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On the edge-reconstruction of graphs

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A graph X is edge-reconstructible if it is uniquely determined up to isomorphism by the set of graphs $X - e$ obtained by deleting one edge e. The graphs of a comparatively rich class are shown to be edge-reconstructible. This class contains all non-trivial strong products and certain lexicographic products.

An outstanding and unsolved problem in graph theory is the Ulam problem [6] of reconstructing graphs. The vertex-form of this problem can be formulated as follows. If X and Y are two graphs with more than 3 vertices such that there exists a bijection $\varphi : V(X) \to V(Y)$ with $X - x \cong Y - \varphi x$ for all $x \in V(X)$, is it then true that $X \cong Y$? This problem has been treated in a series of papers, see [1, 2, 3, 4], but only partial results have been obtained. The same situation prevails for the edge-problem. There is a bijection $\varphi : E(X) \to E(Y)$ such that $X - e \cong Y - \varphi e$ for all $e \in E(X)$ and the question is whether the existence of such a φ implies $X \cong Y$. The reference [4] is also concerned with the edge-problem. We shall call a graph X edge-reconstructible if every Y for which a φ as above exists is isomorphic with X.

Before turning to the main result we recall some graph-theoretic definitions. Graph-theoretic notions not defined in this paper can be found in [5].

A graph X is a pair $\big(V(X), E(X)\big)$ where $V(X)$ is the set of vertices and $E(X)$ is a set of unordered pairs $[x, y]$ of different elements of $V(X)$. The elements of $E(X)$ are the edges of X. By $|X|$ is meant the cardinal number of $V(X)$. A graph X is called complete if

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$E(X)$ consists of all possible edges. A clique in a graph is a maximal complete subgraph. A k-clique is a clique with k vertices.

The next definition is crucial for this paper.

DEFINITION. A clique C of the graph X is said to have property (D) if there does not exist a clique $C' \neq C$ such that $|C-C'| = 1$.

THEOREM 1. *Let X be a finite connected graph with the following two properties:*

(I) *for some natural number $k \geq 3$ in X exist cliques with k vertices but no cliques with $k-1$ vertices;*

(II) *all cliques with k vertices have property (D).*

Then X is edge-reconstructible.

Proof. For X being a complete graph with $n > 3$ vertices the assertion is trivial. Therefore we assume X to be not complete. At first we shall study the influence of deleting an edge on the cliques of the graph X. Let C be a clique in X and $e = [x, y]$ an edge joining two vertices x, y of C. The deletion of e gives rise to complete subgraphs on the vertex sets $V(C) - x$ and $V(C) - y$. $V(C) - x$ will be a clique in $X - e$ iff in X does not exist a clique $C' \neq C$ with $V(C) - x \subset V(C')$; analogously for $V(C) - y$. If therefore C has property (D) then for all $x, y \in V(C)$, $x \neq y$, $V(C) - x$ and $V(C) - y$ in $X - e$, $e = [x, y]$, span two cliques C_1 and C_2 for which holds $|C_1-C_2| = |C_2-C_1| = 1$. Two cliques with this property in the following will be called related.

Now let be given the graphs $X_1, \ldots, X_m$, $m = |E(X)|$ obtained by deleting one edge from the graph X which is supposed to meet the conditions of the theorem. From the above considerations it follows that among these graphs such X_i must exist which have pairs of related cliques with $k-1$ vertices. Further these X_i only can result from deleting an edge from X which joins two vertices of a clique of X with k vertices. Let us choose arbitrarily one of the X_i with cliques consisting of $k-1$ vertices. If C_1, C_2 are two related cliques with

$|C_1| = |C_2| = k - 1$ the endpoints x, y of the edge, the deletion of which resulted in X_i, are easily determined as $\{x\} = C_1 - C_2$ and $\{y\} = C_2 - C_1$. Therefore it is clear how to reconstruct the graph X from the selected X_i. We should remark that there may be several pairs of related $(k-1)$-cliques in X_i. But all these pairs determine the same vertices x, y as the endpoints of the deleted edge. It remains to show that by adding any edge $e \neq [x, y]$ to X_i one obtains a graph Y such that for at least one edge $e_1 \in E(Y)$ the graph $Y - e_1$ is not isomorphic with any X_i, $1 \leq i \leq m$. Suppose we have added an edge $e = [x_1, y_1] \neq [x, y]$, thereby obtaining the graph Y. First let neither x_1 nor y_1 belong to any $(k-1)$-clique of X_i. Then in the graph Y the pairs of related $(k-1)$-cliques are the same as in X_i. Let us select one of these pairs C_1, C_2 and let $z \in V(C_1) \cap V(C_2)$. Then by deleting the edge $e_1 = [x, z]$ from Y the clique C_1 is destroyed and C_2 remains unchanged. In $Y - e_1$ the $(k-1)$-clique C_2 is not related to any other $(k-1)$-clique. On the other hand from our assumptions and the considerations at the beginning of the proof we have that in an X_i with $(k-1)$-cliques to every $(k-1)$-clique exists a uniquely determined related $(k-1)$-clique. This accomplishes the considered case. Now let us consider the possibility that the endpoint x_1 of e belongs to $V(C_1) \cup V(C_2)$ and y_1 does not. From our assumption about the k-cliques in X it follows that there are at least two vertices in $V(C_1) \cup V(C_2)$ to which y_1 is not adjacent. Therefore by adding $e = [x_1, y_1]$ at most one of the cliques C_1, C_2 of X_i can be destroyed, that is, is properly contained in another clique C_3 of Y. If neither C_1 nor C_2 is destroyed by adding e then the same procedure as above directly will lead to the same contradiction. If for instance C_1 is destroyed then y_1 in X_i is adjacent to all vertices of C_1 besides $x_1 \in V(C_1)$ but not adjacent to $y \in V(C_2)$. Then the resulting clique C_3 of Y with $C_3 \supset C_1$, $C_3 \neq C_1$

must be of cardinality $|C_3| > k$ because otherwise in X would exist a $(k-1)$-clique. But this means that in X_i and therefore in X exists a vertex v, $v \notin V(C_1) \cup V(C_2)$, adjacent to all vertices of C_1 but not to y, which contradicts our assumption about the k-cliques in X. This completes the proof of the theorem.

REMARK 1. It should be mentioned that Theorem 1 holds also if $V(X)$ has infinite cardinality.

REMARK 2. It can be shown that the conditions of the theorem for some $k \geq 3$ are met by a graph X if and only if in all the subgraphs X_i $(k-1)$-cliques occur only as related pairs. If therefore the graphs X_i are known one can decide whether the unknown graph X meets the conditions of the theorem.

Now we shall apply this result to the strong and to the lexicographic product and we start with the first one.

DEFINITION. Let two graphs $X = (V(X), E(X))$ and $Y = (V(Y), E(Y))$ be given. The strong product $X * Y$ of X and Y is the graph Z defined by

$V(Z) = V(X) \times V(Y)$ (set-theoretic cartesian product),

$$E(Z) = \{[(x_1, y_1), (x_2, y_2)] \mid x_1 = x_2 \wedge [y_1, y_2] \in E(Y) \text{ or } y_1 = y_2 \wedge [x_1, x_2] \in E(X) \text{ or } [x_1, x_2] \in E(X) \wedge [y_1, y_2] \in E(Y)\} .$$

The following two lemmas are rather obvious consequences of the definition.

LEMMA 1. *Every clique C in $X * Y$ is of the form $C = C_1 * C_2$, where C_1 is a clique in X and C_2 is a clique Y.*

LEMMA 2. *For any two cliques C and C' in $X * Y$ holds $|C-C'| \geq 2$.*

The second lemma can be formulated as: every clique in a strong product has property (D). Now it is easy to prove the following theorem.

THEOREM 2. *A finite strong product $X * Y$ where $|X| \geq 2$ and $|Y| \geq 2$ is edge-reconstructible.*

Proof. From Lemma 1 it follows that every clique in $X * Y$ has at least four vertices and therefore $X * Y$ meets the first condition of Theorem 1. The second condition of this theorem is met because of Lemma 2.

REMARK 3. Theorem 2 still holds for infinite products $X * Y$ if both X and Y possess finite cliques.

DEFINITION. The lexicographic product $Z = X \circ Y$ of the graphs X and Y is defined by

$$V(Z) = V(X) \times V(Y) ,$$

$$E(Z) = \{[(x_1, y_1), (x_2, y_2)] \mid x_1 = x_2 \wedge [y_1, y_2] \in E(Y) \text{ or } [x_1, x_2] \in E(X)\} .$$

To be able to apply Theorem 1 we have to study the cliques in a lexicographic product.

LEMMA 3. *The vertex-set of a clique* C *in* $X \circ Y$ *is of the form* $\bigcup_{x \in V(C_1)} \{x\} \times V(C_x)$ *where* C_1 *is a clique in* X *and the* C_x *are cliques in* Y .

This is an immediate consequence of the definition.

LEMMA 4. *All cliques in* $X \circ Y$, $|Y| \geq 2$, *have property* (D) *if the cliques in* Y *have this property.*

Proof. Let C and C' be different cliques in $X \circ Y$ and $V(C) = \bigcup_{x \in V(C_1)} \{x\} \times V(C_x)$, $V(C') = \bigcup_{x \in V(C_1')} \{x\} \times V(C_x')$. If $C_1 \neq C_1'$ then clearly C and C' have property (D) . Otherwise for some $x \in V(C_1)$ we must have $V(C_x) \neq V(C_x')$ and therefore $|C_x - C_x'| \geq 2$ which proves the lemma.

Now our last theorem follows easily.

THEOREM 3. *A finite lexicographic product* $X \circ Y$ *with* $|X| \geq 2$, $|Y| \geq 2$ *is edge-constructible if the cliques in* Y *have property* (D) .

Proof. From Lemma 3 we have that every clique in $X \circ Y$ has at least four vertices and therefore condition (I) of Theorem 1 is met. The other condition holds because of Lemma 4.

Having proved these theorems the reader might be interested in previous results on the edge-reconstruction. In [4] the following classes of graphs are shown to be edge reconstructible:

(1) disconnected graphs with at least two non-trivial components;

(2) trees;

(3) connected graphs with bridges but without twigs (see [4]) of length more than one;

(4) regular graphs.

From this it can be seen that our results bring some real progress to the problem considered.

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