AN n-DIMENSIONAL ANALOGUE OF CAUCHY'S INTEGRAL THEOREM

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1. Introduction

As in [5] a parametric n-surface in R^k (where $k \geqq n$) will be a pair (f, M^n), consisting of a continuous mapping f of an oriented topological manifold M^n into the euclidean k-space R^k. (f, M^n) is said to be closed if M^n is compact. The main purpose of this paper is to use the method of [4] to prove a general form of Cauchy's Integral Theorem (Theorem 5.3) for those closed parametric n-surfaces (f, M^n) in R^{n+1}, which have bounded variation in the sense of [5] and for which $f(M^n)$ has a finite Hausdorff n-measure. As in [4], the proof is carried out by approximating the surface with a simpler type of surface. However, when $n > 1$, a difficulty arises in that there are entities, which occur in a natural way, but are not parametric surfaces. We therefore introduce a concept which we call an S-system and which forms a generalisation (see 2.2) of the type of closed parametric n-surface that was studied in [5] II, 3 in connection with a proof of a Gauss-Green Theorem. The surfaces of [5] II, 3 include those that are studied in this paper.

Approximation theorems (4.2 and 4.3) are obtained for S-systems and these are used to prove Cauchy's Theorem for S-systems. Cauchy's Theorem for parametric surfaces is then derived by showing that the relevant closed parametric n-surface in R^{n+1} is a particular case of an S-system.

The definitions used for parametric surfaces and their integrals are those of [5]. It is regretted that on p. 616 of [5] we mentioned the possibility, that a certain case of the surface integral of [5], might be equivalent to the integral defined by L. Cesari in [1]. This is incorrect, because equivalence could occur with at most a particular case of the Cesari surface integral.

The following notational conventions are adopted. The interior, closure and Frontier (or boundary) of a set A are denoted by, Int (A), $\bar{A}$ and $\mathrm{Fr}(A)$. Set complementation is denoted by $\sim$. $\varnothing$ denotes the empty set. Distance is denoted by d. R^k denotes the real euclidean k-space. If $x \in R^k$, then x_i represents the ith coordinate of x; $(x)_i$ is thus a mapping from R^k to R^1.

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The norm $\sqrt{(x_1^2 + \cdots + x_k^2)}$ of the point x of R^k is denoted by $||x||$. P_i, $(i = 1, \cdots, k+1)$ denotes the projection from R^{k+1} to R^k given by

$$P_i(x) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1})$$

The term 'integrable' will be used in the sense that a function f is integrable if it is measurable and $|f|$ has a finite integral. Throughout the entire paper n will be a fixed positive integer.

2. S-systems

2.1. Definition. We denote by $\mathscr{F}$ the Banach space whose points are those real-valued functions on R^{n+1} each of which is bounded and continuous and whose norm is the norm of uniform convergence; i.e.,

$$||f|| = \underset{x \in R^{n+1}}{\text{L.u.b.}} |f(x)|.$$

2.2. Definition. By an S-system we mean a pair consisting of a compact subset K of R^{n+1} and an integral-valued function u on $R^{n+1} \sim K$ and with (K, u) possessing the following properties.

(i) The $(n+1)$-dimensional Lebesgue measure of K is zero.

(ii) u is constant on each component of $R^{n+1} \sim K$ and is zero on the unbounded component.

(iii) For each $i = 1, \cdots, n+1$, there exists a non-negative, extended real valued, integrable function, $e_i(y)$ on R^n, such that:

for every $y \in R^n$ and every finite sequence of points

$$x^{(0)}, x^{(1)}, \cdots, x^{(r)} \quad \text{of} \quad P_i^{-1}(y) \cap (R^{n+1} \sim K) \text{ with}$$

$$x_i^{(0)} < x_i^{(1)} < \cdots < x_i^{(r)},$$

one always has

$$\sum_{j=1}^{r} |u(x^{(j-1)}) - u(x^{(j)})| \leqq e_i(y).$$

We will say that a function $e_i(y)$, satisfying 2.2 (iii), bounds the ith multiplicity of the S-system.

Whenever a symbol — say E — is used to denote an S-system, then the compact set and the integral valued function that comprise E will be denoted by $K(E)$ and u_E respectively, or sometimes just by K and u.

If (f, M^n) is a closed parametric n-surface in R^{n+1} with bounded variation and the $(n+1)$-dimensional Lebesgue measure of $f(M^n)$ equal to zero, then it follows from [5] I 2.5 and 2.6 and II 3.5 and 1.10 that

$$\{f(M^n), \quad u(f, M^n, x)\}$$

is an S-system. Thus an S-system forms a generalisation of this closed parametric n-surface in R^{n+1}.

2.3. Definition. If E is an S-system and $i = 1, \cdots, n+1$, then we denote by $Y_i(E)$, the subset of R^n consisting of those points y that have the following property.

(i) For each point x of $P_i^{-1}(y)$, there exists a $\lambda > 0$ such that x is a point of accumulation of each of the two sets

$$[R^{n+1} \sim K(E)] \cap \{\xi; \xi \in P_i^{-1}(y) \quad \text{and} \quad x_i - \lambda < \xi_i < x_i\},$$
$$[R^{n+1} \sim K(E)] \cap \{\xi; \xi \in P_i^{-1}(y) \quad \text{and} \quad x_i < \xi_i < x_i + \lambda\}$$

and u_E has constant values

$$\alpha_i(E, x), \quad \beta_i(E, x)$$

on each of these two sets.

It follows from 2.3 (i), 2.2 (ii) and the compactness of $K(E)$, that

2.3.1. *if* $y \in Y_i(E)$, *then* $\alpha_i(E, x) = \beta_i(E, x)$ *for all points* x *of* $P_i^{-1}(y)$ *except at most a finite number.*

2.4. Theorem. *If* E *is an* S*-system, then for each* $i = 1, 2, \cdots, n+1$, $R^n \sim Y_i(E)$ *has zero measure.*

Proof. Assume i fixed and let e_i be an integrable function bounding the ith multiplicity of E. Denote by Z_i the set of all those points y of R^n for which the subset $\{K(E) \cap P_i^{-1}(y)\}_i$ of R^1 has zero 1-mesaure. By 2.2 (i) and Fubini's theorem, the n-measure of $R^n \sim Z_i$ is zero. Let $y' \in Z_i \sim (Y_i \cap Z_i)$ and take an arbitrarily large positive integer r. For each point x of $P_i^{-1}(y')$ and each $\lambda > 0$, x is a point of accumulation of each of the two sets

$$N_-(x, \lambda) = [R^{n+1} \sim K(E)] \cap \{\xi; \xi \in P_i^{-1}(y') \quad \text{and} \quad x_i - \lambda < \xi_i < x_i\}$$
$$N_+(x, \lambda) = [R^{n+1} \sim K(E)] \cap \{\xi; \xi \in P_i^{-1}(y') \quad \text{and} \quad x_i < \xi_i < x_i + \lambda\}$$

Therefore by 2.3, there exists a point x' of $P_i^{-1}(y')$ and a $j = +, -$ such that for no $\lambda > 0$ is u constant on $N_j(x', \lambda)$. Hence one can choose a sequence $x^{(0)}, x^{(1)}, \cdots, x^{(r)}$ of points of $P_i^{-1}(y') \cap \{R^{n+1} \sim K(E)\}$ such that $x_i^{(0)} < x_i^{(1)} < \cdots < x_i^{(r)}$ and $u(x^{(j-1)}) \neq u(x^{(j)})$ for $j = 1, \cdots, r$. By 2.2. (iii)

$$e_i(y') \geqq \sum_{j=1}^{r} |u(x^{(j-1)}) - u(x^{(j)})|$$

which is evidently $\geqq r$. Hence $e_i(y') = \infty$ and the set $Z_i \sim (Y_i \cap Z_i)$ has zero measure. Then $R^n \sim Y_i$ has zero measure.

2.5. Definition. If E is an S-system, then for each $i = 1, \cdots, n+1$ and each $y \in Y_i$, we define

$$a_i(E, y) = \sum_{x \in P_i^{-1}(y)} |\alpha_i(E, x) - \beta_i(E, x)|.$$

2.6. Theorem. *If* E *is an* S*-system,* $1 \leqq i \leqq n+1$ *and* e_i *bounds the* i*th* *multiplicity of* E, *then*

$$a_i(E, y) \leqq e_i(y)$$

for all $y \in Y_i(E)$.

PROOF. Let y be an arbitrary point of $Y_i(E)$. Let

$$x^{(1)}, \cdots, x^{(r)}$$

be the finite set of points of $P_i^{-1}(y)$ at which $\alpha_i(x) \neq \beta_i(x)$. We can assume that $r \geqq 1$, because otherwise $a_i(y) = 0$ and is certainly less than or equal to $e_i(y)$. We can also assume that

$$x_i^{(1)} < x_i^{(2)} < \cdots < x_i^{(r)}.$$

It follows from 2.3 that one can now choose points

$$v^{(1)}, w^{(1)}, \cdots, v^{(r)}, w^{(r)}$$

of $P_i^{-1}(y) \cap \{R^{n+1} \sim K(E)\}$ such that

$$v_i^{(1)} < x_i^{(1)} < w_i^{(1)} < v_i^{(2)} < x_i^{(2)} < w_i^{(2)} < \cdots < v_i^{(r)} < x_i^{(r)} < w_i^{(r)}$$

and

$$u(v^{(j)}) = \alpha_i(x^{(j)}),\ u(w^{(j)}) = \beta_i(x^{(j)})$$

for $j = 1, \cdots, r$. Then

$$\begin{aligned} a_i(y) &= \sum_{j=1}^{r} |u(v^{(j)}) - u(w^{(j)})| \\ &\leqq \sum_{j=1}^{r} |u(v^{(j)}) - u(w^{(j)})| \\ &\quad + \sum_{j=1}^{r-1} |u(w^{(j)}) - u(v^{(j+1)})|, \end{aligned}$$

which by 2.2 (iii)

$$\leqq e_i(y).$$

2.7 LEMMA. *If* E *is an S-system and*

$$I = \{x; c_1 \leqq x_1 < d_1,\ c_2 \leqq x_2 < d_2, \cdots,\ c_{n+1} \leqq x_{n+1} < d_{n+1}\}$$

is a half-open interval of R^{n+1}, *then for each* $i = 1, 2, \cdots, n+1$, *the expression*

$$\varphi_i(I, y) = \sum\{\alpha_i(E, x) - \beta_i(E, x)\},$$

where the summation is taken over all $x \in I \cap P_i^{-1}(y)$, *is integrable over* $Y_i(E)$ *with respect to* y. (*Empty sums being regarded as zero*).

PROOF. Assume i fixed. Denote by C the subset of R^1 consisting of all those real numbers c for which the set

$$P_i\{x; x \in K(E) \text{ and } x_i = c\}$$

has its n-dimensional measure equal to zero. By 2.2 (i), $R^1 \sim C$ has zero measure.

Let e_i be a function, integrable on R^n and bounding the ith multiplicity of E.

(i) Suppose first of all, that $c_i, d_i \in C$.

Put

$$B = P_i\{Fr(I) \cap K(E)\} \cup \text{Fr}\{P_i(I)\}.$$

Then B has its n-dimensional measure equal to zero. For each point y of R^n, let $\xi(y)$, $\eta(y)$ be the points of $P_i^{-1}(y)$ whose ith coordinates are c_i, d_i respectively. Then, for all $y \in Y_i \cap \{P_i(\bar{I}) \sim B\}$,

$$\varphi_i(I, y) = u\{\xi(y)\} - u\{\eta(y)\}. \tag{1}$$

Now $P_i(\bar{I}) \sim B$ is an open set of R^n and it follows from 2.2 (ii), that $u\{\xi(y)\}$ and $u\{\eta(y)\}$ are both constant on each component of $P_i(\bar{I}) \sim B$. Then $u\{\xi(y)\}$ and $u\{\eta(y)\}$ are both measurable on $P_i(\bar{I}) \sim B$. Therefore by 2.4 and (1), $\varphi_i(I, y)$ is measurable on $Y_i \cap \{P_i(\bar{I}) \sim B\}$ and hence on $Y_i \cap P_i(I)$. But $\varphi_i(I, y) = 0$ when $y \notin P_i(I)$, so that $\varphi_i(I, y)$ is measurable on Y_i. It follows immediately from 2.5 and 2.6, that

$$|\varphi_i(I, y)| \leqq a_i(y) \leqq e_i(y) \tag{2}$$

for all $y \in Y_i$. Then $\varphi_i(I, y)$ is integrable on Y_i.

(ii) Now suppose that c_i, d_i are arbitrary. Since $R^1 \sim C$ has zero measure, one can choose a monotone increasing sequence $\{c^{(r)}\}$ of members of C such that

$$\lim_{r\to\infty} c^{(r)} = c_i$$

and a monotone increasing sequence $\{d^{(r)}\}$ of members of C such that

$$c_i < d^{(r)}$$

for every r and

$$\lim_{r\to\infty} d^{(r)} = d_i.$$

Define

$$I_r = \{x;\ P_i(x) \in P_i(I) \quad \text{and} \quad c^{(r)} \leqq x_i < d^{(r)}\}.$$

By (i), each $\varphi_i(I_r, y)$ is integrable over Y_i and we evidently have

$$\lim_{r\to\infty} \varphi_i(I_r, y) = \varphi_i(I, y)$$

for all $y \in Y_i$. Since by (2),

$$|\varphi_i(I_r, y)| \leqq e_i(y)$$

for all $y \in Y_i$ and all r, it follows that $\varphi_i(I, y)$ is integrable over Y_i.

2.8 Theorem. *If E is an S-system, then for each $i = 1, \cdots, n+1$, $a_i(E, y)$ is integrable over $Y_i(E)$ with respect to y.*

Proof. Assume i fixed. For each positive integer r, denote by $\mathscr{S}_r$, the (countable) collection of all those half-open cubes of R^{n+1}, that have the form

$$\{x; s_1 2^{-r} \leqq x_1 < (s_1+1)2^{-r}, \cdots, s_{n+1}2^{-r} \leqq x_{n+1} < (s_{n+1}+1)2^{-r}\}$$
$$s_1, \cdots, s_{n+1} = 0, \pm 1, \pm 2, \cdots.$$

For each $y \in Y_i$ and each $I \in \mathscr{S}_r$, define

$$\varphi_i(I, y) = \sum \{\alpha_i(x) - \beta_i(x)\},$$

where the summation is taken over all $x \in I \cap P_i^{-1}(y)$. By 2.7, each $\varphi_i(I, y)$ is integrable over Y_i, hence

$$\psi_r(y) = \sum_{I \in \mathscr{S}_r} |\varphi_i(I, y)|$$

is measurable over Y_i. But

$$\lim_{r\to\infty} \psi_r(y) = a_i(E, y)$$

for all $y \in Y_i$, hence $a_i(E, y)$ is measurable on Y_i. By 2.6 and 2.2 (iii), it is integrable on Y_i.

2.9. Definition. If E is an S-system, then for each $i = 1, \cdots, n+1$ we define

$$A_i(E) = \int_{Y_i(E)} a_i(E, y)dy.$$

2.10. Definition. If E is an S-system and f is a real-valued function on R^{n+1}, then we define for each $i = 1, \cdots, n+1$ and each $y \in Y_i(E)$,

$$H_i(E, f, y) = \sum \{\alpha_i(E, x) - \beta_i(E, x)\} f(x),$$

where the summation is taken over all $x \in P_i^{-1}(y)$.

2.11. Theorem. *If E is an S-system and $f \in \mathscr{F}$, then for each $i = 1, \cdots, n+1$, $H_i(E, f, y)$ is integrable over Y_i with respect to y.*

Proof. Assume i fixed. Since each member of $\mathscr{F}$ is bounded, there exists a positive constant k such that

$$(1) \qquad |f(x)| \leqq k$$

for all $x \in R^{n+1}$. For each positive integer r, let $\mathscr{S}_r$ have the same meaning as in the proof of Theorem 2.8. For each $x \in R^{n+1}$ and each positive integer r, define

$$f_r(x) = \text{L.u.b.}_{\xi \in I}\ f(\xi),$$

where I is the member of $\mathscr{S}_r$ that contains x. Then

$$H_i(E, f_r, y) = \sum_{I \in \mathscr{S}_r} \sum_{x \in I \cap P_i^{-1}(y)} \{\alpha_i(x) - \beta_i(x)\} f_r(x)$$
$$= \sum_{I \in \mathscr{S}_r} [f_r(I) \sum_{x \in I \cap P_i^{-1}(y)} \{\alpha_i(x) - \beta_i(x)\}]$$

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so that by 2.7, $H_i(E, f_r, y)$ is measurable on Y_i. But for each point x of R^{n+1},

$$\lim_{r\to\infty} f_r(x) = f(x),$$

hence by 2.10,

$$\lim_{r\to\infty} H_i(E, f_r, y) = H_i(E, f, y).$$

Therefore $H_i(E, f, y)$ is measurable on Y_i with respect to y. It follows from 2.10 and (1), that

$$|H_i(E, f, y)| \leqq k \sum_{x \in P_i^{-1}(y)} |\alpha_i(x) - \beta_i(x)|;$$

i.e. by 2.5,

$$|H_i(E, f, y)| \leqq ka_i(E, y)$$

for all $y \in Y_i$. Then by 2.8, $H_i(E, f, y)$ is integrable on Y_i with respect to y.

2.12. THEOREM. *If E is an S-system, if r is an integer and if we define*

$$K(F) = K(E)$$

and

$$u_F(x) = r \cdot u_E(x)$$

for all $x \in R^{n+1} \sim K(F)$, then

$$F = \{K(F), u_F\}$$

is an S-system.

PROOF. Properties 2.2 (i) and (ii) are evidently satisfied. If e_i is an integrable function bounding the i th multiplicity of E and we define

$$f_i(y) = re_i(y),$$

then f_i bounds the ith multiplicity of F.

2.13. THEOREM. *If E and F are S-systems and if we define*

$$K(G) = K(E) \cup K(F)$$

and

$$u_G(x) = u_E(x) + u_F(x)$$

for $x \in R^{n+1} \sim K(G)$, then

$$G = \{K(G),\ u_G\}$$

is an S-system.

PROOF. Properties 2.2 (i) and (ii) are evidently satisfied. Let e_i and f_i be integrable functions bounding the ith multiplicities of E and F. Define

$$g_i = e_i + f_i.$$

Then g_i is non-negative and integrable on R^n. If $y \in R^n$ and $x^{(0)}, x^{(1)}, \cdots, x^{(r)}$

is a finite sequence of points of $P_i^{-1}(y) \cap \{R^{n+1} \sim K(G)\}$ with

$$x_i^{(0)} < x_i^{(1)} < \cdots < x_i^{(r)},$$

then

$$\begin{aligned}
&\sum_{j=1}^{r} |u_G(x^{(j-1)}) - u_G(x^{(j)})| \\
&= \sum_{j=1}^{r} |u_E(x^{(j-1)}) + u_F(x^{(j-1)}) - u_E(x^{(j)}) - u_F(x^{(j)})| \\
&\leqq e_i(y) + f_i(y) = g_i(y).
\end{aligned}$$

Thus, the proof is complete.

2.14. THEOREM. *If E and F are S-systems such that u_E and u_F are bounded and if we define*

$$K(G) = K(E) \cup K(F)$$

and

$$u_G(x) = u_E(x) \cdot u_F(x)$$

for all $x \in R^{n+1} \sim K(G)$, then

$$G = \{K(G),\ u_G\}$$

is an S-system.

PROOF. Properties 2.2 (i) and (ii) are satisfied. Let k be a positive real number such that

$$|u_E(x)| \leqq k$$

for all $x \in R^{n+1} \sim K(E)$ and

$$|u_F(x)| \leqq k$$

for all $x \in R^{n+1} \sim K(F)$. Suppose that e_i and f_i are integrable functions bounding the ith multiplicities of E and F. For each $i = 1, \cdots, n+1$, define

$$g_i = k(f_i + e_i).$$

Then g_i is non-negative and integrable on R^n. If $y \in R^n$ and $x^{(0)}, x^{(1)}, \cdots, x^{(r)}$ is a finite sequence of points of $P_i^{-1}(y) \cap \{R^{n+1} \sim K(G)\}$ with

$$x_i^{(0)} < x_i^{(1)} < \cdots < x_i^{(r)},$$

then

$$\sum_{j=1}^{r} |u_G(x^{(j-1)}) - u_G(x^{(j)})|$$
$$= \sum_{j=1}^{r} |u_E(x^{(j-1)}) \cdot u_F(x^{(j-1)}) - u_E(x^{(j)}) \cdot u_F(x^{(j)})|$$
$$= \sum_{j=1}^{r} |u_E(x^{(j-1)}) \{u_F(x^{(j-1)}) - u_F(x^{(j)})\}$$
$$+ \{u_E(x^{(j-1)}) - u_E(x^{(j)})\} u_F(x^{(j)})| \leqq k \sum_{j=1}^{r} |u_F(x^{(j-1)}) - u_F(x^{(j)})|$$
$$+ k \sum_{j=1}^{r} |u_E(x^{(j-1)}) - u_E(x^{(j)})| \leqq k\{f_i(y) + e_i(y)\}$$
$$= g_i(y).$$

This completes the proof.

2.15. Theorem. *If E is an S-system and $f \in \mathscr{F}$, then*

$$\left| \int_{Y_i(E)} H_i(E, f, y) dy \right| \leqq ||f|| \, A_i(E)$$

for each $i = 1, \cdots, n+1$.

Proof. For each $y \in Y_i$ we have

$$|H_i(E, f, y)| = |\sum_{x \in P_i^{-1}(y)} \{\alpha_i(x) - \beta_i(x)\} f(x)| \leqq ||f|| \sum_{x \in P_i^{-1}(y)} |\alpha_i(x) - \beta_i(x)|$$
$$= ||f|| a_i(y).$$

Then

$$\left| \int_{Y_i} H_i(E, f, y) dy \right| \leqq ||f|| \int_{Y_i} a_i(y) dy = ||f|| A_i.$$

2.16. Definition. If E is an S-system, then we define

$$O(E) = \{x; x \in R^{n+1} \sim K(E) \text{ and } u_E(x) \neq 0\}.$$

As a consequence of 2.2 (ii) and the fact that $K(E)$ is closed we have
2.16.1. *$O(E)$ is open.*

2.17. Theorem. *If E is an S-system, then $K(E) \cup O(E)$ is compact.*
Proof. It follows from 2.2, that $K(E) \cup O(E)$ is bounded and since it is the complement of the open set

$$\{x; x \in R^{n+1} \sim K(E) \text{ and } u_E(x) = 0\},$$

it is closed. Hence it is compact.

3. Continuous linear transformations.

In order to prove our approximation theorems in 4, we need to define operations of addition and multiplication and thus construct a ring from the set $\mathscr{S}$ consisting of those S-systems E for which u_E is bounded. It is not possible to make $\mathscr{S}$ itself into a ring, but one can construct a ring by dividing $\mathscr{S}$ into equivalence classes.

Instead of defining the operations between the equivalence classes, we find it more convenient to represent each class by a continuous linear transformation from $\mathscr{F}$ to R^{n+1}.

3.1. DEFINITION. We denote by $\mathscr{L}$, the real vector space of continuous linear transformations from $\mathscr{F}$ to R^{n+1}. We define a norm for $\mathscr{L}$ in the usual way by putting for each L

$$||L|| = \text{L.u.b.}\ ||L(f)||,$$

where the least upper bound is taken over all $f \in \mathscr{F}$ for which $||f|| \leqq 1$. $\mathscr{L}$ thus becomes a Banach space.

If $L \in \mathscr{L}$, then for each $i = 1, \cdots, n+1$, we denote by L_i the real continuous linear functional on $\mathscr{F}$, given by

$$L_i(f) = \{L(f)\}_i.$$

3.2. DEFINITION. Let E be an S-system. For each $f \in \mathscr{F}$, put

$$L_i(f) = (-1)^{i-1} \int_{Y_i(E)} H_i(E, f, y)dy$$

and

$$L(f) = \{L_1(f), \cdots, L_{n+1}(f)\}.$$

Then L is a linear transformation from $\mathscr{F}$ to R^{n+1} and it follows from 2.15 that for each f

$$|L_i(f)| \leqq A_i(E)\ ||f||$$

hence

$$||L(f)|| \leqq \left[\sum_{i=1}^{n+1} \{A_i(E)\}^2\right]^{1/2} ||f||. \tag{1}$$

Thus L is continuous, hence $L \in \mathscr{L}$.

We denote this member of $\mathscr{L}$ by $\tilde{E}$ or $E^{\sim}$.

It follows immediately from (1), that

3.2.1.
$$||\tilde{E}|| \leqq \left[\sum_{i=1}^{n+1} \{A_i(E)\}^2\right]^{1/2}.$$

3.3. THEOREM. *If E and F are S-systems such that*

$$\tilde{E} = \tilde{F},$$

then

$$u_E(x) = u_F(x)$$

for all $x \in R^{n+1} \sim \{K(E) \cup K(F)\}$.

PROOF. Let p be an arbitrary point of $R^{n+1} \sim \{K(E) \cup K(F)\}$. We have to show that

$$u_E(p) = u_F(p). \tag{1}$$

There exists a point q in the unbounded components of $R^{n+1} \sim K(E)$ and $R^{n+1} \sim K(F)$ such that

$$P_1(q) = P_1(p)$$

and

$$p_1 < q_1.$$

Since $K(E)$ and $K(F)$ are closed, there exists an $\varepsilon > 0$ such that the closed spheres S_p, S_q with radii ε and centres p, q do not intersect $K(E)$ or $K(F)$. By 2.2 (ii),

$$u_E(x) = u_E(p), \quad u_F(x) = u_F(p) \tag{2}$$

for all $x \in S_p$ and

$$u_E(x) = u_F(x) = 0 \tag{3}$$

for all $x \in S_q$. Define a function g on R^{n+1} by putting

$$\begin{aligned} g(x) &= \varepsilon - \|P_1(x-p)\| \quad \text{if } \|P_1(x-p)\| \leqq \varepsilon \text{ and } p_1 \leqq x_1 \leqq q_1, \\ &= \varepsilon - \|x-p\| \quad \text{if } \|x-p\| \leqq \varepsilon \text{ and } x_1 \leqq p_1, \\ &= \varepsilon - \|x-q\| \quad \text{if } \|x-q\| \leqq \varepsilon \text{ and } x_1 \geqq q_1, \\ &= 0 \text{ for all other values of } x. \end{aligned}$$

Then $g \in \mathscr{F}$, hence by hypothesis,

$$\int_{Y_1(E)} H_1(E, g, y)dy = \int_{Y_1(F)} H_1(F, g, y)dy. \tag{4}$$

But by 2.10

$$H_1(E, g, y) = \sum_{x \in P_1^{-1}(y)} \{\alpha_1(E, x) - \beta_1(E, x)\}g(x) \tag{5}$$

for all $y \in Y_1(E)$. Let

$$B = \{x; x \in R^{n+1}, \ \|P_1(x-p)\| \leqq \varepsilon \text{ and } p_1 \leqq x_1 \leqq q_1\}. \tag{6}$$

Then

$$g(x) = 0 \tag{7}$$

for all x outside $B \cup S_p \cup S_q$, hence by (5)

$$H_1(E, g, y) = 0 \tag{8}$$

for all $y \in Y_1(E)$ for which $\|y - P_1(p)\| > \varepsilon$.
By (2) and (3),

$$\alpha_1(E, x) - \beta_1(E, x) = 0$$

for all $x \in S_p \cup S_q$, therefore when $||y - P_1(p)|| \leqq \varepsilon$, we have by (5) and (7

$$H_1(E, g, y) = \sum_{x \in B \cap P_1^{-1}(y)} \{\alpha_1(E, x) - \beta_1(E, x)\} \{\varepsilon - ||P_1(x - p)||\},$$

hence by (2) and (3)

$$H_1(E, g, y) = u_E(p) \{\varepsilon - ||y - P_1(p)||\} \tag{9}$$

for all $y \in Y_1(E)$ for which $||y - P_1(p)|| \leqq \varepsilon$.

Define a function h on R^n by putting

$$\begin{aligned} h(y) &= \varepsilon - ||y - P_1(p)|| \quad \text{if} \quad ||y - P_1(p)|| \leqq \varepsilon \\ &= 0 \text{ otherwise.} \end{aligned} \tag{10}$$

By (8) and (9),

$$\int_{Y_1(E)} H_1(E, g, y) dy = u_E(p) \int_{R^n} h(y) dy.$$

Similarly,

$$\int_{Y_1(F)} H_1(F, g, y) dy = u_F(p) \int_{R^n} h(y) dy.$$

By (10), the integral of h is not zero so that by (4),

$$u_E(p) = u_F(p).$$

Thus (1) is true.

3.4. THEOREM. *If E and F are S-systems such that*

$$u_E(x) = u_F(x)$$

for almost all $x \in R^{n+1}$, then $\tilde{E} = \tilde{F}$.

PROOF. Let B be a subset of R^{n+1} with zero measure and such that

$$K(E) \subseteqq B, \quad K(F) \subseteqq B$$

and

$$u_E(x) = u_F(x)$$

for all $x \in R^{n+1} \sim B$. For each $i = 1, \cdots, n+1$, let Z_i be the subset of $Y_i(E) \cap Y_i(F)$ consisting of all those points y for which $\{P_i^{-1}(y) \cap B\}_i$ has its 1-dimensional measure equal to zero. Then $R^n \sim Z_i$ has zero n-measure.

If i is fixed, $y \in Z_i$ and $x \in P_i^{-1}(y)$, then x is a point of accumulation of each of the two sets

$$C = \{\xi; \xi \in P_i^{-1}(y), \ \xi \notin B \quad \text{and} \quad \xi_i < x_i\},$$
$$D = \{\xi; \xi \in P_i^{-1}(y), \ \xi \notin B \quad \text{and} \quad \xi_i > x_i\},$$

hence there exists a point $\xi' \in C$ such that

$$\alpha_i(E, x) = u_E(\xi') = u_F(\xi') = \alpha_i(F, x)$$

and there exists a point $\xi'' \in D$ such that

$$\beta_i(E, x) = u_E(\xi'') = u_F(\xi'') = \beta_i(F, x).$$

Therefore for each $f \in \mathscr{F}$ and $y \in Z_i$,

$$H_i(E, f, y) = \sum_{x \in P_i^{-1}(y)} \{\alpha_i(E, x) - \beta_i(E, x)\} f(x)$$
$$= \sum_{x \in P_i^{-1}(y)} \{\alpha_i(F, x) - \beta_i(F, x)\} f(x) = H_i(F, f, y),$$

hence

$$\tilde{E}_i(f) = (-1)^{i-1} \int_{Z_i} H_i(E, f, y) dy = (-1)^{i-1} \int_{Z_i} H_i(F, f, y) dy = \tilde{F}_i(f)$$

Thus

$$\tilde{E} = \tilde{F}.$$

3.5. Definition. Let $\mathscr{L}_c$ denote the subset of $\mathscr{L}$ consisting of all L for which there exists an S-system E with $\tilde{E} = L$.

The following theorem shows that $\mathscr{L}_c$ is a module.

3.6. Theorem. *Let L, $M \in \mathscr{L}_c$ and r be an integer. Then*

(i) $L + M \in \mathscr{L}_c$;

(ii) $rL \in \mathscr{L}_c$;

(iii) *if E, F and G are S-systems such that*

$$\tilde{E} = L, \ \tilde{F} = M, \ \tilde{G} = L + M,$$

then

$$u_G(x) = u_E(x) + u_F(x)$$

for almost all $x \in R^{n+1}$;

(iv) *if E, H are S-systems such that $\tilde{E} = L$, $\tilde{H} = rL$, then*

$$u_H(x) = r u_E(x)$$

for almost all $x \in R^{n+1}$.

Proof. Let E and F be S-systems such that $\tilde{E} = L$, $\tilde{F} = M$. Define

(1) $$K(T) = K(E) \cup K(F)$$

and

(2) $$u_T(x) = u_E(x) + u_F(x)$$

for all $x \in R^{n+1} \sim K(T)$. By 2.13, T is an S-system. Let

(3) $$Z_i = Y_i(E) \cap Y_i(F) \cap Y_i(T)$$

for each i. If $y \in Z_i$ and $x \in P_i^{-1}(y)$, then by (2)

(4) $$\alpha_i(T, x) = \alpha_i(E, x) + \alpha_i(F, x)$$

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and

$$\beta_i(T, x) = \beta_i(E, x) + \beta_i(F, x). \tag{5}$$

For each $f \in \mathscr{F}$ and each $y \in Z_i$,

$$H_i(T, f, y) = \sum_{x \in P_i^{-1}(y)} \{\alpha_i(T, x) - \beta_i(T, x)\} f(x),$$

which by (4) and (5)

$$= \sum_{x \in P_i^{-1}(y)} \{\alpha_i(E, x) - \beta_i(E, x)\} f(x) + \sum_{x \in P_i^{-1}(y)} \{\alpha_i(F, x) - \beta_i(F, x)\} f(x),$$

so that

$$H_i(T, f, y) = H_i(E, f, y) + H_i(F, f, y). \tag{6}$$

Hence, for each i and each $f \in \mathscr{F}$,

$$\begin{aligned} \tilde{T}_i(f) &= (-1)^{i-1} \int_{Z_i} H_i(T, f, y) dy \\ &= (-1)^{i-1} \int_{Z_i} H_i(E, f, y) dy + (-1)^{i-1} \int_{Z_i} H_i(F, f, y) dy \\ &= \tilde{E}_i(f) + \tilde{F}_i(f) = (L + M)_i(f). \end{aligned}$$

Thus

$$\tilde{T} = L + M \tag{7}$$

so that $L + M \in \mathscr{L}_c$. Thus (i) is true.

If G is an S-system such that $\tilde{G} = L + M$, then by (7), $\tilde{G} = \tilde{T}$. Therefore, by 3.3,

$$u_G(x) = u_T(x)$$

for almost all $x \in R^{n+1}$; i.e., by (2)

$$u_G(x) = u_E(x) + u_F(x)$$

for almost all $x \in R^{n+1}$. Thus (iii) is proved.

To prove (ii) we define

$$K(U) = K(E), \quad \text{and} \tag{8}$$

$$u_U(x) = r \cdot u_E(x) \tag{9}$$

for all $x \in R^{n+1} \sim K(U)$. By 2.12, U is an S-system. Similarly to the way in which (6) was derived, we can show that

$$H_i(U, f, y) = rH_i(E, f, y)$$

for each $f \in \mathscr{F}$ and each $y \in Y_i(E) \cap Y_i(U)$; hence

$$\tilde{U}_i(f) = (-1)^{i-1} \int_{Y_i(E) \cap Y_i(U)} rH_i(E, f, y) dy = r\tilde{E}_i(f) = rL_i(f).$$

Thus

$$\tilde{U} = rL \tag{10}$$

so that $rL \in \mathscr{L}_c$. This completes the proof of (ii).

If H is an S-system such that $\tilde{H} = rL$, then by (10), $\tilde{H} = \tilde{U}$. Hence by 3.3 and (9),

$$u_H(x) = ru_E(x)$$

for almost all $x \in R^{n+1}$. Thus (iv) is proved.

3.7. Definition. We denote by $\mathscr{L}_b$ the subclass of $\mathscr{L}_c$ consisting of all L with the following property:

3.7.1. *there exists an S-system E such that u_E is bounded and $\tilde{E} = L$.*

It follows from 3.7.1 and 3.3, that:

3.7.2. *if $L \in \mathscr{L}_b$ and E is any S-system with $\tilde{E} = L$, then u_E is bounded.*

As a consequence of 3.6 (iii) and (iv), $\mathscr{L}_b$ is a sub-module of $\mathscr{L}_c$.

We define a multiplication for $\mathscr{L}_b$ in the following way. Let $L, M \in \mathscr{L}_b$ and let E, F be S-systems such that $\tilde{E} = L$, $\tilde{F} = M$. Let G be an S-system such that

$$u_G(x) = u_E(x) \cdot u_F(x) \tag{1}$$

for almost all $x \in R^{n+1}$. (By 2.14 at least one such G exists.) Put

$$L \cdot M = \tilde{G}.$$

If E', F', G' are further S-systems such that $E' = L$, $F' = M$ and

$$u_{G'}(x) = u_{E'}(x) \cdot u_{F'}(x) \tag{2}$$

for almost all $x \in R^{n+1}$, then by 3.3

$$u_{E'}(x) = u_F(x),\ u_{F'}(x) = u_F(x)$$

for almost all $x \in R^{n+1}$, hence by (1) and (2),

$$u_{G'}(x) = u_G(x)$$

for almost all $x \in R^{n+1}$; therefore by 3.4,

$$\tilde{G}' = \tilde{G}.$$

Thus the definition of $L \cdot M$ does not depend on the choice of E, F or G.

The following theorem shows that $\mathscr{L}_b$ is a commutative ring. Multiplication in $\mathscr{L}_b$ is not continuous.

3.8. Theorem. *If L, M, $N \in \mathscr{L}_b$, then*

$$L \cdot M = M \cdot L,$$
$$L \cdot (M \cdot N) = (L \cdot M) \cdot N,$$

and

$$L \cdot (M + N) = L \cdot M + L \cdot N.$$

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PROOF. Let E, F and G be S-systems such that $\tilde{E} = L$, $\tilde{F} = M$ and $\tilde{G} = N$.

Let H be an S-system such that

(1) $$u_H(x) = u_E(x) \cdot u_F(x)$$

for almost all $x \in R^{n+1}$. Then

(2) $$u_H(x) = u_F(x) \cdot u_E(x)$$

for almost all $x \in R^{n+1}$. By (1) and (2)

$$L \cdot M = \tilde{H} = M \cdot L.$$

Let T be an S-system such that

(3) $$u_T(x) = u_E(x) \cdot u_F(x) \cdot u_G(x)$$

for almost all $x \in R^{n+1}$. One can easily prove that

$$L \cdot (M \cdot N) = \tilde{T} = (L \cdot M) \cdot N.$$

Let U be an S-system such that

$$\begin{aligned} u_U(x) &= u_E(x)\{u_F(x) + u_{G(x)}\} \\ &= u_E(x)u_F(x) + u_E(x)u_G(x) \end{aligned}$$

for almost all $x \in R^{n+1}$. Then

$$L \cdot (M + N) = \tilde{U} = L \cdot M + L \cdot N.$$

3.9. THEOREM. *If E is an S-system, $f \in \mathscr{F}$ and f is constant on $K(E)$, then*

$$\tilde{E}(f) = 0.$$

PROOF. By 2.10

(1) $$H_i(E, f, y) = \sum_{x \in P_i^{-1}(y)} \{\alpha_i(x) - \beta_i(x)\} f(x)$$

for $i = 1, \cdots, n+1$ and $y \in Y_i(E)$. But by 2.3, $\alpha_i(x) = \beta_i(x)$, when $x \notin K(E)$, hence by (1)

$$H_i(E, f, y) = \sum \{\alpha_i(x) - \beta_i(x)\} f(x),$$

where the summation is taken over all $x \in K(E) \cap P^{-1}(y)$. But, if f has the constant value b on $K(E)$, then

$$\begin{aligned} H_i(E, f, y) &= b \sum \{\alpha_i(x) - \beta_i(x)\} \\ &= 0. \end{aligned}$$

Then

$$\tilde{E}_i(f) = (-1)^{i-1} \int_{Y_i(E)} H_i(E, f, y)\, dy = 0,$$

so that $\tilde{E}(f) = 0$.

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3.10. THEOREM. *If E is an S-system, $f \in \mathscr{F}$ and k is a constant such that*

$$|f(x)| \leqq k$$

for all $x \in K(E)$, then

$$||\tilde{E}(f)|| \leqq ||\tilde{E}|| \cdot k.$$

PROOF. As in the proof of 3.9, we have for each $y \in Y_i(E)$,

$$H_i(E, f, y) = \sum \{\alpha_i(x) - \beta_i(x)\}f(x) \tag{1}$$

where the summation is taken over all $x \in K(E) \cap P_i^{-1}(y)$. But by Tietze's Extension Theorem ([2] p. 80 or [3] p. 28) there exists a $g \in \mathscr{F}$ such that

$$g(x) = f(x) \tag{2}$$

for all $x \in K(E)$ and

$$|g(x)| \leqq k \tag{3}$$

for all $x \in R^{n+1}$. By (1) and (2)

$$\begin{aligned} H_i(E, f, y) &= \sum\{\alpha_i(x) - \beta_i(x)\}g(x) \\ &= H_i(E, g, y). \end{aligned}$$

Thus

$$\tilde{E}_i(f) = \tilde{E}_i(g),$$

hence

$$\tilde{E}(f) = \tilde{E}(g)$$

and by 3.1

$$||\tilde{E}(f)|| \leqq ||\tilde{E}|| \cdot k.$$

3.11. DEFINITION. For each closed interval I, there is an S-system given by

$$\begin{aligned} K &= \mathrm{Fr}\,(I) \\ u(x) &= 1 \quad \text{if} \quad x \in \mathrm{Int}(I) \\ &= 0 \quad \text{if} \quad x \in R^{n+1} \sim I. \end{aligned}$$

We denote this S-system, also by I.

Evidently

3.11.1. $$\tilde{I} \in \mathscr{L}_b.$$

3.12. THEOREM. *If $I^{(1)}, \cdots, I^{(r)}$ $(r \geqq 1)$ are closed intervals with mutually disjoint interiors and if*

$$I = \bigcup_{j=1}^{r} I^{(j)}$$

is also a closed interval, then

$$\tilde{I} = \sum_{j=1}^{r} \tilde{I}^{(j)}.$$

PROOF. We will have

$$(1) \qquad u_I(x) = \sum_{j=1}^{r} u_{I^{(j)}}(x)$$

for almost all $x \in R^{n+1}$. Let E be an S-system such that

$$(2) \qquad \tilde{E} = \sum_{j=1}^{r} I^{(j)}.$$

By 3.6,

$$u_E(x) = \sum_{j=1}^{r} u_{I^{(j)}}(x)$$

for almost ell $x \in R^{n+1}$, hence by (1)

$$(3) \qquad u_I(x) = u_E(x)$$

for almost all x. By (3) and 3.4, $\tilde{I} = \tilde{E}$, hence by (2)

$$\tilde{I} = \sum_{j=1}^{r} I^{(j)}.$$

3.13. THEOREM. *If E is an S-system such that*

$$|u_E(x)| \leqq k$$

for all $x \in R^{n+1} \sim K(E)$ and if $I^{(1)}, \cdots, I^{(r)}$ $(r \geqq 1)$ are closed cubes with mutually disjoint interiors, then

$$\sum_{j=1}^{r} ||\tilde{I}^{(j)}\tilde{E}|| \leqq \sum_{i=1}^{n+1} A_i(E) + 2n^{1/2}k \sum_{j=1}^{r} (\textit{edge of } I^{(j)})^n.$$

PROOF. Since $\tilde{I}^{(j)}\tilde{E} \in \mathscr{L}_b$, there exists for each j an S-system $F^{(j)}$ such that

$$(1) \qquad \tilde{F}^{(j)} = \tilde{I}^{(j)}\tilde{E}.$$

Let f be any member of $\mathscr{F}$ for which

$$(2) \qquad ||f|| \leqq 1.$$

Then

$$(3) \qquad H_i(F^{(j)}, f, y) = \sum_{x \in P_i^{-1}(y)} \{\alpha_i(F^{(j)}, x) - \beta_i(F^{(j)}, x)\} f(x),$$

for all $y \in Y_i(F^{(j)})$. But by 3.11 and 3.7, we have

$$(4) \qquad u_{F^{(j)}}(x) = 0$$

for all $x \in R^{n+1} \sim \{I^{(j)} \cup K(F^{(j)})\}$,

$$(5) \qquad u_{F^{(j)}}(x) = u_E(x)$$

for all $x \in \text{Int}\,(I^{(j)}) \cap [R^{n+1} \sim \{K(F^{(j)}) \cup K(E)\}]$
and hence

$$(6) \qquad |u_{F^{(j)}}(x)| \leqq k$$

for all $x \in R^{n+1} \sim K(F^{(j)})$. By (3) and (4)

$$H_i(F^{(j)}, f, y) = (\sum_1 + \sum_2) \{\alpha_i(F^{(j)}, x) - \beta_i(F^{(j)}, x)\} f(x),$$

where $\sum_1$ and $\sum_2$ denote summation over all $x \in \text{Int}(I^{(j)}) \cap P_i^{-1}(y)$ and $\text{Fr}\,(I^{(j)}) \cap P_i^{-1}(y)$ respectively, hence by (2), (5) and (6)

$$(7) \quad |H_i(F^{(j)}, f, y)| \leqq \sum |a_i(E, x) - \beta_i(E, x)| + \begin{cases} 2k & \text{if} \quad y \in P_i(I^{(j)}) \\ 0 & \text{if} \quad y \notin P_i(I^{(j)}), \end{cases}$$

for all $y \in Y_i(F^{(j)}) \cap Y_i(E)$, where the summation is taken over all $x \in \text{Int}(I^{(j)}) \cap P_i^{-1}(y)$.

Since

$$|\bar{F}_i^{(j)}(f)| \leqq \int_{Y_i(F^{(j)})} |H_i(F^{(j)}, f, y)| dy,$$

it follows from (1) and (7), that

$$(8) \qquad |\{\bar{I}^{(j)}\bar{E}\}_i(f)| \leqq b_i^{(j)} + 2k(\text{edge of } I^{(j)})^n,$$

where

$$(9) \qquad b_i^{(j)} = \int_{Y_i(E)} \{\sum |\alpha_i(E, x) - \beta_i(E, x)|\} dy,$$

the summation being taken over all $x \in \text{Int}\,(I^{(j)}) \cap P_i^{-1}(y)$. Let

$$U = \bigcup_{j=1}^{r} \text{Int}\,(I^{(j)}).$$

Then by (9),

$$\sum_{j=1}^{r} b_i^{(j)} = \int_{Y_i(E)} \{\sum |\alpha_i(E, x) - \beta_i(E, x)|\}\, dy,$$

where the summation is over $x \in U \cap P_i^{-1}(y)$,

$$\leqq \int_{Y_i(E)} a_i(E, y)\, dy,$$

so that

$$(10) \qquad \sum_{j=1}^{r} b_i^{(j)} \leqq A_i(E).$$

It follows from (8) that

$$||\{\bar{I}^{(j)}\bar{E}\}(f)|| \leqq ||b^{(j)} + p||,$$

where $b^{(j)} = (b_1^{(j)}, \cdots, b_{n+1}^{(j)})$ and $p = 2k$ (edge of $I^{(j)})^n \,.\, (1, 1, \cdots, 1)$, hence by (2) and 3.1,

$$||\bar{I}^{(j)}\bar{E}|| \leqq ||b^{(j)}|| + ||p||, \quad \leqq \sum_{i=1}^{n+1} b_i^{(j)} + 2n^{1/2} k\,(\text{edge of } I^{(j)})^n.$$

Then by (10),

$$\sum_{j=1}^{r} ||\bar{I}^{(j)}\bar{E}|| \leqq \sum_{i=1}^{n+1} A_i(E) + 2n^{1/2} k \sum_{j=1}^{r} (\text{edge of } I^{(j)})^n.$$

3.14. THEOREM. *If E is an S-system with $\tilde{E} \in \mathscr{L}_b$ and I is a closed interval of R^{n+1} that does not intersect $K(E)$, then*

$$\tilde{I} \cdot \tilde{E} = u_E(I) \cdot \tilde{I}.$$

PROOF. Let F and G be S-systems such that

$$\tilde{F} = \tilde{I} \cdot \tilde{E}, \quad \tilde{G} = u_E(I) \cdot \tilde{I}.$$

By 3.3 and 3.7,

$$u_F(x) = u_I(x) \cdot u_E(x) \tag{1}$$

for almost all $x \in R^{n+1}$ and by 3.6 (iv)

$$u_G(x) = u_E(I) \cdot u_I(x) \tag{2}$$

for almost all $x \in R^{n+1}$. By (1) and 3.11,

$$u_F(x) = u_E(x) \tag{3}$$

for almost all $x \in \text{Int}(I)$ and

$$u_F(x) = 0 \tag{4}$$

for almost all $x \in R^{n+1} \sim I$. It follows from (2) and 3.11 that

$$u_G(x) = u_E(x) \tag{5}$$

for almost all $x \in \text{Int}(I)$ and

$$u_G(x) = 0 \tag{6}$$

for almost all $x \in R^{n+1} \sim I$. By (3), (4), (5) and (6)

$$u_F(x) = u_G(x)$$

for almost all $x \in R^{n+1}$, so that by 3.4, $\tilde{F} = \tilde{G}$.

3.15. THEOREM *If E is an S-system with $E \in \mathscr{L}_b$ and I is a closed interval of R^{n+1} containing $K(E)$, then*

$$\tilde{I} \cdot \tilde{E} = \tilde{E}.$$

PROOF. Let F be an S-system such that

$$\tilde{F} = \tilde{I} \cdot \tilde{E}. \tag{1}$$

Then by 3.3 and 3.7

$$u_F(x) = u_I(x) \cdot u_E(x)$$

for almost all $x \in R^{n+1}$. Hence by 3.11,

$$u_F(x) = u_E(x) \tag{2}$$

for almost all $x \in \text{Int}(I)$ and

$$u_F(x) = 0 \tag{3}$$

for almost all $x \in R^{n+1} \sim I$.

But since $K(E) \subseteqq I$, it follows from 2.2 (ii) that

(4) $$u_E(x) = 0$$

for all $x \in R^{n+1} \sim I$. As a consequence of (2), (3) and (4) we now have

$$u_F(x) = u_E(x)$$

for almost all $x \in R^{n+1}$, so that by 3.4, $\tilde{F} = \tilde{E}$; i.e., by (1), $\tilde{E} = \tilde{I} \cdot \tilde{E}$.

4. Some approximation theorems

In 4 we prove some theorems which enable us to approximate a particular S-system with a finite number of S-systems, each of which is the product of an integer and an S-system corresponding to a cube. These theorems will be used in 5 to prove Cauchy's theorem.

4.1. THEOREM. *If B is a compact non-empty subset of R^{n+1} with a finite Hausdorff n-measure Λ and if ε is an arbitrary positive number, then there exists a finite set*

$$I^{(1)}, \cdots, I^{(r)} \qquad (r \geqq 1)$$

of closed cubes of R^{n+1} with mutually disjoint interiors and such that:

(i) *the diameter of each $I^{(j)}$ is less than ε;*

(ii) $$B \subseteqq \operatorname{Int}\{\bigcup_{j=1}^{r} I^{(j)}\}$$

and each $I^{(j)}$ intersects B;

(iii) $$\sum_{j=1}^{r} (\text{edge of } I^{(j)})^n < n^{\frac{1}{2}n} 2^{2n+1}\Lambda + 1.$$

PROOF. It follows from the definition of Hausdorff measure that there exists a partition

$$B = B_1 \cup B_2 \cup \cdots$$

of B into a sequence (possibly infinite) of mutually disjoint subsets such that

(1) $$\text{diameter of } B_s \leqq 2^{-2}n^{-\frac{1}{2}}\varepsilon$$

for each s and

$$\sum_s 2^{-n}\alpha(n) \cdot (\text{diameter of } B_s)^n < \Lambda + n^{-\frac{1}{2}n} 2^{-2n-2},$$

where $\alpha(n)$ is the n-measure of the unit n-cell $\{x; x \in R^n \text{ and } ||x|| \leqq 1\}$. Since a cube of R^n with edge $2n^{-\frac{1}{2}}$ can be included in this n-cell, $\alpha(n) \geqq 2^n n^{-\frac{1}{2}n}$ and therefore

$$\sum_s (\text{diameter of } B_s)^n < n^{\frac{1}{2}n}\Lambda + 2^{-2n-2}.$$

Each B_s can now be covered by an open set U_s such that

(2) $$\text{diameter of } U_s < 2^{-1} n^{-\frac{1}{2}} \varepsilon$$

for each s and

$$(\text{diameter of } U_s)^n - (\text{diameter of } B_s)^n < 2^{-s-2n-2}$$

for each s. Then

(3) $$\sum_s (\text{diameter of } U_s)^n < n^{\frac{1}{2}n} \Lambda + 2^{-2n-1}.$$

Since B is compact one can choose a finite non-empty collection $\mathscr{U}$ from the U_s's which covers B. For each $U \in \mathscr{U}$ one can choose an integer $t(U)$ such that

(4) $$2^{-t(U)-1} \leqq \text{diameter of } U < 2^{-t(U)}.$$

For each integer s, let $\mathscr{S}_s$ denote the collection consisting of all those closed cubes of R^{n+1} of the form

$$\{x;\, w_1 2^{-s} \leqq x_1 \leqq (w_1+1)2^{-s}, \cdots, w_{n+1} 2^{-s} \leqq x_{n+1} \leqq (w_{n+1}+1)2^{-s}\}$$
$$w_1, \cdots, w_{n+1} = 0, \pm 1, \pm 2, \cdots.$$

For each $U \in \mathscr{U}$, let $\mathscr{I}(U)$ be the collection consisting of those cubes of $\mathscr{S}_{t(U)}$ that intersect U. By (4) the number of cubes in $\mathscr{I}(U)$ is $\leqq 2^{n+1}$, hence again by (4),

(5) $$\sum_{I \in \mathscr{I}(U)} (\text{edge of } I)^n \leqq 2^{n+1}(2 \cdot \text{diameter of } U)^n,$$

for each $U \in \mathscr{U}$.

From the collection

$$\bigcup_{U \in \mathscr{U}} \mathscr{I}(U)$$

one can now select a (finite) subcollection $\mathscr{I}'$ of closed cubes with mutually disjoint interiors and covering

(6) $$\bigcup_{U \in \mathscr{U}} U.$$

Let $I^{(1)}, \cdots, I^{(r)}$ be the members of $\mathscr{I}'$ that intersect B. Since $\mathscr{I}'$ covers the open set (6), which contains B, it follows that

$$B \subseteqq \text{Int}\,\{\bigcup_{j=1}^{r} I^{(j)}\}.$$

Thus (ii) is true.

Now each $I^{(j)}$ belongs to some $\mathscr{I}(U)$, hence to $\mathscr{S}_{t(U)}$ so that

$$\text{diameter of } I^{(j)} = n^{\frac{1}{2}} 2^{-t(U)},$$

which by (4),

$$\leqq 2n^{\frac{1}{2}} \cdot (\text{diameter of } U)$$

and by (2)

$$< \varepsilon.$$

Thus (i) is true.

Evidently

$$\sum_{j=1}^{r} (\text{edge of } I^{(j)})^n \leqq \sum_{I \in \mathscr{J}'} (\text{edge of } I)^n$$
$$\leqq \sum_{U \in \mathscr{U}} \sum_{I \in \mathscr{J}(U)} (\text{edge of } I)^n,$$

which by (5)

$$\leqq \sum_{U \in \mathscr{U}} 2^{2n+1} (\text{diameter of } U)^n$$
$$\leqq \sum_{s} 2^{2n+1} (\text{diameter of } U_s)^n$$

and by (3)

$$< n^{\frac{1}{2}n} 2^{2n+1} \Lambda + 1.$$

This proves (iii) and completes the proof of the theorem.

4.2. Theorem. *If E is an S-system and ε is an arbitrary positive number, then there exists an S-system F such that*

(i) $$K(F) = K(E),$$

(ii) $$O(F) = O(E),$$

(iii) u_F *is bounded, and*

(iv) $$\|\tilde{E} - \tilde{F}\| < \varepsilon.$$

Proof. For each positive integer s, define

(1) $$K(E_s) = K(E)$$

and, for all $x \in R^{n+1} \sim K(E_s)$, define

(2) $$\begin{aligned} u_{E_s}(x) &= u_E(x) \quad \text{if} \quad -s \leqq u_E(x) \leqq s, \\ &= -s \quad \text{if} \quad u_E(x) \leqq -s, \\ &= \quad s \quad \text{if} \quad u_E(x) \geqq s. \end{aligned}$$

Then E_s evidently satisfies 2.2 (i) and (ii). To prove that it satisfies 2.2 (iii), let e_i be an integrable function bounding the i^{th} multiplicity of E, let y be an arbitrary point of R^n and take a finite sequence

$$x^{(0)}, x^{(1)}, \cdots, x^{(r)}$$

of points of $P_i^{-1}(y) \cap \{R^{n+1} \sim K(E_s)\}$ with

$$x_i^{(0)} < x_i^{(1)} < \cdots < x_i^{(r)}.$$

It follows from (2), that

$$|u_{E_s}(x^{(j-1)}) - u_{E_s}(x^{(j)})| \leqq |u_E(x^{(j-1)}) - u_E(x^{(j)})|,$$

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hence

$$\sum_{j=1}^{r} |u_{E_s}(x^{(j-1)}) - u_{E_s}(x^{(j)})| \leqq e_i(y).$$

Thus 2.2 (iii) is satisfied, hence each E_s is an S-system. Evidently

$$O(E_s) = O(E) \tag{3}$$

for each s.

Define, for each positive integer s,

$$K(G_s) = K(E) \tag{4}$$

and

$$u_{G_s}(x) = u_E(x) - u_{E_s}(x) \tag{5}$$

for all $x \in R^{n+1} \sim K(G_s)$. By 2.12 and 2.13, each G_s is an S-system. Let

$$Z_i = Y_i(E) \cap \bigcap_{s=1}^{\infty} Y_i(G_s). \tag{6}$$

Then $R^n \sim Z_i$ has zero n-measure. Take an arbitrary point y of Z_i. If $x', x'' \in P_i^{-1}(y) \cap \{R^{n+1} \sim K(E)\}$, then

$$|u_{G_s}(x') - u_{G_s}(x'')| = |\{u_E(x') - u_{E_s}(x')\} - \{u_E(x'') - u_{E_s}(x'')\}|$$

and by Theorem 2 on page 3 of [4],

$$\leqq |u_E(x') - u_E(x'')|,$$

hence for each $x \in P_i^{-1}(y)$

$$|\alpha_i(G_s, x) - \beta_i(G_s, x)| \leqq |\alpha_i(E, x) - \beta_i(E, x)|$$

so that by 2.5

$$a_i(G_s, y) \leqq a_i(E, y) \tag{7}$$

for all $y \in Z_i$ and each s. Now it follows from 2.3.1 that $u_E(x)$ is bounded for $x \in P_i^{-1}(y) \cap \{R^{n+1} \sim K(E)\}$, hence by (2) and (5), there exists an s, such that

$$u_{G_s}(x) = 0$$

for all $x \in P_i^{-1}(y) \cap \{R^{n+1} \sim K(E)\}$ and all $s \geqq s_1$. Therefore by 2.5,

$$a_i(G_s, y) = 0$$

for all $s \geqq s_1$; i.e.,

$$\lim_{s\to\infty} a_i(G_s, y) = 0 \tag{8}$$

for all $y \in Z_i$. Since by 2.8, $a_i(E, y)$ is integrable, it follows from (7), (8) and dominated convergence that

$$\lim_{s\to\infty} \int_{Z_i} a_i(G_s, y)\, dy = 0;$$

i.e. by 2.9

$$\lim_{s\to\infty} A_i(G_s) = 0,$$

for each $i = 1, \cdots, n+1$. Hence we can choose an s_0 such that

$$\left[\sum_{i=1}^{n+1} A_i(G_{s_0})^2\right]^{1/2} < \varepsilon,$$

therefore by 3.2.1,

$$||\tilde{G}_{s_0}|| < \varepsilon.$$

But by (5), 3.6 and 3.4,

$$\tilde{E} = \tilde{E}_{s_0} + \tilde{G}_{s_0},$$

hence

(9) $$||\tilde{E} - \tilde{E}_{s_0}|| < \varepsilon.$$

Thus, if we put $F = E_{s_0}$, it follows from (1), (3), (2) and (9), that F has the required properties.

4.3. Theorem. *Let E be an S-system such that $O(E) \neq \emptyset$, $K(E)$ has a finite Hausdorff n-measure Λ and*

$$|u_E(x)| \leqq k$$

for all $x \in R^{n+1} \sim K(E)$. Let ε be an arbitrary positive number.

Then there exists a finite set

$$I^{(1)}, I^{(2)}, \cdots, I^{(r)} \quad (r \geqq 1)$$

of closed intervals of R^{n+1}, corresponding integers

$$i_1, i_2, \cdots, i_r,$$

and a finite set

$$F^{(1)}, F^{(2)}, \cdots, F^{(s)} \quad (s \geqq 1)$$

of S-systems, with the following conditions satisfied.

(i) *Each $I^{(j)}$ is contained in $O(E)$.*

(ii) $\tilde{E} = \sum_{j=1}^{r} i_j \tilde{I}^{(j)} + \sum_{p=1}^{s} \tilde{F}^{(p)}$.

(iii) $\sum_{p=1}^{s} ||\tilde{F}^{(p)}|| < \sum_{i=1}^{n+1} A_i(E) + 2^{2n+2} n^{\frac{n+1}{2}} k\Lambda + 2n^{1/2} k$.

(iv) *The diameter of each $K(F^{(p)})$ is less than ε.*

Proof. It follows from 2.2 (ii), that $O(E)$ is open, hence there exists a δ such that $0 < \delta < \varepsilon$ and no closed interval with diameter less than δ can cover the whole of $O(E)$. By 4.1, there exists a finite set

$$J^{(1)}, \cdots, J^{(s)} \quad (s \geqq 1)$$

of closed cubes with mutually disjoint interiors and such that

(1) $$\text{diameter of } J^{(p)} < \delta$$

for each $p = 1, \cdots, s$,

(2) $$K(E) \subseteqq \operatorname{Int} \{ \bigcup_{p=1}^{s} J^{(p)} \},$$

(3) $$J^{(p)} \cap K(E) \neq \emptyset$$

for each p, and

(4) $$\sum_{p=1}^{s} (\text{edge of } J^{(p)})^n < n^{\frac{1}{2}n} 2^{2n+1} \Lambda + 1.$$

Let I be a closed interval that contains all the $J^{(p)}$'s, hence also $K(E)$.
One can choose a finite set

$$I^{(1)}, \cdots, I^{(t)}$$

of closed intervals, whose interiors are disjoint with each other and with the interiors of the $J^{(p)}$'s and for which

(5) $$I = \bigcup_{j=1}^{t} I^{(j)} \cup \bigcup_{p=1}^{s} J^{(p)}.$$

We can assume that

$$I^{(1)}, \cdots, I^{(r)} \qquad (r \geqq 1)$$

are those of the $I^{(j)}$'s that are contained in $O(E)$. By (5) and 3.12,

$$\tilde{I} = \sum_{j=1}^{t} \tilde{I}^{(j)} + \sum_{p=1}^{s} \tilde{J}^{(p)},$$

hence

$$\tilde{I} \cdot \tilde{E} = \sum_{j=1}^{t} \tilde{I}^{(j)} \cdot \tilde{E} + \sum_{p=1}^{s} \tilde{J}^{(p)} \cdot \tilde{E},$$

so that by 3.14 and 3.15,

$$\tilde{E} = \sum_{j=1}^{t} u_E(I^{(j)}) \cdot \tilde{I}^{(j)} + \sum_{p=1}^{s} \tilde{J}^{(p)} \cdot \tilde{E}$$

and, since $u_E(I^{(j)}) = 0$ when $j > r$, we have

(6) $$\tilde{E} = \sum_{j=1}^{r} u_E(I^{(j)}) \cdot \tilde{I}^{(j)} + \sum_{p=1}^{s} \tilde{J}^{(p)} \cdot \tilde{E}.$$

Define

(7) $$i_j = u_E(I^{(j)}) \qquad (j = 1, \cdots, r).$$

For each $p = 1, \cdots, s$, there exists an S-system $G^{(p)}$ such that

(8) $$\tilde{G}^{(p)} = \tilde{J}^{(p)} \cdot \tilde{E}.$$

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Define

$$K(F^{(p)}) = J^{(p)} \cap K(G^{(p)}),$$
$$u_{F^{(p)}}(x) = u_{G^{(p)}}(x) \quad \text{if} \quad x \in J^{(p)} \cap \{R^{n+1} \sim K(F^{(p)})\}$$
$$= 0 \quad \text{if} \quad x \in R^{n+1} \sim J^{(p)}.$$

It is not difficult to verify that, for each p, $F^{(p)}$ is an S-system,

(9) $$u_{F^{(p)}}(x) = u_{G^{(p)}}(x)$$

for almost all $x \in R^{n+1}$ and

(10) $$\text{diameter of } K(F^{(p)}) \leqq \text{diameter of } J^{(p)}.$$

It follows from (9) and 3.4 that $\tilde{F}^{(p)} = \tilde{G}^{(p)}$, hence by (8)

(11) $$\tilde{F}^{(p)} = \tilde{J}^{(p)} \cdot \tilde{E} \qquad (p = 1, \cdots, s).$$

By (6), (7) and (11),

$$\tilde{E} = \sum_{j=1}^{r} i_j \cdot \tilde{I}^{(j)} + \sum_{p=1}^{s} \tilde{F}^{(p)};$$

thus (ii) is true. We have already proved (i).

It follows from 3.13, that

$$\sum_{p=1}^{s} ||\tilde{J}^{(p)} \cdot \tilde{E}|| \leqq \sum_{i=1}^{n+1} A_i(E) + 2n^{1/2} k \sum_{p=1}^{s} (\text{edge of } J^{(p)})^n$$

hence by (4) and (11),

$$\sum_{p=1}^{s} ||\tilde{F}^{(p)}|| < \sum_{i=1}^{n+1} A_i(E) + 2^{2n+2} n^{\frac{n+1}{2}} k\Lambda + 2n^{1/2} k.$$

Thus (iii) is true. (iv) follows immediately from (1) and (10).

5. Cauchy's Theorem

We now make use of 4.2 and 4.3 in proving Cauchy's integral theorem, first of all for S-systems (5.1 and 5.2) and then for closed parametric n-surfaces in R^{n+1} (5.3).

5.1. Theorem. *If E is an S-system such that $K(E)$ has a finite Hausdorff n-measure Λ and if $f_1, \cdots, f_{n+1} \in \mathscr{F}$ and have the property: for each closed interval I of R^{n+1} that is contained in $O(E)$,*

$$\sum_{i=1}^{n+1} \tilde{I}_i(f_i) = 0;$$

then

$$\sum_{i=1}^{n+1} \tilde{E}_i(f_i) = 0.$$

Proof. If $O(E)$ is empty, then $u_E(x) \equiv 0$, hence $\tilde{E} = 0$ and the theorem is trivial. Hence we can assume that

$$O(E) \neq \emptyset. \tag{1}$$

(a) Assume to begin with that there exists a constant $k > 0$ such that

$$|u_E(x)| \leqq k \tag{2}$$

for all $x \in R^{n+1} \sim K(E)$. Take an arbitrary $\eta > 0$. Put

$$c = \sum_{i=1}^{n+1} A_i(E) + 2^{2n+2} n^{\frac{n+1}{1}} k\Lambda + 2n^{1/2} k. \tag{3}$$

By 2.17, there exists a $\rho > 0$ and such that $||x|| < \rho$ for all $x \in K(E) \cup O(E)$. Define for each i,

$$\begin{aligned} g_i(x) &= f_i(x) \quad \text{if} \quad ||x|| \leqq \rho, \\ &= \frac{f_i(x)}{1 + ||x|| - \rho} \quad \text{if} \quad ||x|| \geqq \rho. \end{aligned}$$

Then

$$\begin{aligned} & g_i \in \mathscr{F} && i = 1, \cdots, n+1, \\ & g_i(x) = f_i(x) && i = 1, \cdots, n+1 \end{aligned} \tag{4}$$

for all $x \in K(E) \cup O(E)$ and

$$g_i(x) \to 0 \qquad i = 1, \cdots, n+1 \tag{5}$$

as $x \to \infty$. By (5) and continuity, each g_i is uniformly continuous on R^{n+1}. Hence we can choose an $\varepsilon > 0$ so that

$$|g_i(x') - g_i(x'')| < \frac{\eta}{(n+1)c} \qquad i = 1, \cdots, n+1, \tag{6}$$

for all $x', x'' \in R^{n+1}$ with

$$||x' - x''|| < \varepsilon. \tag{7}$$

Let $I^{(j)}$, i_j, $F^{(p)}$ be defined as in 4.3. By (4) and 3.9,

$$\tilde{E}(f_i - g_i) = 0,$$

hence

$$\sum_{i=1}^{n+1} \tilde{E}_i(f_i) = \sum_{i=1}^{n+1} \tilde{E}_i(g_i),$$

which by 4.3 (ii)

$$= \sum_{j=1}^{r} i_j \sum_{i=1}^{n+1} I_i^{(j)}(g_i) + \sum_{i=1}^{n+1} \sum_{p=1}^{s} F_i^{(p)}(g_i),$$

so that by 4.3 (i), (4) and hypothesis,

$$\sum_{i=1}^{n+1} \tilde{E}_i(f_i) = \sum_{i=1}^{n+1} \sum_{p=1}^{s} \tilde{F}_i^{(p)}(g_i). \tag{8}$$

We will now prove that

$$|\tilde{F}_i^{(p)}(g_i)| \leqq ||\tilde{F}^{(p)}|| \frac{\eta}{(n+1)c} \qquad \begin{matrix} i = 1, \cdots, n+1 \\ p = 1, \cdots, s \end{matrix} \tag{9}$$

When $K(F^{(p)}) = \emptyset$, $u_{F^{(p)}} \equiv 0$, hence $\tilde{F}^{(p)} = 0$ and (9) is trivial. Suppose therefore that $K(F^{(p)}) \neq \emptyset$. Choose a point $b^{(p)} \epsilon K(F^{(p)})$ and define

$$g_i^{(p)}(x) \equiv g_i(x) - g_i(b^{(p)}),$$
$$h_i^{(p)}(x) \equiv g_i(b^{(p)}).$$

Then

$$|\tilde{F}_i^{(p)}(g_i)| = |\tilde{F}_i^{(p)}(g_i^{(p)}) + \tilde{F}_i^{(p)}(h_i^{(p)})|,$$

hence by 3.9,

$$|\tilde{F}_i^{(p)}(g_i)| = |\tilde{F}_i^{(p)}(g_i^{(p)})|. \tag{10}$$

But by 4.3 (iv), (6) and (7),

$$|g_i^{(p)}(x)| < \frac{\eta}{(n+1)c}$$

for all $x \epsilon K(F^{(p)})$, so that by 3.10

$$|\tilde{F}_i^{(p)}(g_i^{(p)})| \leqq ||\tilde{F}^{(p)}|| \frac{\eta}{(n+1)c}. \tag{11}$$

(10) and (11) evidently imply (9).

It now follows from (3), (9) and 4.3 (iii), that

$$\sum_{p=1}^{s} |\tilde{F}_i^{(p)}(g_i)| \leqq \frac{\eta}{n+1} \qquad i = 1, \cdots, n+1,$$

hence

$$\sum_{i=1}^{n+1} \sum_{p=1}^{s} |\tilde{F}_i^{(p)}(g_i)| \leqq \eta$$

and therefore by (8),

$$|\sum_{i=1}^{n+1} \tilde{E}_i(f_i)| \leqq \eta.$$

Thus

$$\sum_{i=1}^{n+1} \tilde{E}_i(f_i) = 0.$$

(b) Suppose now that there is no restriction on u_E. Since each $f_i \epsilon \mathscr{F}$ there exists a constant $\Gamma > 0$ such that

$$|f_i(x)| \leqq \Gamma \qquad (i = 1, \cdots, n+1)$$

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for all $x \in R^{n+1}$. Take an arbitrary $\eta > 0$ and put

$$\varepsilon = \frac{\eta}{(n+1)\Gamma} \tag{12}$$

Let F be defined as in Theorem 4.2.
By (a)

$$\sum_{i=1}^{n+1} \tilde{F}_i(f_i) = 0. \tag{13}$$

But it follows from 4.2 (iv), that

$$|\tilde{E}_i(f_i) - \tilde{F}_i(f_i)| < \varepsilon\Gamma = \frac{\eta}{n+1}$$

so that by (13)

$$|\sum_{i=1}^{n+1} \tilde{E}_i(f_i)| < \eta.$$

Thus

$$\sum_{i=1}^{n+1} \tilde{E}_i(f_i) = 0.$$

5.2. THEOREM. *Let E be an S-system such that $K(E)$ has finite Hausdorff n-measure. Let $f_i, \cdots, f_{n+1} \in \mathscr{F}$ and have the properties:*

(i) *each of the partial derivatives*

$$\frac{\partial f_i}{\partial x_i} \qquad (i = 1, \cdots, n+1)$$

exists and is continuous on $O(E)$;

(ii) $$\sum_{i=1}^{n+1} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} = 0$$

at all points of $O(E)$.

Then

$$\sum_{i=1}^{n+1} \tilde{E}_i(f_i) = 0.$$

PROOF. Let

$$I = \{x; c_1 \leqq x_1 \leqq d_1, \cdots, c_{n+1} \leqq x_{n+1} \leqq d_{n+1}\}$$

be an arbitrary closed interval that is contained in $O(E)$. It follows from 2.10, 3.2 and 3.11, that for each $f \in \mathscr{F}$,

$$\tilde{I}_i(f) = (-1)^{i-1} \int_{P_i(I)} [f\{\eta^{(i)}(y)\} - f\{\xi^{(i)}(y)\}]\, dy, \tag{1}$$

where $\xi^{(i)}(y)$, $\eta^{(i)}(y)$ denote the points of $P_i^{-1}(y)$ whose ith coordinates are c_i, d_i respectively. It is well known that (i) and (ii) of the hypothesis imply

$$\sum_{i=1}^{n+1} (-1)^{i-1} \int_{P_i(I)} [f_i\{\eta^{(i)}(y)\} - f_i\{\xi^{(i)}(y)\}]\, dy = 0;$$

i.e., by (1)

$$\sum_{i=1}^{n+1} \tilde{I}_i(f_i) = 0.$$

Hence by 5.1,

$$\sum_{i=1}^{n+1} \tilde{E}_i(f_i) = 0.$$

5.3. THEOREM. *Let (f, M^n) be a closed parametric n-surface in R^{n+1} with bounded variation and such that $f(M^n)$ has a finite Hausdorff n-measure. Let $g_i, \cdots, g_{n+1}$ be real-valued functions on $f(M^n) \cup O(f, M^n)$ with the following properties*:

(i) *each g_i is continuous on $f(M^n) \cup O(f, M^n)$;*

(ii) *each of the partial derivatives*

$$\frac{\partial g_i}{\partial x_i}$$

exists and is continuous on $O(f, M^n)$;

(iii)
$$\sum_{i=1}^{n+1} (-1)^{i-1} \frac{\partial g_i}{\partial x_i} = 0$$

at all points of $O(f, M^n)$.

Then

$$\sum_{i=1}^{n+1} \int_{(f, M^n)} g_i(x)\, dP_i(x) = 0.$$

PROOF. Put

$$K(E) = f(M^n)$$

and

$$u_E(x) = u(f, M^n, x)$$

for all $x \in R^{n+1} \sim K(E)$. Then we have shown in 2.2 that E is an S-system. It follows from 3.4, 3.7 and 3.10 of [5] II, that for each $g \in \mathscr{F}$,

(1)
$$\tilde{E}_i(g) = \int_{(f, M^n)} g(x) dP_i(x)$$

By 2.17, $K(E) \cup O(E)$ is compact, hence each g_i is bounded on $K(E) \cup O(E)$. By Tietze's Extension Theorem ([2] p. 80 or [3] p. 28) each g_i can be extended to a bounded continuous function on R^{n+1}. Then each $g_i \in \mathscr{F}$ so that by (1)

$$\sum_{i=1}^{n+1} \int_{(f, M^n)} g_i(x)\, dP_i(x) = \sum_{i=1}^{n+1} \tilde{E}_i(g_i)$$

and by 5.2 is equal to zero.

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