Re}\,(x) = \mathrm{Re}\,(\pm x) \leqq |x|$.

(vi) If $x \in X^+$, then $\mathrm{Re}\,(e^{i\theta}x) = (\cos\theta)x \leqq x$, and so $|x| \leqq x$. But also $x = \mathrm{Re}\,(e^{i0}x) \leqq |x|$. Thus $x = |x|$. On the other hand, by definition, $|x| \in X^+$ for all x. Thus $x \in X^+$ whenever $x = |x|$.

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(vii) If $x = 0$, then $\mathrm{Re}\,(e^{i\theta}x) = 0$ for all real θ, and so $|x| \leqq 0$. Therefore $|x| = 0$. On the other hand, if $|x| = 0$, and $x = u+iv$ with $u, v \in X_R$, we have $(\cos\theta)u-(\sin\theta)v \leqq 0$ for all real θ. By taking $\theta = 0, \pi, \pi/2, 3\pi/2$, we obtain $u = v = 0$, $x = 0$.

Definition. A complex linear lattice (X, X_R, X^+) with modulus $|\,.\,|$, is called a *complex Banach lattice* if and only if the following axioms are also satisfied.

L3. X is a complex Banach space.

L4. If $|x| \leqq |y|$, then $\|x\| \leqq \|y\|$.

Lemma 7. *Let (X, X_R, X^+) be a complex Banach lattice. Then we have:*

(i) $\|\,|x|\,\| = \|x\| \quad (x \in X)$*;*

(ii) $\|\mathrm{Re}\,(x)\| \leqq \|x\| \quad (x \in X)$*;*

(iii) *X_R is a closed real linear subspace of X;*

(iv) *X^+ is a closed cone in X_R, and $X_R = X^+ - X^+$;*

(v) *(X, X_R, X^+) is a partially ordered Banach space in the sense of Section* 4.

Proof. (i) Lemma 6 (iv) and L4.

(ii) Lemma 6 (v) and L4.

(iii) This follows from (ii) and the linearity of the mapping $x \to \mathrm{Re}\,(x)$.

(iv) By Lemma 6 (iii), we have $\|\,|x|-|y|\,\| \leqq \|x-y\|$. Therefore the mapping $x \to |x|$ is continuous. Lemma 6 (vi) now shows that X^+ is closed. For $x \in X_R$, we have $x \leqq |x|$, and so $x \in X^+ - X^+$.

(v) This follows from (iii) and (iv).

Lemma 8. *Let x be an element of a complex Banach lattice such that* $\mathrm{Re}\,(x) = |x|$. *Then $x = |x|$.*

Proof. Let $x = u+iv$ with $u, v \in X_R$. We are given that $u = |x|$, and so

$$(\cos\theta)u+(\sin\theta)v = \mathrm{Re}\,(e^{-i\theta}(u+iv)) \leqq |u+iv| = u \quad (\theta \in \boldsymbol{R}).$$

Therefore

$$(\sin\theta)v \leqq (1-\cos\theta)u \quad (\theta \in \boldsymbol{R}),$$

$$\left(\cos\frac{\theta}{2}\right)v \leqq \left(\sin\frac{\theta}{2}\right)u \quad (0<\theta<\pi).$$

Letting $\theta \to 0$ and using the fact that X^+ is closed, we obtain $v \leqq 0$. A similar argument, starting with $e^{i\theta}(u+iv)$ gives $-v \leqq 0$. Thus $v = 0$, and $x = u = |x|$.

Definition. A linear mapping d of a complex linear lattice X on to X such that

$$|dx| = |x| \quad (x \in X)$$

is here called a *rotation* of X. A complex linear lattice X is said to be *alignable* if and only if for each $x \in X$ there exists a rotation d of X such that $x = d|x|$.

Since $|x| = 0$ if and only if $x = 0$, a rotation d is a linear isomorphism.

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Also, its inverse d^{-1} is a rotation, for we have

$$|d^{-1}x| = |d(d^{-1}x)| = |x| \quad (x \in X).$$

It is clear that a rotation of a complex Banach lattice is an isometry.

Theorem 15. *Let X be an alignable complex Banach lattice. Let t be a compact irreducible positive operator in X with $r(t) = 1$, and let a be a linear mapping of X into itself such that*

$$|ax| \leqq t|x| \quad (x \in X).$$

If there exists a non-zero vector y such that $ay = \zeta y$ and $|\zeta| = 1$, then $t|y| = |y|$, and there exists a rotation d of X such that

(i) $d|y| = y$,

(ii) $a = \zeta d t d^{-1}$.

Proof. Suppose that $ay = \zeta y$ with $y \neq 0$ and $|\zeta| = 1$, and let $w = t|y| - |y|$. Then

$$|y| = |\zeta y| = |ay| \leqq t|y|,$$

and so $w \geqq 0$. Let p be the minimal idempotent in $A(t)$. Then $pt = p$, and so

$$pw = pt|y| - p|y| = p|y| - p|y| = 0.$$

But p is strictly positive, and so $w = 0$, $t|y| = |y|$.

Since X is alignable, there exists a rotation d such that $d|y| = y$. Let $b = \zeta^{-1}d^{-1}ad$. Then

$$b|y| = \zeta^{-1}d^{-1}ad|y| = \zeta^{-1}d^{-1}ay = \zeta^{-1}d^{-1}\zeta y = d^{-1}y = |y| = t|y|,$$

$$(t-b)|y| = 0. \quad \text{......(1)}$$

Also

$$|bx| = |\zeta^{-1}d^{-1}adx| = |adx| \leqq t|dx| = t|x| \quad (x \in X). \quad \text{......(2)}$$

Let c be defined by $cx = \mathrm{Re}\,(bx)\,(x \in X)$. By (2), $\| bx \| \leqq \| t|x| \| \leqq \| t \| \| x \|$, and so b and c are bounded (complex and real) linear operators. For $x \in X^+$, (2) gives

$$cx = \mathrm{Re}\,(bx) \leqq |bx| \leqq t|x| = tx, \quad \text{.....................(3)}$$

and so $t-c$ is a non-negative operator. By (1), since $t|y| \in X_R$,

$$(t-c)|y| = 0.$$

Since p has rank 1 and $t|y| = |y|$, the range of p is spanned by $|y|$, and so

$$(t-c)p = 0. \quad \text{.......................................(4)}$$

But $p(t-c)$ is non-negative, and so $p(t-c)-(t-c)p$ is non-negative and Lemma 2 is applicable. By (4), $t-c$ is not strictly positive, and so $t-c = 0$. Therefore, by (3),

$$\mathrm{Re}\,(bx) = tx = |bx| \quad (x \in X^+).$$

By Lemma 8, it follows that $bx = |bx| = tx$ $(x \in X^+)$. But X is the complex linear hull of X^+, and so $b = t$, $a = \zeta d t d^{-1}$.

Lemma 9. *Let d be a rotation of a complex linear lattice X, let $Y = X$, $Y_R = dX_R$, $Y^+ = dX^+$, and let $|y|_Y$ be defined by*

$$|y|_Y = d|y| \quad (y \in Y).$$

Then (Y, Y_R, Y^+) *is a complex linear lattice with modulus given by* $|\,.\,|_Y$, *and the mapping d is an order isomorphism of* (X_R, X^+) *on to* (Y_R, Y^+).

If (X, X_R, X^+) *is a complex Banach lattice, then so is* (Y, Y_R, Y^+).

Proof. Since d is a complex linear isomorphism of X on to Y and $dX_R = Y_R$, we have $Y = Y_R \oplus iY_R$, and the real part operation in Y is given by

$$\mathrm{Re}_Y(dx) = d\,\mathrm{Re}(x) \quad (x \in X). \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Moreover Y^+ is a cone in Y_R and $x \in X^+$ if and only if $dx \in Y^+$. Therefore d is an order isomorphism of (X_R, X^+) on to (Y_R, Y^+). Let $\leqq_Y$ denote the partial order relation given by Y^+ in Y_R. Then we have $\mathrm{Re}(e^{i\theta}x) \leqq u$ if and only if $d\,\mathrm{Re}(e^{i\theta}x) \leqq_Y du$, i.e. (by (1)) if and only if $\mathrm{Re}_Y(e^{i\theta}dx) \leqq_Y du$. Therefore (Y, Y_R, Y^+) is a complex linear lattice with the modulus given by $|\,y\,|_Y = d\,|\,y\,|$.

Suppose now that (X, X_R, X^+) is a complex Banach lattice, and that $|\,y\,|_Y \leqq_Y |\,z\,|_Y$. Then $d\,|\,y\,| \leqq_Y d\,|\,z\,|$, and, since d is an order isomorphism, $|\,y\,| \leqq |\,z\,|$, $\|\,y\,\| \leqq \|\,z\,\|$. Thus (Y, Y_R, Y^+) is a complex Banach lattice.

Lemma 10. *Let t be a non-negative operator in a complex Banach lattice X. Then*

$$|\,tx\,| \leqq t\,|\,x\,| \quad (x \in X).$$

Proof. Since $tX^+ \subset X^+$, we have $tX_R \subset X_R$, and therefore $t\,\mathrm{Re}(x) = \mathrm{Re}(tx)$. Therefore, for all $x \in X$ and $\theta \in \boldsymbol{R}$, we have

$$\mathrm{Re}(e^{i\theta}tx) = \mathrm{Re}(t(e^{i\theta}x)) = t(\mathrm{Re}(e^{i\theta}x)) \leqq t\,|\,x\,|;$$

and so, by L2, $|\,tx\,| \leqq t\,|\,x\,|$.

Theorem 16. *Let X be an alignable complex Banach lattice. Let t be a compact irreducible positive operator in X with* $r(t) = 1$, *and let z be the eigenvector of t with* $z > 0$, $\|\,z\,\| = 1$, *and* $tz = z$. *Let* ζ *be an eigenvalue of t with* $|\,\zeta\,| = 1$, *y a corresponding eigenvector,* $ty = \zeta y$, $\|\,y\,\| = 1$.

Then $|\,y\,| = z$, *and there exists a rotation d with the following properties.*

(i) $y = dz$.

(ii) $t = \zeta dtd^{-1}$.

(iii) $\zeta^{-1}t$ *is an irreducible positive operator with respect to the complex Banach lattice* (Y, Y_R, Y^+), *where* $Y = X$, $Y_R = dX_R$, $Y^+ = dX^+$.

Corollary 1. *The spectral projection corresponding to the eigenvalue* ζ *has rank* 1.

Corollary 2. *If* $\lambda \in \mathrm{spec}\,(t)$, *then* $\zeta\lambda$, $\zeta^{-1}\lambda \in \mathrm{spec}\,(t)$.

Proof. Lemma 10 shows that we can apply Theorem 15 with $a = t$. We have $t\,|\,y\,| = |\,y\,|$, $\|\,|\,y\,|\,\| = \|\,y\,\| = 1$. Since, by Theorems 7 and 13, the eigenspace corresponding to 1 has dimension 1, it follows that $|\,y\,| = z$. Also, by Theorem 15, there exists a rotation d such that $d\,|\,y\,| = y$, and $t = \zeta dtd^{-1}$. By Lemma 9, (Y, Y_R, Y^+), given by $Y = X$, $Y_R = dX_R$, $Y^+ = dX^+$, is a complex Banach lattice. We have

$$(\zeta^{-1}t)Y^+ = (\zeta^{-1}t)dX^+ = (dt)X^+ \subset dX^+ = Y^+,$$

and so $\zeta^{-1}t$ is a positive operator with respect to (Y, Y_R, Y^+).

Let K be a full cone with respect to the positive cone Y^+ such that $K \neq 0$, K is not dense in Y^+, and $\zeta^{-1}tK \subset K$. Let $J = d^{-1}K$. Since d^{-1} is an order isomorphism and isometry, J is a full cone with respect to X^+, $J \neq 0$, J is not dense in X^+.

$$tJ = td^{-1}K = d^{-1}(\zeta^{-1}t)K \subset d^{-1}K = J.$$

But this is impossible, and so $\zeta^{-1}t$ is irreducible.

To prove Corollary 1, we note that the spectral projection corresponding to the eigenvalue ζ of t is the spectral projection corresponding to the eigenvalue 1 of $\zeta^{-1}t$. Since $\zeta^{-1}t$ is irreducible, this projection has rank 1.

To prove Corollary 2, suppose that λ is a non-zero point of spec (t). Then λ is an eigenvalue of t and there exists $v \neq 0$ with $tv = \lambda v$. We have

$$tdv = \zeta dtv = \zeta\lambda dv, \quad td^{-1}v = \zeta^{-1}d^{-1}tv = \zeta^{-1}\lambda d^{-1}v.$$

Since $dv \neq 0$ and $d^{-1}v \neq 0$, this shows that $\zeta\lambda$ and $\zeta^{-1}\lambda$ belong to spec (t).

Theorem 17. *Let X be an alignable complex Banach lattice, let t be a compact irreducible positive operator in X with $r(t) = 1$, and let $\zeta_1, \ldots, \zeta_m$ be the eigenvalues of t on the unit circle $|\zeta| = 1$. Then the following conclusions hold.*

(i) *Each of the eigenvalues $\zeta_1, \ldots, \zeta_m$ is simple.*

(ii) *$\zeta_1, \ldots, \zeta_m$ are the roots of the equation $\zeta^m = 1$.*

(iii) $\exp(2\pi i/m)$ spec (t) = spec (t).

Proof. (i) This follows at once from Theorems 9, 14 and 16, Corollary 1.

(ii) By Theorem 16, Corollary 2, $\zeta_i\zeta_j$ and $\zeta_i^{-1}\zeta_j$ belong to spec (t) for all i, j $(1, 2, \ldots, m)$, and they are therefore members of the set $(\zeta_1, \zeta_2, \ldots, \zeta_m)$. Thus this set is a group, and since it has exactly m elements, each element satisfies the equation $\zeta^m = 1$.

(iii) This now follows from Theorem 16, Corollary 2, if we take $\zeta = \exp(2\pi i/m)$.

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