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The Lang–Weil Estimate for Cubic Hypersurfaces

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Abstract. An improved estimate is provided for the number of $\mathbb{F}_q$-rational points on a geometrically irreducible, projective, cubic hypersurface that is not equal to a cone.

1 Introduction

Let $k = \mathbb{F}_q$ be a finite field of order q and characteristic p and let $X \subset \mathbb{P}^n_k$ be a geometrically irreducible projective variety of dimension r and degree d. A well-known result of Lang and Weil [8] shows that

$$|\#X(k) - q^r| \leqslant (d-1)(d-2)q^{r-\frac{1}{2}} + c(d,r)q^{r-1}$$

for a constant $c(d,r) > 0$ depending only on d and r. In the generality with which it is stated this estimate is essentially best possible, as the consideration of a cone over a non-singular plane curve shows. Our aim in this note is to discuss available augmentations in the setting of cubic hypersurfaces $X \subset \mathbb{P}^n_k$, where $r = n-1$.

When X has small singular locus, one can do much better. Let σ denote the projective dimension of this singular locus, with $\sigma = -1$ if X is non-singular. Then it follows from work of Hooley [7] that

$$|\#X(k) - q^{n-1}| \leqslant c_1(n)q^{\frac{n+\sigma}{2}} \tag{1.1}$$

for a constant $c_1(n) > 0$ depending only on n. Hooley's argument relies crucially on the resolution of the Weil conjectures due to Deligne [6] and is not specific to cubic hypersurfaces.

An allied estimate is available through work of Davenport and Lewis [4], which uses Weyl differencing to estimate certain cubic exponential sums. The methods here yield

$$|\#X(k) - q^{n-1}| \leqslant c_2(n)q^{n-\frac{h}{4}} \tag{1.2}$$

for a constant $c_2(n) > 0$, where h is defined to be such that $n-h$ is the greatest dimension of any linear space contained in X. This is the so-called "h-invariant" of

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the cubic hypersurface and means that X can be defined by a cubic form of the shape $x_0Q_0 + \cdots + x_{h-1}Q_{h-1}$ for quadratic forms $Q_i \in \bar{k}[x_0, \dots, x_n]$. The estimates in (1.1) and (1.2) are not entirely straightforward to compare, but it is clear that for the latter to supersede (1.1) one needs $h > 2(n - \sigma)$. One always has $1 \leqslant h \leqslant n$ and $h \geqslant \frac{1}{2}(n - \sigma)$ since any points with $x_i = Q_i = 0$ for $0 \leqslant i \leqslant h - 1$ are contained in the singular locus of X.

Our goal in the present investigation is to administer an improvement of the Lang–Weil estimate for cubic hypersurfaces under the most general conditions possible. This is the object of the following result.

Theorem *Suppose that $X \subset \mathbb{P}^n_k$ is a geometrically irreducible cubic hypersurface that is not a cone. Then for $n \geqslant 3$ we have*

$$|\#X(k) - q^{n-1}| \leqslant c_3(n)q^{n-2}$$

for a constant $c_3(n) > 0$ depending only on n.

The work of Bombieri [1] allows one to take $c_3(n) \leqslant 21^{2n+1}$ in the theorem. In the light of (1.1) and (1.2) one might try to establish the theorem by studying cubic hypersurfaces with singular locus of dimension $\sigma \geqslant n - 3$, or those for which the h-invariant does not exceed 7. We shall follow a different approach, basing our argument on hyperplane sections as in the work of Lang and Weil [8]. However, rather than using codimension one slices to reduce the analysis to curves, we will instead stop the induction procedure at surfaces.

2 Proof of the Theorem

Suppose that $X \subset \mathbb{P}^n_k$ is a geometrically irreducible cubic hypersurface that is not a cone, with $n \geqslant 3$. We will establish the theorem by induction on n. Bertini's theorem is valid for $q \geqslant c_4(n) > 0$ and ensures that generic hyperplane sections of X produce geometrically irreducible cubic hypersurfaces in $\mathbb{P}^{n-1}_k$ that are not cones. Thus the induction argument developed in [8] goes through unchanged, rendering it sufficient to focus on the case $n = 3$ of surfaces.

Let $X \subset \mathbb{P}^3_k$ be a geometrically irreducible cubic surface that is not equal to a cone over a cubic curve. Our aim is to establish the existence of an absolute constant $c > 0$ such that

$$|\#X(k) - q^2| \leqslant cq, \tag{2.1}$$

which will then suffice for the deduction of the theorem. To achieve this we will take advantage of the fact that $\#X(k) = \#Y(k) + O(q)$, for any surface Y that is birationally equivalent to X over k, since then X and Y may be identified on a non-empty Zariski open subset. Appealing to the identity $\#\mathbb{P}^2_k(k) = 1 + q + q^2$ we surmise that (2.1) holds true for k-rational cubic surfaces.

When X is a non-singular cubic surface, (2.1) follows directly from work of Davenport and Lewis [5], which was later refined by Swinnerton-Dyer [9] to affirm the Weil conjectures for non-singular cubic surfaces. We may therefore suppose that X

is singular and invoke the classification delineated by Bruce and Wall [2]. Thus it follows from our hypotheses on X that either it contains $\delta \leqslant 4$ isolated double points or else it contains a double line. In the latter case the double line must be defined over k, and so X is birational to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ over k. Hence (2.1) holds for such X. When X contains isolated double points we may appeal to work of Coray and Tsfasman [3, Lemma 1.1] to conclude that X is k-rational if any of the following come to pass:

- there is a singular point defined over k;
- $\delta = 1$ or 4;
- $\delta = 3$ and $X(k) \neq \varnothing$.

Since $X(k) \neq \varnothing$ by the Chevalley–Warning theorem, we deduce that (2.1) holds unless $\delta = 2$ and X contains a pair of double points that are conjugate to each other over a quadratic extension of k. The cubic surface need not be k-rational in this case, but it follows from [3, Proposition 4.8] that X is birational over k to a non-singular cubic surface with three coplanar k-rational lines and a further pair of coplanar conjugate lines. Since (2.1) holds for non-singular cubic surfaces, we therefore conclude that the estimate holds in every case.

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